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THE LAURENT EXPANSION WITHOUT CAUCHY'S INTEGRAL THEOREM

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1. **Introduction.** Since Cauchy's time the theory of analytic functions of a complex variable has depended on complex integration theory, and in particular on the fundamental integral theorem (1825) and integral formulas bearing his name. Cauchy defined an analytic function to be one which had a *continuous* first derivative in a region D, and showed that an analytic function had derivatives of all orders in D. It was not until 1900, with E. Goursat's famous proof of Cauchy's integral theorem, that the continuity of the first derivative could be inferred from its mere existence at all points of D. Although obviously not the first to observe this, it was remarked in 1939 by Titchmarsh [12, p. 71] that, although all of these results concerned only the complex *differential* calculus, "they all depend on the complex integral calculus".

This aesthetically rather unsatisfactory situation noted by Titchmarsh persisted until 1959 when Plunkett [8] succeeded in proving the continuity of the derivative without complex integration theory. His proof was topological, and depended ultimately on the fact that a maximum modulus principle could be proved for analytic functions without complex integration. (The underlying background in analytic topology had been developed in the years 1950–1955 by Whyburn [15], [16], Eggleston and Ursell [3], and Titus and Young [13].) Further results were obtained in the years 1961–62 by Connell [1] who proved the existence of derivatives of second (hence of all higher) order, Connell and Porcelli [2], Read [9], and the late G. T. Whyburn [17], [18]. In addition to the infinite differentiability property, these results included Cauchy's inequalities and the Taylor series expansion and were all proved without complex integration theory. In the revised edition of *Topological Analysis* published in 1964, Whyburn [19] gave a unified development of all of this material by reasonably elementary (although ingenious) methods; in particular, see his six page appendix "Topological Background for the Maximum Principle". A brief outline of the theory was also given by Whyburn in [20, pp. 30–38].

During the fall term of the 1969–70 academic year, the present author lectured on these topics to a small class of senior undergraduate students. This was their

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second half-course in complex analysis, the first having been at the usual introductory level and covering the standard topics in the traditional way. After having dealt with the relevant material in [19] without using complex integrals, we turned to a study of isolated singularities. Although this is usually dealt with by means of the Laurent expansion, I was reluctant to do so under the circumstances, this expansion not having been dealt with in [19]. (The students had, of course, all seen the Laurent expansion the previous year.) Instead, I used the well-known alternative approach wherein one defines and characterizes the kind of singularity by the behavior of the function in a deleted neighbourhood of the singular point. Nevertheless, it appeared to me that one should be able to obtain the general Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n, \qquad R_1 < |z-z_0| < R_2, \tag{1}$$

in the same way as the Taylor expansion, that is without the apparatus of Cauchy's integral theorem and formulas, and preferably without *any complex integration theory*. Unfortunately the course was almost over before I succeeded in doing so. The method is not quite in the spirit of Whyburn [19] inasmuch as we make use of the Riemann theory of integration of (complex-valued) functions of a real variable. In fact, the method consists simply in noting that if one sets $z=z_0+re^{i\theta}$, $R_1<r<R_2$, in (1), then (1) is just the Fourier expansion of the periodic function $F_r(\theta)=f(z_0+re^{i\theta})$. (This remark is not original with the author; see for example Whittaker and Watson [14, pp. 161, 162], Zygmund [21, p. 2], and especially Titchmarsh [12, pp. 401, 402]. In these references essentially only the *formal* connection of Laurent and Fourier series is noted, and no significant use is made of this fact.) The remainder of this paper is devoted to an elaboration of this remark, and to certain consequences of it, all in the general spirit of [19].

It is worth mentioning that Plunkett's theorem permits the replacement of Goursat's proof of Cauchy's integral theorem by the earlier proof based on Green's theorem, expressing a line integral in terms of a real double integral. Historically, we should also mention Stoïlow (see, for example [11]) whose point of view and specific results provided the orientation subsequently developed by Whyburn and his students.

Since completing the draft of this paper, the author has learned of some (as of this date, unpublished) work by K. O. Leland [4], [5], [6], [7] dealing with the Laurent expansion more strictly from the point of view of topological analysis. In [5] the Laurent expansion is obtained under the same assumptions used in this paper by using the Stone-Weierstrass theorem coupled with a Cauchy inequality for polynomials. In [7] the Laurent expansion and other results are obtained by this same polynomial approach, but assuming only the existence of the derivative in an annulus. The author would like to express his thanks to Professor Porcelli for letting him know of this work, and to Professor Leland for sending him preprints of [5] and [7].

The author also expresses his appreciation to the referee for several useful suggestions which have been adopted.

2. **The Laurent expansion.** As was hinted at in the introduction, the entire development of complex analysis without complex integration theory can be based on the following two theorems. (See Whyburn [19, pp. 74, 76].)

THEOREM 1. (*Weak maximum modulus principle for rectangles*). *If f is continuous on a rectangle C and its interior R, and differentiable on R-F where F is a finite set of points, then $|f(z)| \leq M$ on C implies $|f(z)| \leq M$ on $R \cup C$.*

THEOREM 2. (*Plunkett*). *If f is continuous in a region D and differentiable on D-F where F is a finite set, then f is differentiable and f' is continuous at all points of D.*

As can be seen in [19], the conclusions of Theorems 1 and 2 are valid under much weaker hypotheses. In order to obtain our main results we shall not even require the full strength of Theorem 2. *Indeed, we emphasize that all we shall use is the fact that if f is differentiable (analytic) in a region D, then f' is continuous in D.* Our first step is to note that if f is analytic in D and $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$, then (by the usual proof), u and v are *continuously* differentiable in D and satisfy the Cauchy-Riemann equations

$$u_x(x, y) = v_y(x. y), \qquad u_y(x, y) = -v_x(x, y), \qquad \forall z = x+iy \in D.$$

Writing $z = re^{i\theta}$, so $x = r\cos\theta$, $y = r\sin\theta$ and

$$f(z) = u(r\cos\theta, r\sin\theta) + iv(r\cos\theta, r\sin\theta) = U(r, \theta) + iV(r, \theta),$$

it follows by elementary calculus that U, V are also *continuously* differentiable for all (r, θ) with $re^{i\theta} \in D$, and that the Cauchy-Riemann equations in polar form,

$$U_\theta(r, \theta) = -rV_r(r, \theta), \qquad V_\theta(r, \theta) = rU_r(r, \theta), \qquad z = re^{i\theta} \in D, \tag{2}$$

are satisfied.

THEOREM 3. (*Laurent expansion*). *Let f be analytic in the annulus*

$$D: R_1 < |z - z_0| < R_2,$$

where $0 \leq R_1 < R_2 \leq +\infty$. Then f has a unique expansion in D of the form

$$f(z) = \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n, \tag{3}$$

where the two (power) series on the right side of (3) *are almost uniformly convergent* *in* $|z-z_0|>R_1$ *and* $|z-z_0|<R_2$ *respectively. The coefficients* a_n *are given by*

$$(4)\quad a_n = \frac{1}{2\pi}\int_0^{2\pi}(re^{i\theta})^{-n}f(z_0+re^{i\theta})\,d\theta \qquad -\infty < n < +\infty,\quad R_1 < r < R_2,$$

the value of the integrals (4) *being independent of* r.

Proof. Setting $z-z_0=\zeta=re^{i\theta}$; $g(\zeta)=f(z_0+\zeta)=F_r(\theta)=U(r,\theta)+iV(r,\theta)$ for $R_1<r<R_2$, $0\le\theta\le 2\pi$, it follows from the preceding remarks that F_r is periodic with period 2π, and has the *continuous* derivative $F_r'=U_r+iV_r$. By an elementary result for Fourier series relative to the orthogonal system

$$\left\{\frac{1}{2\pi}e^{in\theta}\,|\,n=0,\pm 1,\pm 2,\dots\right\},$$

F_r has a Fourier expansion which converges *to* $F_r(\theta)$ on $[0, 2\pi]$, given by

$$(5)\qquad \sum_{n=-\infty}^{\infty} A_n(r)e^{in\theta} = F_r(\theta) = f(z_0+re^{i\theta}),$$

where

$$(6)\qquad A_n(r) = \left(F_r, \frac{1}{2\pi}e^{in\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})e^{-in\theta}\,d\theta.$$

Hence, if we define $a_n(r)$ by (4), so $A_n(r)=r^n a_n(r)$, then (5) reduces to (3). The proof of the theorem thus reduces essentially to proving that the integrals in (4) are independent of r.

We shall show that for all n,

$$(7)\qquad \frac{d}{dr}\int_0^{2\pi} r^n e^{in\theta}\{U(r,\theta)+iV(r,\theta)\}\,d\theta = 0, \qquad R_1 < r < R_2$$

which will prove the main part of the theorem. If $n=0$, then using the *continuity* of the partial derivatives U_r and V_r on $(R_1, R_2)\times[0, 2\pi]$, the left side of (7) becomes

$$\frac{d}{dr}\int_0^{2\pi}(U+iV)\,d\theta = \int_0^{2\pi}(U_r+iV_r)\,d\theta,$$

so by (2),

$$r\frac{d}{dr}\int_0^{2\pi}(U+iV)\,d\theta = \int_0^{2\pi}(V_\theta - iU_\theta)\,d\theta = V(r,\theta)-iU(r,\theta)\Big|_0^{2\pi} = 0,$$

proving (7) for $n=0$. Similarly, if $n\neq 0$,

$$\begin{aligned} r\frac{d}{dr}&\int_0^{2\pi} r^n e^{in\theta}(U+iV)\,d\theta \\ &= n\int_0^{2\pi} r^n e^{in\theta}(U+iV)\,d\theta + \int_0^{2\pi} r^n e^{in\theta}(V_\theta - iU_\theta)\,d\theta \\ &= n\int_0^{2\pi} r^n e^{in\theta}(U+iV)\,d\theta + r^n e^{in\theta}(V-iU)\Big|_0^{2\pi} - in\int_0^{2\pi} r^n e^{in\theta}(V-iU)\,d\theta \\ &= 0, \end{aligned}$$

completing the proof of (7).

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Because of the way we obtained (3) as a Fourier series expansion, what we have actually proved is that

$$\lim_{n\to\infty} \sum_{k=-n}^{n} a_k(z-z_0)^k = f(z) \quad \text{for} \quad R_1 < |z-z_0| < R_2.$$

It is, however, easy to see that both series on the right side of (3) *converge, the first almost uniformly in* $|z-z_0|>R_1$, *the second almost uniformly in* $|z-z_0|<R_2$. For, in the first case, given any r, $R_1<r<\infty$, we have on choosing r_1 so that $R_1<r_1<\min(r, R_2)$,

$$|a_{-n}(z-z_0)^{-n}| \le |a_{-n}|\, r^{-n} \le \frac{r^{-n}}{2\pi}\int_0^{2\pi} r_1^n\, |f(z_0+r_1e^{i\theta})|\, d\theta \le M(r_1)\left(\frac{r_1}{r}\right)^n$$

for $|z-z_0|\ge r$, $n\ge 1$, where $M(r_1)=\max_{|z-z_0|=r_1} |f(z)|$. By the Weierstrass M-test, the series $\sum_1^\infty a_{-n}(z-z_0)^{-n}$ is uniformly convergent on $|z-z_0|\ge r$. Similarly, the other series in (3) converges uniformly on $|z-z_0|\le r$ for each $r<R_2$.

It only remains to prove the *uniqueness* of the Laurent expansion (3). Suppose we also had

$$(3') \qquad f(z) = \sum_{n=1}^{\infty} a_{-n}^1(z-z_0)^{-n} + \sum_{n=0}^{\infty} a_n^1(z-z_0)^n \quad \text{for} \quad R_1 < |z-z_0| < R_2.$$

Setting $z=z_0+re^{i\theta}$, $0\le\theta\le 2\pi$, it follows from the (necessarily) *uniform* convergence of the two series on the right side of (3′) on the circle $|z-z_0|=r$, that the trigonometric series

$$(8) \qquad \sum_{n=-\infty}^{\infty} a_n^1 r^n e^{in\theta} = f(z_0+re^{i\theta}) = F_r(\theta), \qquad 0\le\theta\le 2\pi,$$

is *uniformly* convergent on $[0, 2\pi]$. But then it follows from an elementary result in the theory of Fourier series that the coefficients $a_n^1 r^n$ in (8) are necessarily the Fourier coefficients $A_n(r)$ defined by (6). Hence $a_n^1\equiv a_n$, and the uniqueness of the Laurent expansion of f in D is proved.

REMARK. By using the deeper Heine-Cantor theorem on trigonometric series, (see, for example, Rogosinski [10, p. 144]) the conclusion $a_n^1\equiv a_n$ could be deduced from the weaker assumption that (8) holds for even a single $r\in(R_1, R_2)$.

3. **Consequences of the Laurent expansion theorem.** In this section we shall show that all of the results obtained by a variety of ingenious arguments in Whyburn [19, pp. 77–82] follow easily and naturally from Theorem 3. In particular we shall obtain the usual Taylor series expansion, deduce the existence of derivatives of all orders (with an "integral formula"), Riemann's theorem on removable singularities, Cauchy's inequalities, and a strong form of the maximum modulus principle.

Suppose that f is analytic in the disk $|z-z_0|<R_2$. Then we may take $R_1=0$ in Theorem 3, and may let $r\to 0+$ in (4). Since f is analytic at z_0, it is continuous, and

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hence bounded in some neighbourhood $N_\varepsilon(z_0)$. But then, if $|f(z)|\leq M$ for $z\in N_\varepsilon(z_0)$, it follows that for $n=1, 2, \ldots,$

$$|a_{-n}| = \frac{1}{2\pi}\left|\int_0^{2\pi}(re^{i\theta})^n f(z_0+re^{i\theta})\,d\theta\right| \leq Mr^n, \qquad 0<r<\varepsilon.$$

That is, $a_{-n}=0$ for $n=1, 2, \ldots,$ so (3) reduces to

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \qquad |z-z_0| < R_2, \tag{9}$$

where the coefficients a_n are given by (4) for any $r\in(0, R_2)$. From (9) it follows by elementary theorems on power series that for $k\geq 0$,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-z_0)^{n-k}, \qquad |z-z_0| < R_2,$$

and hence that f has derivatives of all orders, which at z_0 are given by

$$f^{(k)}(z_0) = k!a_k = \frac{k!}{2\pi}\int_0^{2\pi}(re^{i\theta})^{-k} f(z_0+re^{i\theta})\,d\theta \qquad (0<r<R_2). \tag{10}$$

We thus have the result that if f is analytic at z_0, then f has the Taylor expansion

$$f(z) = \sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k, \tag{11}$$

this series being convergent in the largest disk (centered at z_0) in which f is analytic. Incidentally, we note that in order to obtain the representation (9) we only required the *boundedness* of f in a deleted neighborhood of z_0. That is, we also obtain (see Whyburn [19, p. 76])

Riemann's Theorem on Removable Singularities: If f is bounded and differentiable on $D-z_0$, where D is a region containing z_0, then $a_0=\lim_{z\to z_0} f(z)$ exists, and if we define $f(z_0)=a_0$, then f is differentiable at z_0.

Because of the uniqueness proved for the expansion (3) and inherited by the expansions (9) and (11), we also obtain *Cauchy's inequalities* from (10):

$$|a_n| = \frac{|f^{(n)}(z_0)|}{n!} \leq \frac{M(r)}{r^n} \quad \textit{for} \quad 0<r<R_2, \qquad n=0, 1, 2, \ldots, \tag{12}$$

where $M(r)=\max_{|z-z_0|=r}|f(z)|$. One can, in fact (see Titchmarsh [12, p. 84]) proceed directly from (9) without using the integral formula (10) to obtain the more powerful inequality

$$\sum_{n=0}^{\infty}|a_n|^2 r^{2n} \leq M^2(r), \qquad 0<r<R_2. \tag{13}$$

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Now, to obtain the maximum modulus principle, suppose f is analytic in a region D. By using the *Cauchy mean value theorem*

$$f(z_0)=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta, \quad \text{where } N_r(z_0)\subset D, \tag{14}$$

(which is just (10) for $k=0$) the standard argument (see, for example [12, p. 165]) shows that $|f|$ does not attain a maximum value at any point $z_0\in D$ unless f is constant in D. It then follows at once that if D is bounded and $|f|$ is continuous in $\bar{D}$, then $|f(z)|<M=\max_{\zeta\in\partial D}|f(\zeta)|$ holds for all $z\in D$ unless f is constant in D, and this is the usual strong form of the maximum modulus principle (cf. Whyburn [19, p. 79]).

As a final application, this time to the complex *integral* calculus, let us suppose that line integrals $\int_\gamma g(\zeta)\,d\zeta$ have been defined along appropriate (for example, rectifiable) *paths* γ, and shown to exist for functions g which are *continuous* on γ. (γ need not consist of a single continuous curve but must be compact and have finite length). We shall prove

THEOREM 4. *Let γ be a path and D a region, in the complex plane* (*not necessarily having points in common*). *Suppose that $f(z,\zeta)$ is continuous on $D\times\gamma$ and that for each $\zeta\in\gamma$, $f(z,\zeta)$ is analytic in D. Then the function F defined by*

$$F(z)=\int_\gamma f(z,\zeta)\,d\zeta, \qquad z\in D,$$

is analytic in D, and for each $n\geq 1$,

$$F^{(n)}(z)=\int_\gamma \frac{\partial^n f(z,\zeta)}{\partial z^n}\,d\zeta, \qquad z\in D. \tag{15}$$

Proof. For any $z\in D$, $\exists R=R_z>0$ such that the disk $K:|t-z|\leq R$, lies in D. By (10), with $k=1$, we have, for each $\zeta\in\gamma$,

$$f_1(z,\zeta)\equiv\frac{\partial f(z,\zeta)}{\partial z}=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(z+re^{i\theta},\zeta)}{re^{i\theta}}\,d\theta, \qquad 0<r<R.$$

Hence, if $\zeta+k\in\gamma$ and $|h|\leq r<R/2$,

$$f_1(z+h,\zeta+k)-f_1(z,\zeta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{i}{re^{i\theta}}\{f(z+h+re^{i\theta},\zeta+k)-f(z+re^{i\theta},\zeta)\}\,d\theta.$$

But since f is uniformly continuous on the compact set $K\times\gamma$, it now follows at once that f_1 is continuous at (z,ζ), and hence on $D\times\gamma$. In particular, it follows that for each $z\in D$, the integral $\int_\gamma f_1(z,\zeta)\,d\zeta$ *exists*. Now, if $z\in D$, $z+h\in D$, and $h\neq 0$, we have

$$\frac{F(z+h)-F(z)}{h}-\int_\gamma f_1(z,\zeta)\,d\zeta=\int_\gamma\left\{\frac{f(z+h,\zeta)-f(z,\zeta)}{h}-f_1(z,\zeta)\right\}d\zeta.$$

Taking K and r as above, it follows that for $0<|h|\le r/2$,

$$\left|\frac{f(z+h,\zeta)-f(z,\zeta)}{h}-f_1(z,\zeta)\right| = \left|h\sum_{n=2}^{\infty}\frac{f_n(z,\zeta)}{n!}h^{n-2}\right|$$

$$\le |h|\sum_{n=2}^{\infty} Mr^{-n}|h|^{n-2} \le \frac{2M}{r^2}|h|, \qquad \forall\zeta\in\gamma,$$

where $M=\max_{K\times\gamma}|f(z,\zeta)|$, and $f_n(z,\zeta)\equiv\partial^n f(z,\zeta)/\partial z^n$. Since γ has finite length, it now follows at once that $F'(z)$ exists, so that F is analytic in D, and moreover that (15) is valid for $n=1$. Since we have shown that the function f_1 satisfies the same hypotheses on $D\times\gamma$ as f, we also obtain (15) for $n=2$. The result follows for all n by induction.

REMARK. The usual proof of this result (see, for example, Titchmarsh [12, p. 99]) uses the general form of Cauchy's integral formulas, and a (complete) proof by this method is neither so simple nor so transparent as the above proof.

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