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JACOBSON RADICAL ALGEBRAS WITH QUADRATIC GROWTH

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Abstract. We show that over every countable algebraically closed field $\mathbb{K}$ there exists a finitely generated $\mathbb{K}$-algebra that is Jacobson radical, infinite-dimensional, generated by two elements, graded and has quadratic growth. We also propose a way of constructing examples of algebras with quadratic growth that satisfy special types of relations.

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1. Introduction. Algebras with linear growth were described by Small et al. [**5**]. Bergman [**2**, p. 18] proved that algebras with growth function smaller than $f(n) = \frac{n(n+1)}{2}$ have linear growth. This raises the following question: What properties should algebras with a growth function close to $f(n) = \frac{n(n+1)}{2}$ satisfy? Examples of primitive algebras with very small growth functions were constructed by Uzi Vishne using Morse trajectories [**9**]. Bartholdi [**1**] constructed self-similar algebras with very small growth functions over the field $\mathbb{F}_2$ which are graded nil. In fact, all algebras constructed in [**1**] are primitive and hence not Jacobson radical (as mentioned in [**8**]).

In this paper, we will construct an example with growth function bounded above by $n^2 + 4n + 3$, which is both infinite dimensional and Jacobson radical. It is unclear whether this algebra is nil. We will also present a way to construct other examples that are bounded above by the same growth function.

Recall that non-nil Jacobson radical algebras with the Gelfand–Kirillov dimension two were constructed in [**8**], and nil algebras with the Gelfand–Kirillov dimension not exceeding three were constructed in [**4**]. It is not known if there are nil algebras with quadratic growth, or more generally with the Gelfand–Kirillov dimension two.

Our main result is as follows:

THEOREM 1.1. *Over every countable, algebraically closed field $\mathbb{K}$ there exists a finitely generated $\mathbb{K}$ algebra that is Jacobson radical, infinite dimensional, generated by two elements, graded and has quadratic growth.*

In addition, we also propose a new way of constructing examples of algebras with quadratic growth satisfying special types of relations (see Theorem 6.3).

2. Notation and proof outline. In what follows, $\mathbb{K}$ is a countable field, and $A = \mathbb{K}\langle x, y\rangle$ is the free $\mathbb{K}$-algebra in two non-commuting indeterminates x and y. The monomials in this algebra will be the products of the form $x_1 \cdots x_n$, with each $x_i \in \{x, y\}$ (whereas the monomials *with coefficient* will be of the form $kx_1 \cdots x_n$ with $k \in \mathbb{K}$). The degree of a monomial is the length of this product. For any $n \geq 0$, $H(n)$ will denote the homogeneous subspace of degree n: the $\mathbb{K}$-space generated by the degree-n monomials. Finally, $\bar{A} = \sum_{n=1}^{\infty} H(n)$ will be the $\mathbb{K}$-space of polynomials with no constant term.

Proof outline for Theorem 1.1 is as follows:

- In Section 6, an increasing sequence of natural numbers N_i is fixed and subspaces $F_i \subseteq H(2^{N_i})$ are constructed such that for every element $f \in \bar{A}$ there is $g \in \bar{A}$ such that $f + g - fg \in \mathcal{E}(F_i)$ for some i. The set $\mathcal{E}(F_i)$ is defined in Section 5.
- In Section 3, for fixed subspaces F_i, subspaces $U(2^n)$, $V(2^n) \subseteq H(2^n)$ are constructed inductively for $n = 1, 2, \ldots$ This part bears resemblance to results from [**3**]. Properties that the $V(2^n)$ spaces exhibit include $V(2^n) \subseteq V(2^{n-1})^2$ and $\dim V(2^n) = 2$, the latter being instrumental in establishing quadratic growth. Our conditions guarantee that each set F_i is in $U(2^{N_i})$.
- In Section 4, we introduce the ideal E, whose construction uses the sets $U(2^n)$ and $V(2^n)$ in order to arrive at our desired quotient, A/E. Note that the ideal E is defined differently than defined in [**3**]. We then find an upper bound of the growth of A/E.
- In Sections 5 and 6 we show that the algebra A/E is Jacobson radical.
- The proof of Theorem 1.1 is concluded in Section 6.

3. Constructing sets $U(2^n)$ and $V(2^n)$. Suppose we have a strictly increasing sequence of natural numbers

$$\{N_i\}_{i=0}^{\infty}$$

with $N_0 = 1$, and a sequence of homogeneous subspaces $\{F_i\}_{i=0}^{\infty}$ with each

$$F_i \subseteq H\left(2^{N_i}\right)$$

and $F_0 = (0)$.

In this section we will show that, for every $i \geq 0$, there exists a subspace $U_i \subset H(2^i)$ and two monomials (with non-zero coefficient) $v_{i,1}, v_{i,2} \in H(2^i)$ such that for each $i \geq 0$:

1. $U_i \oplus \mathbb{K}v_{i,1} \oplus \mathbb{K}v_{i,2} = H(2^i)$.
2. There exists $v \in \mathbb{K}v_{i,1} + \mathbb{K}v_{i,2}$ such that $U_{i+1} = H(2^i)U_i + U_iH(2^i) + vH(2^i)$.
3. $F_i \subseteq U_{N_i}$.

We will eventually set $V_i = \mathbb{K}v_{i,1} + \mathbb{K}v_{i,2}$ so that $U_i \oplus V_i = H(2^i)$.

We shall attack the problem with induction. For the base case, set U_0 as an arbitrary subspace of $H(1)$ with $\dim U_0 = \dim H(1) - 2$, and set $v_{0,1}, v_{0,2}$ as two linearly independent monomials such that $U_0 + \mathbb{K}v_{0,1} + \mathbb{K}v_{0,2} = H(1)$.

For the inductive step, assume the existence of $U_{N_i}, v_{N_i,1}, v_{N_i,2}$ for some $i \geq 0$, and find possible $U_k, v_{k,1}, v_{k,2}$ for all $N_i < k \leq N_{i+1}$.

Let

$$W \cong \mathbb{K}^{2(N_{i+1}-N_i)}$$

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be a $\mathbb{K}$-space with indices $\{x_{k,1}, x_{k,2}\}_{k=N_i}^{N_{i+1}-1}$, W_k be the subspace of all elements where $(x_{k,1}, x_{k,2}) = (0, 0)$ and

$$\overline{W} = W \backslash \bigcup_{k=N_i}^{N_{i+1}-1} W_k.$$

Given some vector $\vec{w} \in \overline{W}$, define a subspace $U_k(\vec{w})$ and elements $v_{k,1}(\vec{w})$, $v_{k,2}(\vec{w})$ in $H(2^k)$ recursively for each $N_i \leq k \leq N_{i+1}$ as follows: First, set $U_{N_i}(\vec{w}) = U_{N_i}$, $v_{N_i,1}(\vec{w}) = v_{N_i,1}$, $v_{N_i,2}(\vec{w}) = v_{N_i,2}$. Then, assuming $U_k(\vec{w})$, $v_{k,1}(\vec{w})$, $v_{k,2}(\vec{w})$ are defined for some $N_i \leq k < N_{i+1}$:

$$U_{k+1}(\vec{w}) = H(2^k)U_k(\vec{w}) + U_k(\vec{w})H(2^k) + (x_{k,2}(\vec{w})v_{k,1}(\vec{w}) - x_{k,1}(\vec{w})v_{k,2}(\vec{w}))H(2^k).$$

If $x_{k,1}(\vec{w}) \neq 0$, set:

$$v_{k+1,1}(\vec{w}) = x_{k,1}(\vec{w})^{-1}v_{k,1}^2(\vec{w}),$$
$$v_{k+1,2}(\vec{w}) = x_{k,1}(\vec{w})^{-1}v_{k,1}(\vec{w})v_{k,2}(\vec{w}),$$

and if $x_{k,1}(\vec{w}) = 0$, then $x_{k,2}(\vec{w}) \neq 0$, so set:

$$v_{k+1,1}(\vec{w}) = x_{k,2}(\vec{w})^{-1}v_{k,2}(\vec{w})v_{k,1}(\vec{w}),$$
$$v_{k+1,2}(\vec{w}) = x_{k,2}(\vec{w})^{-1}v_{k,2}^2(\vec{w}).$$

For any $\vec{w} \in \overline{W}$, this clearly satisfies conditions (1) and (2).

LEMMA 3.1. *Let $k \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $N_i \leq k < N_{i+1}$. If $a, b \in \{1, 2\}$ and $\vec{w} \in \overline{W}$, then:*

$$v_{k,a}(\vec{w})v_{k,b}(\vec{w}) \in x_{k,a}(\vec{w})v_{k+1,b}(\vec{w}) + U_{k+1}(\vec{w}).$$

Proof. If $x_{k,1}(\vec{w}) \neq 0$ and $a = 1$, $v_{k,a}(\vec{w})v_{k,b}(\vec{w}) = x_{k,a}(\vec{w})v_{k+1,b}(\vec{w})$. Similarly, if $x_{k,1}(\vec{w}) \neq 0$ and $a = 2$, then $v_{k,a}(\vec{w})v_{k,b}(\vec{w}) = x_{k,a}(\vec{w})v_{k+1,b}(\vec{w}) - x_{k,1}(\vec{w})^{-1}(x_{k,2}(\vec{w})v_{k,1}(\vec{w}) - x_{k,1}(\vec{w})v_{k,2}(\vec{w}))v_{k,b}(\vec{w})$. If $x_{k,1}(\vec{w}) = 0$ and $a = 1$, then $v_{k,a}(\vec{w})v_{k,b}(\vec{w}) = x_{k,2}(\vec{w})^{-1}(x_{k,2}(\vec{w})v_{k,1}(\vec{w}) - x_{k,1}(\vec{w})v_{k,2}(\vec{w}))v_{k,b}(\vec{w})$. If $x_{k,1}(\vec{w}) = 0$ and $a = 2$, $v_{k,a}(\vec{w})v_{k,b}(\vec{w}) = x_{k,2}(\vec{w})v_{k+1,b}(\vec{w})$. □

Let

$$P = \mathbb{K}[x_{k,1}, x_{k,2}]_{k=N_i}^{N_{i+1}-1},$$

i.e. the (commutative) algebra of polynomial functions $W \to \mathbb{K}$. Let

$$Q = \prod_{k=N_i}^{N_{i+1}-1} (\mathbb{K}x_{k,1} + \mathbb{K}x_{k,2})^{2^{N_{i+1}-k-1}}$$

be a homogenous subspace of P.

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THEOREM 3.2. *For any sequence* $\{s_k\}_{k=1}^{2^{N_{i+1}-N_i}}$ *taking values in* $\{1,2\}$*, there exists some* $p_s \in Q$ *such that for any* $\vec{w} \in \overline{W}$,

$$\prod_{k=1}^{2^{N_{i+1}-N_i}} v_{N_i,s_k} \in p_s(\vec{w}) v_{N_{i+1},s_{2^{N_{i+1}-N_i}}}(\vec{w}) + U_{N_{i+1}}(\vec{w}).$$

Proof. We will use induction to show that, for any $0 \leq h \leq N_{i+1} - N_i$ and any sequence $\{s_k\}_{k=1}^{2^h}$ taking values in $\{1,2\}$,

$$\prod_{k=1}^{2^h} v_{N_i,s_k} \in \left(\prod_{j=0}^{h-1} \prod_{k=1}^{2^{h-j-1}} x_{N_i+j,s_{2^j(2k-1)}}(\vec{w}) \right) v_{N_i+h,s_{2^h}}(\vec{w}) + U_{N_i+h}(\vec{w}),$$

with the end result of the theorem proven when $h = N_{i+1} - N_i$.

The base case is simply $v_{N_i,s_1} \in v_{N_i,s_1}(\vec{w}) + U_{N_i}(\vec{w})$.

For the inductive step, let $\{s_k\}_{k=1}^{2^{h+1}}$ be a sequence taking values in $\{1,2\}$ and assume the inductive statement is true for $\{s_k\}_{k=1}^{2^h}$ and $\{s_k\}_{k=2^h+1}^{2^{h+1}}$. Lemma 3.1 shows that:

$$v_{N_i+h,s_{2^h}}(\vec{w}) v_{N_i+h,s_{2^{h+1}}}(\vec{w}) \in x_{N_i+h,s_{2^h}}(\vec{w}) v_{N_i+h+1,s_{2^{h+1}}}(\vec{w}) + U_{N_i+h+1}(\vec{w}).$$

Therefore,

$$\begin{aligned}
\prod_{k=1}^{2^{h+1}} v_{N_i,s_k} \in & \left(\left(\prod_{j=0}^{h-1} \prod_{k=1}^{2^{h-j-1}} x_{N_i+j,s_{2^j(2k-1)}}(\vec{w}) \right) v_{N_i+h,s_{2^h}}(\vec{w}) + U_{N_i+h}(\vec{w}) \right) \cdot \\
& \left(\left(\prod_{j=0}^{h-1} \prod_{k=1}^{2^{h-j-1}} x_{N_i+j,s_{2^j(2k-1)+2^h}}(\vec{w}) \right) v_{N_i+h,s_{2^{h+1}}}(\vec{w}) + U_{N_i+h}(\vec{w}) \right) \\
\subseteq & \left(\prod_{j=0}^{h-1} \prod_{k=1}^{2^{h-j}} x_{N_i+j,s_{2^j(2k-1)}}(\vec{w}) \right) x_{N_i+h,s_{2^h}}(\vec{w}) v_{N_i+h+1,s_{2^{h+1}}}(\vec{w}) + U_{N_i+h+1}(\vec{w}) \\
= & \left(\prod_{j=0}^{h} \prod_{k=1}^{2^{h-j}} x_{N_i+j,s_{2^j(2k-1)}}(\vec{w}) \right) v_{N_i+h+1,s_{2^{h+1}}}(\vec{w}) + U_{N_i+h+1}(\vec{w}).
\end{aligned}$$

□

COROLLARY 3.3. *For any* $f \in H(2^{N_{i+1}})$*, there exist* $p, q \in Q$ *such that, for all* $\vec{w} \in \overline{W}$, $f \in p(\vec{w}) v_{N_{i+1},1}(\vec{w}) + q(\vec{w}) v_{N_{i+1},2}(\vec{w}) + U_{N_{i+1}}(\vec{w})$.

Proof. First, note that:

$$\begin{aligned}
H(2^{N_{i+1}}) &= (U_{N_i} + \mathbb{K} v_{N_i,1} + \mathbb{K} v_{N_i,2})^{2^{N_{i+1}-N_i}} \\
&= (\mathbb{K} v_{N_i,1} + \mathbb{K} v_{N_i,2})^{2^{N_{i+1}-N_i}} + \sum_{k=1}^{2^{N_{i+1}-N_i}} H((k-1)2^{N_i}) U_{N_i} H\big(2^{N_{i+1}} - k2^{N_i}\big)
\end{aligned}$$

and that for each $f \in H(2^{N_{i+1}})$ there exists $f' \in (\mathbb{K} v_{N_i,1} + \mathbb{K} v_{N_i,2})^{2^{N_{i+1}-N_i}}$ such that for any $\vec{w} \in \overline{W}$, $f \in f' + U_{N_{i+1}}(\vec{w})$.

Since f' can be written as a linear combination of the elements of the form $\prod_{k=1}^{2^{N_{i+1}}} v_{N_i,s_k}$, it is sufficient to prove that the corollary holds when f is one of these elements, which is done in Theorem 3.2. □

Let

$$d = \dim F_{i+1}$$

and $\{f_k\}_{k=1}^{d}$ be elements that generate F_{i+1} and let

$$\{p_k, q_k\} \subseteq Q$$

be such that, for all $\vec{w} \in \overline{W}$,

$$f_k \in p_k(\vec{w})v_{N_{i+1},1}(\vec{w}) + q_k(\vec{w})v_{N_{i+1},2}(\vec{w}) + U_{N_{i+1}}(\vec{w}),$$

as detailed in Corollary 3.3. If there exists a $\vec{w} \in \overline{W}$ such that each $p_k(\vec{w}) = q_k(\vec{w}) = 0$, then we can set $(U_k, v_{k,1}, v_{k,2}) = (U_k(\vec{w}), v_{k,1}(\vec{w}), v_{k,2}(\vec{w}))$, and Condition (3) can be satisfied.

Let

$$G = \sum_{k=1}^{d} \mathbb{K}p_k + \mathbb{K}q_k \subseteq Q$$

be the vector space generated by $\{p_k, q_k\}$. Our remaining goal is to show that there exists $\vec{w} \in \overline{W}$ such that $G(\vec{w}) = (0)$.

Let R be the algebra generated by Q, i.e.

$$R = \sum_{k=1}^{\infty} Q^k.$$

LEMMA 3.4. *If G, P are defined as above, then:*

$$R \cap GP \subseteq G + GR.$$

Proof. Let M be the set of all monomials of P (without coefficient). Let M_Q be the monomials that generate Q, $M_R = \bigcup_{j=1}^{\infty} M_Q^j$ be the monomials that generate R and $M'_R = M \backslash (M_R \cup \{1\})$. P can be decomposed: $P = \mathbb{K} \oplus R \oplus \mathbb{K}M'_R$.

Note that for any $m \in M_Q$ and any $m' \in M'_R$, $mm' \in M'_R$. As R is generated by monomials, $R \cap QM'_R = (0)$.

Let $g \in G$, and let $p \in P$ have the decomposition $p = k + r + s$, with $k \in \mathbb{K}$, $r \in R$ and $s \in \mathbb{K}M'_R$. Suppose that $gp \in R$. Since $gk + gr \in R$, $gs \in R \cap QM'_R = (0)$. Therefore, $gp \in \mathbb{K}g + gR$, and $R \cap GP \subseteq G + GR$. □

THEOREM 3.5. *If $\{\vec{w} \in W : G(\vec{w}) = (0)\} \subseteq W \backslash \overline{W} = \bigcup_{k=N_i}^{N_{i+1}-1} W_k$, then $d \geq \frac{1}{2}(N_{i+1} - N_i + 1)$.*

Proof. Given an ideal I of P, we define $Z(I) = \{\vec{w} \in W : I(\vec{w}) = (0)\}$. This is an affine subvariety of W. It is our goal to show that if $Z(GP) \subseteq \bigcup_{k=N_i}^{N_{i+1}-1} W_k$, then $d \geq \frac{1}{2}(N_{i+1} - N_i + 1)$.

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Since Q annihilates each W_k, it must annihilate $Z(GP)$ as well. Hilbert's Nullstellensatz states that since $\mathbb{K}$ is algebraically closed, for each $q \in Q$, there must be an exponent $q^\pi \in GP$.

Using Lemma 3.4, $q^\pi \in R \cap GP \subseteq G + GR$, and so the quotient algebra $R/(G + GR)$ is nil. Since $G^2 \subseteq GR$, R/GR is nil as well. All finitely generated commutative nil algebras are finite-dimensional, so applying Lemma 4.2 in [7] several times gives $2d \geq \operatorname{GKdim} R$. Recall that Lemma 4.2 [7] says that if R is a commutative finitely generated graded algebra of Gelfand–Kirillov dimension t, and I is a principal ideal generated by a homogeneous element, then R/I has the Gelfand–Kirillov dimension at least $t - 1$.

Recall that for any $j \geq 0$, $Q^j = \prod_{k=N_i}^{N_{i-1}-1}(\mathbb{K}x_{k,1} + \mathbb{K}x_{k,2})^{j2^{N_{i+1}-k-1}}$, and that:

$$\dim Q^j = \prod_{k=N_i}^{N_{i+1}-1} (j2^{N_{i+1}-k-1} + 1) \geq 2^{\frac{1}{2}(N_{i+1}-N_i-1)(N_{i+1}-N_i)} j^{N_{i+1}-N_i}.$$

Therefore, $\operatorname{GKdim} R \geq N_{i+1} - N_i + 1$. □

We can thus conclude that, as long as $\dim F_{i+1} < \frac{1}{2}(N_{i+1} - N_i + 1)$, there is a $\vec{w} \in \overline{W}$ such that $G(\vec{w}) = 0$, and we have appropriate spaces $\{U_k\}$ and monomials $\{v_{k,1}, v_{k,2}\}$ for all $k \leq N_{i+1}$. If this holds for all $i \geq 0$, the induction can proceed.

4. Constructing the ideal E. For any $i \geq 0$, let $V_i = \mathbb{K}v_{i,1} + \mathbb{K}v_{i,2}$ (where $v_{i,1}$, $v_{1,2}$ are as in Property (1), Section 3), let $v_i \in V_i$ be such that $U_{i+1} = H(2^i)U_i + U_iH(2^i) + v_iH(2^i)$ and let $Q_i = U_i + \mathbb{K}v_i$ (v_i exists by Property (2), Section 3). If $v_{i,1} \notin \mathbb{K}v_i$, let $W_i = \mathbb{K}v_{i,1}$, otherwise $W_i = \mathbb{K}v_{i,2}$. This way $Q_i \oplus W_i = H(2^i)$, $U_{i+1} = H(2^i)U_i + Q_iH(2^i)$ and $V_{i+1} = W_iV_i$.

PROPOSITION 4.1. *For any $j > i$ and any $k \leq 2^{j-i} - 1$,*

$$H(k2^i)U_iH(2^j - (k+1)2^i) \subseteq U_j.$$

Proof. Apply induction on the value of j by using $H(2^i)U_i + U_iH(2^i) \subseteq U_{i+1}$. □

For any $n > 0$, let $m \geq 0$ be maximal such that $2^m \leq n$, and define:

$$R(n) = \{x \in H(n) : xH(2^{m+1} - n) \subseteq U_{m+1}\},$$

$$L(n) = \{x \in H(n) : H(2^{m+1} - n)x \subseteq U_{m+1}\}.$$

Also, set $R(0) = L(0) = (0)$.

PROPOSITION 4.2. *For any $n > 0$ and any M such that $2^M > n$,*

$$R(n)H(2^M - n) \subseteq U_M,$$
$$H(2^M - n)L(n) \subseteq U_M.$$

Proof. Apply induction on M, using the fact that $H(2^M)U_M + U_MH(2^M) \subseteq U_{M+1}$. □

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PROPOSITION 4.3. *For any* $n > 0$, $R(n)H(1) \subseteq R(n+1)$ *and* $H(1)L(n) \subseteq L(n+1)$.

Proof. Let $m \geq 0$ be maximal such that $2^m \leq n$. If $2^{m+1} - 1 < n$ then:

$$R(n)H(1) \cdot H(2^{m+1} - n - 1) = R(n)H(2^{m+1} - n) \subseteq U_{m+1},$$

and $R(n)H(1) \subseteq R(n+1)$.

If $2^{m+1} - 1 = n$, then:

$$R(n)H(1) \cdot H(2^{m+2} - n - 1) \subseteq U_{m+1}H(2^{m+1}) \subseteq U_{m+2},$$

and $R(n)H(1) \subseteq R(n+1)$.

By symmetry, $H(1)L(n) \subseteq L(n+1)$. □

Define the space $R'(n) \subseteq H(n)$ recursively: if $n = 0$, set $R(0) = \mathbb{K}$, and otherwise let m be maximal such that $2^m \leq n$ and set:

$$R'(n) = W_m R'(n - 2^m).$$

Note that $\dim R'(n) = 1$.

PROPOSITION 4.4. *For any* $n \geq 0$, $R(n) \oplus R'(n) = H(n)$.

Proof. Use induction on n. The base case $n = 0$ is trivial.

For the inductive step, $n \geq 0$, let m be maximal such that $2^m \leq n$, and assume that $R(n - 2^m) \oplus R'(n - 2^m) = H(n - 2^m)$. Proposition 4.2 can be used to confirm that:

$$\begin{aligned} Q_m H(n - 2^m) \cdot H(2^{m+1} - n) &= Q_m H(2^m) \subseteq U_{m+1}, \\ H(2^m)R(n - 2^m) \cdot H(2^{m+1} - n) &\subseteq H(2^m)U_m \subseteq U_{m+1}, \\ R(n) + R'(n) &\supseteq Q_m H(n - 2^m) + H(2^m)R(n - 2^m) + W_m R'(n - 2^m) = H(n). \end{aligned}$$

Since $\dim R'(n) = 1$, either $R(n) \oplus R'(n) = H(n)$ or $R'(n) \subseteq R(n)$. However, the latter option implies $R(n) = H(n)$ and that $H(n) \cdot H(2^{m+1} - n) \subseteq U_{m+1}$, a clear contradiction. Therefore, $R(n) \oplus R'(n) = H(n)$. □

PROPOSITION 4.5. *For any* $n \geq 0$,

$$0 < \dim H(n)/L(n) \leq 2.$$

Proof. Let m be maximal such that $2^m \leq n$.

If $H(n)/L(n)$ were zero, then $L(n) = H(n)$ and $H(2^{m+1} - n)H(n) \subseteq U_{m+1}$, a contradiction.

Using Proposition 4.2, $R(2^{m+1} - n)H(n) \subseteq U_{m+1}$. By Proposition 4.4,

$$L(n) = \{x \in H(n) : R'(2^{m+1} - n)x \in U_{m+1}\}.$$

Let $p \in H(2^{m+1} - n)$ be an element that generates $R'(2^{m+1} - n)$, and let $\phi : H(n) \to H(2^{m+1})/U_{m+1}$ be the $\mathbb{K}$-linear transformation:

$$\phi : x \mapsto px/U_{m+1}$$

so that $L(n) = \ker \phi$. The image of ϕ has at most dimension 2, and so $\dim H(n)/L(n) \leq 2$. □

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Let $L'(n) \subseteq H(n)$ be a space such that $L(n) \oplus L'(n) = H(n)$. Proposition 4.5 shows that $\dim L'(n)$ is either 1 or 2.

Define the space $E(n) \subseteq H(n)$ as:

$$E(n) = \bigcap_{i=0}^{n} L(i)H(n-i) + H(i)R(n-i).$$

LEMMA 4.1. *For any $n > 0$, $E(n)H(1) + H(1)E(n) \subseteq E(n+1)$.*

Proof. Using Proposition 4.3,

$$\begin{aligned} E(n)H(1) &= \bigcap_{i=0}^{n} L(i)H(n-i) \cdot H(1) + H(i)R(n-i)H(1) \\ &\subseteq \bigcap_{i=0}^{n} L(i)H(n+1-i) + H(i)R(n+1-i). \end{aligned}$$

It remains to show that $E(n)H(1) \subseteq L(n+1)H(0) + H(n+1)R(0) = L(n+1)$.

Let $m \geq 0$ be maximal such that $2^m \leq n+1$.

$$\begin{aligned} &H(2^{m+1}-n-1)E(n)H(1) \\ &\quad \subseteq H(2^{m+1}-n-1)L(n-2^m+1)H(2^m) + H(2^m)R(2^m-1)H(1) \\ &\quad \subseteq U_m H(2^m) + H(2^m)U_m \subseteq U_{m+1}. \end{aligned}$$

Therefore, by definition, $E(n)H(1) \subseteq L(n+1)$.

We can prove $H(1)E(n) \subseteq E(n+1)$ by symmetry. □

Let $E = \sum_{n=1}^{\infty} E(n)$.

THEOREM 4.2. *E is an ideal of A.*

Proof. Apply Lemma 4.1 to the definition of E. □

PROPOSITION 4.6. *A/E is infinite dimensional.*

Proof.

$$\dim A/E = \sum_{n=1}^{\infty} \dim H(n)/E(n) > \sum_{n=1}^{\infty} \dim H(n)/R(n) = \sum_{n=1}^{\infty} \dim R'(n) = \infty.$$

□

PROPOSITION 4.7. *A/E has quadratic or linear growth.*

Proof. Using the fact that $(L(i)H(n-i) + H(i)R(n-i)) \oplus L'(i)R'(n-i) = H(n)$, and recalling Proposition 4.5,

$$\begin{aligned} \dim H(n)/E(n) &\leq \sum_{i=0}^{n} \dim L'(i)R'(n-i) \leq \sum_{i=0}^{n} 2 = 2(n+1), \\ &\sum_{i=0}^{n} \dim H(i)/E(i) \leq n^2 + 3n + 1. \end{aligned}$$

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Proposition 4.6 shows that the algebra A/E is not finite-dimensional. Bergman's Gap Theorem [**2**] proves that the only types of growth strictly slower than quadratic are linear and finite, so A/E must have quadratic or linear growth. □

5. $E \supseteq \mathcal{E}(F_i)$**.** In this section we introduce the set $\mathcal{E}(F)$ and prove that $\mathcal{E}(F)$ is an ideal in $\bar{A}$ (and in A). We also show that $\mathcal{E}(F) \subseteq E$. We start with the following result:

THEOREM 5.1. *For any* $n > 0$, *let* m *be maximal such that* $2^m \leq n$, *the following holds*:

$$\bigcap_{i=0}^{2^{m+1}-n} \{x \in H(n) : H(i)xH(2^{m+1} - n - i) \subseteq U_m H(2^m) + H(2^m)U_m\} \subseteq E(n).$$

Proof. It is sufficient to show that for any $0 \leq i \leq 2^{m+1} - n$ and any $x \in H(n)$ such that $x \notin L(2^m - i)H(n - 2^m + i) + H(2^m - i)R(n - 2^m + i)$,

$$H(i)xH(2^{m+1} - n - i) \nsubseteq U_m H(2^m) + H(2^m)U_m.$$

We can uniquely decompose x into $x_1 + x_L x_R$ with:

$$x_1 \subseteq L(2^m - i)H(n - 2^m + i) + H(2^m - i)R(n - 2^m + i),$$
$$x_L \subseteq L'(2^m - i), \;\; x_R \in R'(n - 2^m + i).$$

Under our assumption, $x_L x_R \neq 0$. However,

$$\begin{aligned} &H(i)x_1 H(2^{m+1} - n - i) \\ &\quad \in H(i)L(2^m - i)H(2^m) + H(2^m)R(n - 2^m + i)H(2^{m+1} - n - i) \\ &\quad \subseteq U_m H(2^m) + H(2^m)U_m. \end{aligned}$$

Therefore, it is sufficient to show there exist $y \in H(i)$ and $z \in H(2^{m+1} - n - i)$ such that $y x_L x_R z \notin U_m H(2^m) + H(2^m)U_m$.

As $x_L \notin L(2^m - i)$, there must exist a $y \in H(i)$ such that $y x_L \notin U_m$. Let $y x_L = x_{LU} + x_{LV}$, with $x_{LU} \in U_m$ and $0 \neq x_{LV} \in V_m$. Symmetrically, there is a $z \in H(2^{m+1} - n - i)$ with $x_R = x_{RU} + x_{RV}$, $x_{RU} \in U_m$ and $0 \neq x_{RV} \in V_m$. We see that

$$y x_L x_R z = x_{LU} x_R z + x_{LV} x_{RU} + x_{LV} x_{RV} \notin U_m H(2^m) + H(2^m)U_m.$$

□

For any non-zero homogeneous space $F \subseteq H(n)$, let $\mathcal{E}(F)$ denote the space:

$$\mathcal{E}(F) = \bigcap_{j=0}^{n-1} \sum_{k=0}^{\infty} H(kn + j)FA.$$

PROPOSITION 5.1. *For any non-zero homogeneous space* $F \subseteq H(n)$, $\mathcal{E}(F)$ *is an ideal in* $\bar{A}$.

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Proof. By the definition, it is clear that $\mathcal{E}(F)$ is right ideal. To prove that it is a left ideal, it is sufficient to show that $H(1)\mathcal{E}(F) \subseteq \mathcal{E}(F)$.

$$\begin{aligned} H(1)\mathcal{E}(F) &= \bigcap_{j=0}^{n-1} \sum_{k=0}^{\infty} H(kn+j+1)FA \\ &= \bigcap_{j=1}^{n-1} \sum_{k=0}^{\infty} H(kn+j)FA \cap \sum_{k=0}^{\infty} H(kn+n)FA \\ &= \bigcap_{j=1}^{n-1} \sum_{k=0}^{\infty} H(kn+j)FA \cap \sum_{k=1}^{\infty} H(kn)FA \subseteq \bigcap_{j=0}^{n-1} \sum_{k=0}^{\infty} H(kn+j)FA = \mathcal{E}(F). \end{aligned}$$

□

COROLLARY 5.2. *For any $i \geq 0$, $\mathcal{E}(F_i) \subseteq E$.*

Proof. Since it is graded, $\mathcal{E}(F_i)$ can decompose into homogeneous subspaces. If $n < 2^{N_i}$, $\mathcal{E}(F_i) \cap H(n) = (0)$, and if $n \geq 2^{N_i}$,

$$\mathcal{E}(F_i) \cap H(n) = \bigcap_{j=0}^{n-1} \sum_{k=0}^{\lfloor (n-j)2^{-N_i}-1 \rfloor} H(k2^{N_i}+j)F_i H(n-(k+1)2^{N_i}-j).$$

Let $n \geq 2^{N_i}$ and m be maximal such that $2^m \leq n$. For any $0 \leq j \leq 2^{m+1}-n$,

$$\begin{aligned} &H(j)(\mathcal{E}(F_i) \cap H(n))H(2^{m+1}-n-j) \\ &\qquad \subseteq \sum_{k=1}^{\lfloor (n+j)2^{-N_i}-1 \rfloor} H(k2^{N_i})F_i H(2^{m+1}-(k+1)2^{N_i}) \\ &\qquad \subseteq H(k2^{N_i})U_{N_i} H(2^{m+1}-(k+1)2^{N_i}). \end{aligned}$$

Using Proposition 4.1, this is contained in U_{m+1}, and so by Theorem 5.1, $\mathcal{E}(F_i) \cap H(n) \subseteq E(n)$. □

6. Enumerating elements. To construct a Jacobson radical algebra using the above method, we use an approach very similar to that used in Theorem 9 in **[6]**, but adapted for our constraints. First, we require that the field $\mathbb{K}$ be countable so that we can enumerate the polynomials of $\bar{A}$. For each such $f \in \bar{A}$, we will find a $g \in \bar{A}$ and a sufficiently 'small' F such that $f+g-fg \in \mathcal{E}(F)$.

Let $f \subseteq \bar{A}$ be any polynomial with no constant term, and let d be minimal such that $f \in \sum_{n=1}^{d} H(n)$. We can decompose f into $f_{(1)} + \cdots + f_{(d)}$ with each $f_{(i)} \in H(i)$, and recursively define the spaces $s(n) \subseteq H(n)$ for each $n \geq 0$ with:

- $s(0) = 1$,
- $s(n) = \sum_{i=1}^{\min\{n,d\}} f_{(i)} s(n-i)$ for $n > 0$.

This way,

$$s(n) = \sum_{k=0}^{n} \sum_{1 \leq i_1,\ldots,i_k \leq d,\, i_1+\cdots+i_k=n} f_{(i_1)} \cdots f_{(i_k)}.$$

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Lemma 8 from [**8**] can be used to prove the following simple property:

LEMMA 6.1. *For any* $m_1, m_2 \geq 0$ *and any* $n \geq m_1 + m_2 + 2d$,

$$s(n) \subseteq \sum_{a,b=1}^{d} H(m_1 + a)s(n - m_1 - m_2 - a - b + 1)H(m_2 + b - 1).$$

Using s, we can build our subspace F. Recall that $|X|$ is the number of generators of A.

THEOREM 6.2. *For any* $N \geq 2d$, *there exists a homogeneous subspace* $F \subseteq H(N)$ *with* $\dim F \leq (\frac{|X|^d-1}{|X|-1})^2$ *and a polynomial* $g \in \bar{A}$ *such that* $f + g - fg \in \mathcal{E}(F)$.

Proof. Let $g = -\sum_{n=1}^{2N+d} s(n)$, and let P be the two-sided ideal generated by $\{s(2N + i)\}_{i=1}^{d}$. By the recursive construction of s,

$$\begin{aligned} g &= -\sum_{n=1}^{2N+d} s(n) = -\sum_{n=1}^{2N+d} \sum_{i=1}^{\min\{n,d\}} f_{(i)}s(n-i) \\ &= -\sum_{n=1}^{d} f_{(n)} - \sum_{n=1}^{2N+d} \sum_{i=1}^{\min\{n-1,d\}} f_{(i)}s(n-i) = -f - \sum_{i=1}^{d} \sum_{n=i+1}^{2N+d} f_{(i)}s(n-i) \\ &= -f - \sum_{i=1}^{d} \sum_{n=1}^{2N} f_{(i)}s(n) - \sum_{i=1}^{d} \sum_{n=2N+1}^{2N+d-i} f_{(i)}s(n) \in -f + fg + P. \end{aligned}$$

Now set $F = \sum_{a,b=0}^{d-1} H(a)s(N - a - b)H(b)$. It is our goal to show that $P \subseteq \mathcal{E}(F)$. Thanks to Proposition 5.1, it is sufficient to show that for any $1 \leq i \leq d$, $s(2N + i) \in \mathcal{E}(F)$. Consequently, it is sufficient to show that for any $0 \leq j < N$,

$$s(2N + i) \in H(j)FH(N + i - j) = \sum_{a,b=0}^{d-1} H(j + a)s(N - a - b)H(N + i + b - j),$$

which can be extracted easily from Lemma 6.1.

Finally, recall that $\dim H(n) = |X|^n$, where $|X|$ is the number of generators of A,

$$\dim F \leq \sum_{a,b=0}^{d-1} \dim H(a)s(N - a - b)H(b) = \sum_{a,b=0}^{d-1} |X|^{a+b} = \left(\frac{|X|^d - 1}{|X| - 1}\right)^2.$$

□

Proof of Theorem 1.1. In order to make our quotient algebra $\bar{A}/E$ Jacobson radical, for every $f \in \bar{A}$ there needs to be a $g \in \bar{A}$ such that $f + g - fg \in E$. As $\bar{A}$ is countable, we can make an enumeration $f_1, f_2, \ldots$ For each f_m, let d_m be minimal such that $f_m \in \sum_{n=1}^{d_m} H(n)$. For any $N_m \geq 1 + \log_2 d_m$, Theorem 6.2 can give us a $g_m \in \bar{A}$ and an $F_m \subseteq H(2^{N_m})$ such that $f_m + g_m - f_m g_m \in \mathcal{E}(F_m)$ and $\dim F_m \leq (\frac{|X|^{d_m}-1}{|X|-1})^2$.

If each $\dim F_m < \frac{1}{2}(N_m - N_{m-1} + 1)$, then we can construct sets $U(2^n)$ and $V(2^n)$ as in Section 3 (see last four lines of Section 3), and hence we can construct the ideal E as detailed in Section 4. The algebra A/E is infinite-dimensional (Proposition 4.6),

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has quadratic growth (because affine algebras with linear growth are PI by Small–Stafford–Warfield Theorem [**5**]) with each $\dim H(n)/E(n) \leq 2(n+1)$ (Proposition 4.7) and contains each $\mathcal{E}(F_m)$ (Corollary 5.2). Fortunately, each N_m can be set arbitrarily high in relation to N_{m-1}. The needed upper bound of dimension of F_m depends on d_m, $|X|$, N_m and N_{m-1}, so if each N_m is set to $\lceil \sup\{1 + \log_2 d_m, 2(\frac{|X|^{d_m}-1}{|X|-1})^2 + N_{m-1}\}\rceil$, each F_m will be 'small enough' for the construction of E.

In other words, there is a graded ideal $E \lhd A$ such that $\bigcup_{i\in\mathbb{N}} \mathcal{E}(F_i) \subseteq E$ and A/E is infinite-dimensional, Jacobson radical and has quadratic growth. Specifically, $1 \leq H(n)/(E \cap H(n)) \leq 2n+2$ for each $n \geq 1$.

The following more general theorem can be proved in a similar way.

THEOREM 6.3. *Let $\mathbb{K}$ be an algebraically closed field. Let $A = \mathbb{K}\langle x, y\rangle$ be the free non-commutative algebra generated (in degree one) by the elements x, y. Let $H(n) \subset A$ be the homogeneous subspace of degree $n \geq 0$. Finally, for any $F \subseteq H(n)$, let:*

$$\mathcal{E}(F) = \bigcap_{j=0}^{n-1} \sum_{k=0}^{\infty} H(kn + j)FA.$$

For any sequence $\{N_i\}_{i\in\mathbb{N}}$ of strictly increasing natural numbers, and any sequence $\{F_i\}_{i\in\mathbb{N}}$ of homogeneous subspaces such that $F_i \subseteq H(2^{N_i})$ and $\dim F_i < \frac{1}{2}(N_i - N_{i-1} + 1)$, the quotient algebra $A/\langle\mathcal{E}(F_i)\rangle_{i\in\mathbb{N}}$ can be mapped homomorphically onto an infinite-dimensional graded algebra B of linear or quadratic growth; moreover, the dimension of B_n, in other words the homogeneous subspace of degree n elements of B, is at most $2n+2$ for each n.

Proof. By assumption, $\dim F_m < \frac{1}{2}(N_m - N_{m-1} + 1)$, hence we can construct sets $U(2^n)$, $V(2^n)$ as in Section 3 (see last four lines of Section 3), and hence we can construct the ideal E as detailed in Section 4. The algebra A/E is infinite-dimensional (Proposition 4.6), has at most quadratic growth with each $\dim H(n)/E(n) \leq 2(n+1)$ (Proposition 4.7) and contains each $\mathcal{E}(F_m)$ (Corollary 5.2).

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