Can. J. Math., Vol. XXXII, No. 5, 1980, pp. 1021–1044

ON THE $\mathscr{Y}_\nu$ AND $\mathscr{H}_\nu$ TRANSFORMATIONS

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1. Introduction. Denote by C_0 the collection of complex-valued functions which are continuous and compactly supported on $(0, \infty)$. The transformations of the title are defined on C_0 by

$$(1.1) \qquad (\mathscr{Y}_\nu f)(x) = \int_0^\infty (xt)^{1/2} Y_\nu(xt) f(t) dt,$$

and

$$(1.2) \qquad (\mathscr{H}_\nu f)(x) = \int_0^\infty (xt)^{1/2} \mathbf{H}_\nu(xt) f(t) dt,$$

respectively, where $Y_\nu(x)$ is the Bessel function of the second kind, and $\mathbf{H}_\nu(x)$ is the Struve function; see [**1**; 7.5.4(55)]. The two transformations are studied briefly in [**6**; § 8.4]; tables of transform pairs are given in [**2**; Chapters IX and XI], where it is also stated that, for $-\frac{1}{2} < \nu < \frac{1}{2}$, each of the transformations is the inverse of the other.

These transformations are of importance in many axially symmetric problems. When solutions that are regular on the axis of symmetry are wanted, the solution often involves the Hankel transformation H_ν, defined for $f \in C_0$ by

$$(H_\nu f)(x) = \int_0^\infty (xt)^{1/2} J_\nu(xt) f(t) dt.$$

However when solutions to corresponding problems that are singular on the axis of symmetry are wanted, the solution will involve $\mathscr{Y}_\nu$, with its coefficient determined by $\mathscr{H}_\nu$. For example, in generalized axially-symmetric potential theory (GASP theory), one studies the partial differential equation

$$\Delta_\lambda u = u_{rr} + \frac{2\lambda}{r} u_r + u_{zz} = 0$$

in $r > 0$, $z > 0$. The solution of this equation such that $u(r, 0+) = f(r)$, which is regular on $r = 0$, is given formally by

$$u(r, z) = r^{-\lambda}(H_{\lambda-1/2} h_z H_{\lambda-1/2} f_\lambda)(r),$$

Received March 23, 1979. This research was supported by the National Research Council of Canada, grant no. A4048. It was undertaken while the author was on sabbatical leave at the California Institute of Technology.

where $h_z(t) = e^{-zt}$, and $f_\lambda(t) = t^\lambda f(t)$, and a solution that is singular on $r = 0$ is given formally by

$$u(r, z) = r^{-\lambda}(\mathscr{Y}_{\lambda-1/2}h_z\mathscr{H}_{\lambda-1/2}f_\lambda)(r).$$

Since GASP theory is perhaps the most important application of the Hankel transformation, it thus seems worthwhile to obtain the basic facts of boundedness, range, and inverses, about the $\mathscr{Y}_\nu$ and $\mathscr{H}_\nu$ transformations in approximately the same detail as for the Hankel transformation.

Thus our objective in this paper is to study the boundedness and ranges of the two transformations on the spaces $\mathscr{L}_{\mu,p}$, defined for real μ and $1 \leqq p < \infty$ to consist of those complex-valued functions f, measurable on $(0, \infty)$, and such that $\|f\|_{\mu,p} < \infty$, where

$$(1.3) \qquad \|f\|_{\mu,p} = \left\{\int_0^\infty |x^\mu f(x)|^p dx/x\right\}^{1/p}.$$

(For further information on these spaces, see [**3**; § 3], but notice that the spaces $L_{\mu,p}$ of that paper are slightly different from $\mathscr{L}_{\mu,p}$ here.) We shall also look into the question of whether the transformations are inverse to each other.

The results will be largely derived from our results in [**4**] on the boundedness of the Hankel transformation. However we shall also need an integral representation for the Hankel transformation and information about its inverse. This we shall develop in Section 2. In addition we shall need considerable information about the even and odd Hilbert transformations, and we shall develop this in Section 3. The results of these two sections may be of independent interest.

In Section 4 we shall determine the boundedness and characterize the range of $\mathscr{Y}_\nu$ on the $\mathscr{L}_{\mu,p}$ spaces, while in Section 5 we shall do the same for $\mathscr{H}_\nu$. In Section 6 we shall show that in some circumstances $\mathscr{Y}_\nu$ and $\mathscr{H}_\nu$ are inverse to each other, both in the regular sense and in an extended sense.

A notation we shall use frequently is $[X, Y]$ for the collection of bounded linear operators with domain X and range in Y; here X and Y are Banach spaces. $[X, X]$ is abbreviated to $[X]$. If $\xi \in \mathbf{R}$, M_ξ will denote the operator on complex-valued functions on $(0, \infty)$ defined by

$$[1.4] \qquad (M_\xi f)(x) = x^\xi f(x).$$

Clearly, if $\xi \in \mathbf{R}$, $\eta \in \mathbf{R}$,

$$(1.5) \qquad M_\xi M_\eta = M_{\xi+\eta} \quad \text{and } M_0 = I,$$

where I is the identity operator. It is also easy to see that M_ξ is an isometric isomorphism of $\mathscr{L}_{\mu,p}$ onto $\mathscr{L}_{\mu-\xi,p}$.

One of our main tools will be the Mellin transformation $\mathfrak{M}$, defined as follows: if $f \in \mathscr{L}_{\mu,p}$ $1 \leqq p \leqq 2$, then

(1.6) $\quad (\mathfrak{M}f)(\mu + it) = (\mathscr{C}_\mu f)^\wedge(t),$

where $(\mathscr{C}_\mu f)(t) = e^{\mu t}f(e^t)$, and $\hat{F}$ is the Fourier transformation of F, that is if $F \in L_1(-\infty, \infty)$,

$$\hat{F}(t) = \int_{-\infty}^{\infty} e^{itx}f(x)dx.$$

Using standard results about the Fourier transformation, it is easy to see that for $1 \leqq p \leqq 2$, $\mathfrak{M} \in [\mathscr{L}_{\mu,p}, L_{p'}(-\infty, \infty)]$, where

(1.7) $\quad (1/p) + (1/p') = 1.$

We shall often write $(\mathfrak{M}f)(\mu + it)$ as $(\mathfrak{M}f)(s)$, with $\operatorname{Re} s = \mu$. This is justified by the obvious fact that if $f \in \mathscr{L}_{\mu,1}$,

(1.8) $$(\mathfrak{M}f)(s) = \int_0^\infty t^{s-1}f(t)dt, \quad \operatorname{Re} s = \mu.$$

It follows easily from the standard inversion theorems for the Fourier transformation that if $f \in \mathscr{L}_{\mu,p}$, $1 < p \leqq 2$, then

(1.9) $$f(x) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\mu-iR}^{\mu+iR} x^{-s}(\mathfrak{M}f)(s)ds,$$

where the limit is in the topology of $\mathscr{L}_{\mu,p}$.

One further fact we will need is the relation between $\mathfrak{M}$ and M_ξ.

It is easy to show that if $f \in \mathscr{L}_{\mu,p}$, $1 < p \leqq 2$, then for $\operatorname{Re} s = \mu - \xi$,

(1.10) $\quad (\mathfrak{M}M_\xi f)(s) = (\mathfrak{M}f)(s + \xi).$

2. The Hankel transformation. The Hankel transformation H_ν is defined for $\nu > -1$ on C_0 by

(2.1) $$(H_\nu f)(x) = \int_0^\infty (xt)^{1/2}J_\nu(xt)f(t)dt,$$

where $J_\nu(x)$ is the Bessel function of the first kind. In [**4**; § 7] we showed that if $1 < p < \infty$, $\gamma(p) \leqq \mu < \nu + 3/2$, where

(2.2) $\quad \gamma(p) = \max\,(1/p, 1/p'),$

then for all $q \geqq p$ such that $q' \geqq 1/\mu$, $H_\nu \in [\mathscr{L}_{\mu,p}, \mathscr{L}_{1-\mu,q}]$, while in [**5**], we characterized the range of H_ν on $\mathscr{L}_{\mu,p}$. A fact about H_ν that we shall make considerable use of is that from [**3**; § 8] or [**4**; § 7] if $f \in \mathscr{L}_{\mu,p}$, $1 < p \leqq 2$, $\gamma(p) \leqq \mu < \nu + 3/2$, then for $\operatorname{Re} s = 1 - \mu$

(2.3) $\quad (\mathfrak{M}H_\nu f)(s) = m_\nu(s)(\mathfrak{M}f)(1 - s),$

where

$$(2.4)\qquad m_\nu(s) = 2^{s-1/2}\Gamma(\tfrac{1}{2}(\nu + s + \tfrac{1}{2}))/\Gamma(\tfrac{1}{2}(\nu - s + 3/2)).$$

In this section we shall develop an integral representation for H_ν, and also discuss the inversion of the transformation. To these ends, we first need a product theorem for the Hankel transformation, and a lemma.

THEOREM 2.1. *If* $f \in \mathscr{L}_{\mu,p}$, $g \in \mathscr{L}_{\mu,q}$, *where* $1 < p < \infty$, $1 < q < \infty$, $p^{-1} + q^{-1} \geqq 1$, *and* $\max(\gamma(p), \gamma(q)) \leqq \mu < \nu + 3/2$, *then*

$$(2.5)\qquad \int_0^\infty (H_\nu f)(x)g(x)dx = \int_0^\infty f(x)(H_\nu g)(x)dx.$$

Proof. If $f \in C_0$ and $g \in C_0$, then from (2.1)

$$\int_0^\infty (H_\nu f)(x)g(x)dx = \int_0^\infty g(x)dx\int_0^\infty (xt)^{1/2}J_\nu(xt)f(t)dt$$
$$= \int_0^\infty f(t)dt\int_0^\infty (tx)^{1/2}J_\nu(tx)f(x)dx = \int_0^\infty f(t)(H_\nu g)(t)dt$$
$$= \int_0^\infty f(x)(H_\nu g)(x)dx,$$

the interchange of the orders of the integrations being easily justified by Fubini's theorem. Thus (2.5) is true if $f \in C_0$ and $g \in C_0$, and hence, since from [**3**; Lemma 2.2], C_0 is dense in $\mathscr{L}_{\mu,p}$ and $\mathscr{L}_{\mu,q}$, the general result will be true if we show that both sides of (2.5) represent bounded bilinear functionals on $\mathscr{L}_{\mu,p} \times \mathscr{L}_{\mu,q}$.

Now since $p^{-1} + q^{-1} \geqq 1$, $p' \geqq q$; also since $p^{-1} \leqq \gamma(p) \leqq \mu$, $(p')' = p \geqq 1/\mu$, and hence $H_\nu \in [\mathscr{L}_{\mu,q}, \mathscr{L}_{1-\mu,p'}]$, and thus using Hölder's inequality

$$\left|\int_0^\infty f(x)(H_\nu g)(x)dx\right| \leqq \int_0^\infty |x^\mu f(x)|\,|x^{1-\mu}(H_\nu g)(x)|dx/x$$
$$\leqq ||f||_{\mu,p}||H_\nu g||_{1-\mu,p'} \leqq K_\nu||f||_{\mu,p}||g||_{\mu,q},$$

where K_ν is a bound for H_ν as an element of $[\mathscr{L}_{\mu,q}, \mathscr{L}_{1-\mu,p'}]$, so that the right hand side of (2.3) is a bounded bilinear functional on $\mathscr{L}_{\mu,p} \times \mathscr{L}_{\mu,q}$, as is the left hand side of (2.3) by a similar calculation, and the result follows.

Definition 2.1. For $x > 0$, ν real, let

$$(2.6)\qquad q_{\nu,x}(t) = \begin{cases} t^{\nu+1/2}, & 0 < t \leqq x \quad \text{and} \\ 0\ , & t > x \end{cases}$$

$$(2.7)\qquad r_{\nu,x}(t) = x^{\nu+1}t^{-1/2}J_{\nu+1}(xt).$$

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LEMMA 2.1. *Suppose* $1 < p < \infty$. *Then* $q_{\nu,x} \in \mathscr{L}_{\mu,p}$ *if and only if* $\mu > -(\nu + \frac{1}{2})$. *Also, if* $\nu > -3/2$, $r_{\nu,x} \in \mathscr{L}_{\mu,p}$ *if and only if* $-(\nu + \frac{1}{2}) < \mu < 1$. *Further, if* $\nu > -1$,

$$H_\nu q_{\nu,x} = r_{\nu,x}, \tag{2.8}$$

and

$$H_\nu r_{\nu,x} = q_{\nu,x}. \tag{2.9}$$

Proof.

$$\|q_{\nu,x}\|_{\mu,p} = \left\{\int_0^x t^{p(\mu+\nu+1/2)-1} dt\right\}^{1/p} < \infty$$

if and only if $\mu > -(\nu + \frac{1}{2})$. Since, from the series for the Bessel function, if $\nu > -2$

$$r_{\nu,x}(t) \sim x^{2\nu+1} t^{\nu+1/2} / \Gamma(\nu + 1) \text{ as } t \to 0+,$$

and from [**1**; 7.13.1(3)]

$$r_{\nu,x}(t) \sim (2/\pi)^{1/2} x^{\nu+1/2} \cos(xt - \tfrac{1}{2}(\nu + \tfrac{1}{2})\pi)/t \text{ as } t \to \infty,$$

$r_{\nu,x} \in L_{\mu,p}$ if and only if

$$\int_0^\delta t^{p(\mu+\nu+1/2)-1} dt < \infty \quad \text{and} \quad \int_R^\infty t^{p(\mu-1)-1} dt < \infty$$

for some positive δ and R, and thus for $-(\nu + \frac{1}{2}) < \mu < 1$; but $-(\nu + \frac{1}{2}) < 1$ implies $\nu > -3/2$. In particular, $q_{\nu,x} \in \mathscr{L}_{1/2,2}$ if $-(\nu + \frac{1}{2}) < \frac{1}{2}$, that is if $\nu > -1$, and then from [**6**; Theorem 129 and § 8.4, Example (1)], and [**1**; 7.7.1(2)], for almost all $t > 0$

$$\begin{aligned}
(H_\nu q_{\nu,x})(t) &= \frac{d}{dt}\int_0^\infty q_{\nu,x}(u) \frac{du}{u} \int_0^{tu} v^{1/2} J_\nu(v) dv \\
&= \frac{d}{dt}\int_0^x u^{\nu-1/2} du \int_0^{tu} v^{1/2} J_\nu(v) dv = \frac{d}{dt}\int_0^x u^{\nu+1} du \int_0^t v^{1/2} J_\nu(uv) dv \\
&= \frac{d}{dt}\int_0^t v^{1/2} dv \int_0^x u^{\nu+1} J_\nu(vu) du \\
&= t^{1/2}\int_0^x u^{\nu+1} J_\nu(tu) du = t^{-(\nu+3/2)} \int_0^{xt} u^{\nu+1} J_\nu(u) du \\
&= x^{\nu+1} t^{-1/2} J_{\nu+1}(xt) = r_{\nu,x}(t).
\end{aligned}$$

Also, from [**6**; Theorem 129, and § 8.4, Example 1], on $\mathscr{L}_{1/2,2}$, $H_\nu^2 = I$, and thus since if $\nu > -1$, $q_{\nu,x} \in \mathscr{L}_{1/2,2}$,

$$H_\nu r_{\nu,x} = H_\nu^2 q_{\nu,x} = q_{\nu,x}.$$

THEOREM 2.2. *If* $f \in \mathscr{L}_{\mu,p}$, *where* $1 < p < \infty$, $\gamma(p) \leqq \mu < \nu + 3/2$, *then for almost all $x > 0$,*

$$(2.10)\quad (H_\nu f)(x) = x^{-(\nu+1/2)} \frac{d}{dx} x^{\nu+1/2} \int_0^\infty (xt)^{1/2} J_{\nu+1}(xt) f(t) dt/t.$$

Proof. Since $q_{\nu,x} \in \mathcal{L}_{\mu,p'}$, from Theorem 2.1 and Lemma 2.1, for $x > 0$

$$\int_0^x t^{\nu+1/2}(H_\nu f)(t)dt = \int_0^\infty q_{\nu,x}(t)(H_\nu f)(t)dt = \int_0^\infty (H_\nu q_{\nu,x})(t)f(t)dt$$
$$= \int_0^\infty r_{\nu,x}(t)f(t)dt = x^{\nu+1/2}\int_0^\infty (xt)^{1/2}J_{\nu+1}(xt)f(t)dt/t,$$

and the result follows on differentiation.

If instead of taking $g = q_{\nu,x}$ in (2.3) we had taken $g = \chi_{(0,x)}$, the characteristic function of $(0, x)$, we would obtain

$$(H_\nu f)(x) = \frac{d}{dx}\int_0^\infty j_\nu(xt)f(t)dt,$$

where

$$j_\nu(x) = \int_0^x t^{1/2}J_\nu(t)dt.$$

This formula seems less useful than (2.10), firstly because it requires the evaluation of two integrals, and secondly because it is less well posed for using tables of Hankel transforms; for the integral appearing in (2.10) can often be evaluated using, say, [**2**; Chapter VIII] by changing ν to $\nu + 1$ there and adjusting f.

We will now obtain an inverse for H_ν on $\mathcal{L}_{\mu,p}$ for $\mu < 1$.

THEOREM 2.3. *If $f \in \mathcal{L}_{\mu,p}$ where $1 < p < \infty$, $\gamma(p) \leqq \mu < \min(1, \nu + 3/2)$, then for almost all $x > 0$,*

$$(2.11)\quad f(x) = x^{-(\nu+1/2)}\frac{d}{dx}x^{\nu+1/2}\int_0^\infty (xt)^{1/2}J_{\nu+1}(xt)(H_\nu f)(t)dt/t.$$

Proof. Since $\nu > -1$, $-(\nu + \frac{1}{2}) < \frac{1}{2} \leqq \gamma(p) \leqq \mu$, and hence by Lemma 2.1, $r_{\nu,x} \in \mathcal{L}_{\mu,p'}$. But then by Theorem 2.1 and Lemma 2.1,

$$x^{\nu+1/2}\int_0^\infty (xt)^{1/2}J_{\nu+1}(xt)(H_\nu f)(t)dt/t = \int_0^\infty r_{\nu,x}(t)(H_\nu f)(t)dt$$
$$= \int_0^\infty H_\nu r_{\nu,x}(t)f(t)dt = \int_0^x t^{\nu+1/2}f(t)dt,$$

and the result follows on differentiating.

COROLLARY 1. *If $1 < p < \infty$, $\gamma(p) \leqq \mu < \min(1, \nu + 3/2)$ then on $\mathcal{L}_{\mu,p}$, H_ν is one-to-one.*

The reader will note that the right hand sides of (2.10) and (2.11) are the same except that in (2.11) f is replaced by $H_\nu f$, so that formally $H_\nu^{-1} = H_\nu$ or $H_\nu^2 = I$. However, except on $\mathscr{L}_{1/2,2}$, this is purely formal, for if $f \in L_{\mu,p}$, $1 < p < \infty$, $\gamma(p) \leqq \mu < \nu + 3/2$, then $H_\nu f \in \mathscr{L}_{1-\mu,p}$ and thus for H_ν^2 to be defined we require that $\gamma(p) \leqq 1 - \mu < \nu + 3/2$; but since $\gamma(p) \geqq \frac{1}{2}$, with equality only if $p = 2$, $1 - \mu \leqq \frac{1}{2} \leqq \gamma(p)$, with equality only if $p = 2$, and thus $\mu = \frac{1}{2}$ and $p = 2$.

So far we have not shown that H_ν is one-to-one on $\mathscr{L}_{\mu,p}$ for $\gamma(p) \leqq \mu < \nu + 3/2$, but only for $\gamma(p) \leqq \mu < \min(1, \nu + 3/2)$. The following theorem covers the matter.

THEOREM 2.4. *If $\gamma(p) \leqq \mu < \nu + 3/2$, then H_ν is one-to-one on $\mathscr{L}_{\mu,p}$.*

Proof. Suppose $f \in \mathscr{L}_{\mu,p}$ and $H_\nu f = 0$. Then from [**5**; Lemma 3.4],

$$M_{\mu-\gamma} I_{\mu-\gamma,(\nu-\mu+\gamma+3/2)/2} H_{\nu-\mu+\gamma} M_{\mu-\gamma} f = 0,$$

where $I_{\alpha,\xi}$ is defined by [**5**; Definition 3.1]. Now $M_{\mu-\gamma}$ is clearly one-to-one, and from [**3**; Lemma 3.4], $I_{\mu-\gamma,(\nu-\mu+\gamma+3/2)/2}$ is one-to-one on $\mathscr{L}_{\gamma,p}$. Hence

$$H_{\nu-\mu+\gamma} M_{\mu-\gamma} f = 0.$$

But $M_{\mu-\gamma} f \in \mathscr{L}_{\gamma,p}$, and since $\gamma < 1$ and $\gamma < \nu - \mu + \gamma + 3/2$, by Corollary 1, $M_{\mu-\gamma} f = 0$, and $f = 0$.

[**5**; Lemma 3.4] can be used in conjunction with Theorem 2.3 to produce an inversion formula for H_ν on $\mathscr{L}_{\mu,p}$ for $\gamma(p) \leqq \mu < \nu + 3/2$, yielding

$$H_\nu^{-1} = 2^{\mu-\gamma} M_{\gamma-\mu} (H_{\nu-\mu+\gamma})^{-1} (I_{\mu-\gamma,(\nu-\mu+\gamma+3/2)/2})^{-1} M_{\gamma-\mu}.$$

Since, as can easily be shown,

$$(I_{\alpha,\xi})^{-1} = M_{2(1-\xi)} (\tfrac{1}{2} M_{-1} D)^n M_{2(n+\xi-1)} I_{n-\alpha,\xi+\alpha},$$

where n is an integer $\geqq \alpha$ and $(Df)(x) = f'(x)$, this gives H_ν^{-1}.

3. The even and odd Hilbert transformations. For our purposes here the even and odd Hilbert transformations, H_+ and H_- respectively, will be defined initially on $\mathscr{L}_{1/2,2}$ by

$$H_+ = -\mathscr{F}_s \mathscr{F}_c, \tag{3.1}$$

and

$$H_- = \mathscr{F}_c \mathscr{F}_s, \tag{3.2}$$

where $\mathscr{F}_c$ and $\mathscr{F}_s$ are respectively the Fourier cosine and Fourier sine transformations; that is $\mathscr{F}_c = H_{-1/2}$ and $\mathscr{F}_s = H_{1/2}$. Since $\mathscr{F}_c$ and $\mathscr{F}_s \in [\mathscr{L}_{1/2,2}]$, $H_\pm \in [\mathscr{L}_{1/2,2}]$; since on $\mathscr{L}_{1/2,2}$, $\mathscr{F}_c^2 = \mathscr{F}_s^2 = I$, it follows

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that on $L_{1/2,2}$

(3.3) $$H_+H_- = H_-H_+ = -I.$$

Also, using Theorem 2.1 twice, once with $\nu = -\frac{1}{2}$ and once with $\nu = \frac{1}{2}$, it follows that if f and $g \in \mathscr{L}_{1/2,2}$

(3.4) $$\int_0^\infty (H_+f)(t)g(t)dt = -\int_0^\infty f(t)(H_-g)(t)dt.$$

Taking g to be the characteristic function of $(0, x)$, where $x > 0$, by elementary computations we obtain, for almost all $x > 0$

(3.5) $$(H_+f)(x) = -\frac{1}{\pi}\frac{d}{dx}\int_0^\infty f(t)\log\left|1 - \frac{x^2}{t^2}\right|dt;$$

and similarly

(3.6) $$(H_-f)(x) = -\frac{1}{\pi}\frac{d}{dx}\int_0^\infty f(t)\log\left|\frac{t-x}{t+x}\right|dt.$$

Comparing (3.5) and (3.6) with [**6**; Theorem 90], it is evident that H_+ is the restriction to $(0, \infty)$ of the Hilbert transformation of even functions, while H_- is the restriction to $(0, \infty)$ of the Hilbert transformation of odd functions; hence the names, even and odd Hilbert transformations.

The action of the Mellin transformation on $H_\pm$ on $\mathscr{L}_{1/2,2}$ is easily computed from (2.3). This yields that if $f \in \mathscr{L}_{1/2,2}$, then for $\operatorname{Re} s = \frac{1}{2}$,

(3.7) $$(\mathfrak{M}H_+f)(s) = -\tan\frac{\pi s}{2}(\mathfrak{M}f)(s),$$

and

(3.8) $$(\mathfrak{M}H_-f)(s) = \cot\frac{\pi s}{2}(\mathfrak{M}f)(s).$$

It is known that H_+ and H_- can be extended to $\mathscr{L}_{\mu,p}$ for $1 < p < \infty$ and a range of μ values depending on the operator in question; see [**3**; Corollary 8.1.2]. The properties of the operators on these spaces are given by the following theorem.

THEOREM 3.1. *Suppose* $1 < p < \infty$. *Then:*

(a) $H_+ \in [\mathscr{L}_{\mu,p}]$ *for* $-1 < \mu < 1$; *if* $-1 < \mu < 0$ *or* $0 < \mu < 1$, H_+ *maps* $\mathscr{L}_{\mu,p}$ *one-to-one onto itself; if* $-1 < \mu < 1$, (3.5) *holds; if* $f \in \mathscr{L}_{\mu,p}$, $1 < p \leqq 2$, $-1 < \mu < 1$, (3.7) *holds with* $\operatorname{Re} s = \mu$.

(b) $H_- \in [\mathscr{L}_{\mu,p}]$ *for* $0 < \mu < 2$; *if* $0 < \mu < 1$ *or* $1 < \mu < 2$, H_- *maps* $\mathscr{L}_{\mu,p}$ *one-to-one onto itself; if* $0 < \mu < 2$, (3.6) *holds; if* $f \in \mathscr{L}_{\mu,p}$, $1 < p \leqq 2$, $0 < \mu < 2$, (3.8) *holds with* $\operatorname{Re} s = \mu$.

(c) *if* $f \in \mathscr{L}_{\mu,p}$, $g \in \mathscr{L}_{1-\mu,p'}$, $1 < p < \infty$, $-1 < \mu < 1$, (3.4) *holds; on* $\mathscr{L}_{\mu,p}$, *with* $1 < p < \infty$, $0 < \mu < 1$, (3.3) *holds. On* $\mathscr{L}_{\mu,p}$, *with* $1 < p < \infty$,

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$-1 < \mu < 1$,

(3.9) $\quad H_+ = M_1 H_- M_{-1}$

or equivalently, on $\mathscr{L}_{\mu,p}$ *with* $1 < p < \infty, 0 < \mu < 2$

(3.10) $\quad H_- = M_{-1} H_+ M_1$.

Proof. The function $m(s) = -\tan \pi s/2$ is in the class $\mathscr{A}$ of [**4**; Definition 3.1] with $\alpha(m) = -1$, $\beta(m) = 1$. For (i) m is holomorphic in $-1 < \operatorname{Re} s < 1$; (ii) if $-1 < \sigma_1 \leqq \sigma_2 < 1$, m is bounded in the strip $\sigma_1 \leqq \operatorname{Re} s \leqq \sigma_2$, as an elementary argument shows; and (iii) if $-1 < \sigma < 1$,

$$|m'(\sigma + it)| = \frac{\pi}{2}\left|\sec^2 \frac{\pi}{2}(\sigma + it)\right| = O(|t|^{-1}) \quad \text{as } |t| \to \infty.$$

Hence by [**4**; Theorem 1], and since (3.7) holds on $\mathscr{L}_{1/2,2}$, $H_+ \in [\mathscr{L}_{\mu,p}]$, $-1 < \mu < 1$, and if $f \in L_{\mu,p}$, $1 < p \leqq 2$, $-1 < \mu < 1$ (3.7) holds with $\operatorname{Re} s = \mu$. $1/m(s) = -\cot \pi s/2 = -\tan \pi(1 - s)/2 = m(1 - s)$, and hence $1/m \in \mathscr{A}$ with $\alpha(1/m) = 0$, $\beta(1/m) = 2$, and thus by [**4**; Theorem 1], H_+ maps $\mathscr{L}_{\mu,p}$ one-to-one onto itself if $0 < \mu < 1$. But $m(s - 2) = m(s)$, and thus $1/m(s) = m(-1 - s)$, and hence also $1/m \in \mathscr{A}$ with $\alpha(1/m) = -2$, $\beta(1/m) = 0$, and thus, again by [**4**; Theorem 1], H_+ maps $\mathscr{L}_{\mu,p}$ one-to-one onto itself if $-1 < \mu < 0$. (3.5) follows from (3.4) taking as g the characteristic function of $(0, x)$, and thus once (3.4) is proved, (a) is proved.

The proof of (b) is exactly similar. That (3.4) holds follows from the fact that it holds for f and $g \in \mathscr{L}_{1/2,2}$, and that both sides of (3.4) represent bounded bilinear functionals on $\mathscr{L}_{\mu,p} \times \mathscr{L}_{1-\mu,p'}$. (3.3) holds since it holds on $\mathscr{L}_{1/2,2}$ and both sides represent bounded operators on $\mathscr{L}_{\mu,p}$. (3.9) and (3.10) follow on $\mathscr{L}_{\mu,2}$ on taking Mellin transforms, and then on their respective $\mathscr{L}_{\mu,p}$ since both sides represent bounded operators on those spaces.

4. The boundedness and range of $\mathscr{Y}_\nu$. We shall determine the boundedness properties of $\mathscr{Y}_\nu$ and find its range by showing that a relation exists between $\mathscr{Y}_\nu$ and H_ν. We shall also find an integral representation for $\mathscr{Y}_\nu$. We first need the following lemma.

LEMMA 4.1. *If* $-1 < \nu < 3/2$,

$$(4.1) \quad (H_- M_{\nu-1/2} r_{\nu,x})(t) = -x^{\nu+1} t^{\nu-1} (Y_{\nu+1}(xt) + \Gamma(\nu + 1)(2/(xt))^{\nu+1}/\pi), \text{ a.e.}$$

Proof. Since $\nu < 3/2$, $\nu - \frac{1}{2} < 1$, and since $\nu > -1$, $-(\nu + \frac{1}{2}) < \nu + 3/2$. Hence the intervals $(-(\nu + \frac{1}{2}), 1)$ and $(\nu - \frac{1}{2}, \nu + 3/2)$ intersect. Let μ be any point of their intersection. Since $-(\nu + \frac{1}{2}) < \mu < 1$, $r_{\nu,x} \in \mathscr{L}_{\mu,p}$ and thus

$$M_{\nu-1/2} r_{\nu,x} \in \mathscr{L}_{\mu-\nu+1/2,p}.$$

Since $\nu - \frac{1}{2} < \mu < \nu + 3/2$, $0 < \mu - \nu + \frac{1}{2} < 2$, and $H_- M_{\nu-1/2} r_{\nu,x}$ exists. Also, for $p = 2$, from (3.8), (1.10), (2.3), and [**2**; 6.2(18) and 6.1(2)], with $\operatorname{Re} s = \mu - \nu + \frac{1}{2} = \mu_1$

$$\begin{aligned}(\mathfrak{M} H_- M_{\nu-1/2} r_{\nu,x})(s) &= \cot \frac{\pi s}{2} (\mathfrak{M} M_{\nu-1/2} r_{\nu,x})(s) \\ &= \cot \frac{\pi s}{2} (\mathfrak{M} r_{\nu,x})(s + \nu - \tfrac{1}{2}) = \cot \frac{\pi s}{2} (\mathfrak{M} H_\nu q_{\nu,x})(s + \nu - \tfrac{1}{2}) \\ &= m_\nu(s + \nu - \tfrac{1}{2}) \cot \frac{\pi s}{2} (\mathfrak{M} q_{\nu,x})(3/2 - \nu - s) \\ &= x^{2-s} m_\nu(s + \nu - \tfrac{1}{2}) \cot \frac{\pi s}{2} \Big/ (2 - s) \\ &= 2^{\nu-2} x^2 (x/2)^{-s} \left(\Gamma(\nu + \tfrac{1}{2}s)/((1 - \tfrac{1}{2}s)\Gamma(1 - \tfrac{1}{2}s))\right) \cot \frac{\pi s}{2} \\ &= 2^{\nu-2} x^2 (x/2)^{-s} \cot \frac{\pi s}{2} \left(\Gamma(\nu + \tfrac{1}{2}s)/(\Gamma(2 - \tfrac{1}{2}s))\right).\end{aligned}$$

Hence, from (1.9),

$$(H_- M_{\nu-1/2} r_{\nu,x})(t) = 2^{\nu-2} x^2 \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\mu_1 - iR}^{\mu_1 + iR} \left(\frac{xt}{2}\right)^{-s} \frac{\Gamma(\nu + \frac{1}{2}s)}{\Gamma(2 - \frac{1}{2}s)} \cot \frac{\pi s}{2}\, ds,$$

where the limit is in the topology of $\mathscr{L}_{\mu_1,2}$. But, closing the contour to the left, a long but straightforward residue calculus calculation yields that pointwise a.e.

$$2^{\nu-2} x^2 \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\mu_1 - iR}^{\mu_1 + iR} \left(\frac{xt}{2}\right)^{-s} \frac{\Gamma(\nu + \frac{1}{2}s)}{\Gamma(2 - \frac{1}{2}s)} \cot \frac{\pi s}{2}\, ds = -x^{\nu+1} t^{1-\nu} (Y_{\nu+1}(xt) + \Gamma(\nu + 1)(2/xt)^{\nu+1}/\pi),$$

since $0 < \mu_1 < 2$, and this must equal $(H_- M_{\nu-1/2} r_{\nu,x})(t)$.

THEOREM 4.1. *If* $|\nu| < 1$, *then on* C_0, $\mathscr{Y}_\nu = H_\nu M_{\nu-1/2} H_+ M_{-(\nu-1/2)}$.

Proof. Suppose $f \in C_0$. Then $M_{-(\nu-1/2)} f \in C_0 \subseteq \mathscr{L}_{\nu,2}$, and hence since $-1 < \nu < 1$, by Theorem 3.1, $H_+ M_{-(\nu-1/2)} f \in \mathscr{L}_{\nu,2}$ and thus

$M_{\nu-1/2}H_+M_{-(\nu-1/2)}f \in L_{1/2,2}$. Hence by Theorem 2.2, for almost all $x > 0$

$$(H_\nu M_{\nu-1/2}H_+M_{-(\nu-1/2)}f)(x)$$

$$= x^{-(\nu+1/2)} \frac{d}{dx} x^{\nu+1/2} \int_0^\infty (xt)^{1/2} J_{\nu+1}(xt)t^{\nu-1/2}(H_+M_{-(\nu-1/2)}f)(t)dt/t$$

$$= x^{-(\nu+1/2)} \frac{d}{dx} \int_0^\infty (M_{\nu-1/2}r_{\nu,x})(t)(H_+M_{-(\nu-1/2)}f)(t)dt$$

$$= -x^{-(\nu+1/2)} \frac{d}{dx} \int_0^\infty (H_-M_{\nu-1/2}r_{\nu,x})(t)t^{-(\nu-1/2)}f(t)dt$$

from (3.4), provided $M_{-(\nu-1/2)}f \in \mathscr{L}_{\mu,2}$ and $M_{\nu-1/2}r_{\nu,x} \in \mathscr{L}_{1-\mu,2}$ for some μ, $-1 < \mu < 1$. But since $f \in C_0$, $M_{-(\nu-1/2)}f \in \mathscr{L}_{\mu,2}$ for any μ; also, we saw in the proof of Lemma 4.1 that there was a μ_1, with $0 < \mu_1 < 2$, such that $M_{\nu-1/2}r_{\nu,x} \in \mathscr{L}_{\mu_1,2}$. Letting $\mu = 1 - \mu_1$,

$$M_{-(\nu-1/2)}f \in \mathscr{L}_{\mu,2},\ M_{\nu-1/2}r_{\nu,x} \in \mathscr{L}_{1-\mu,2} \quad \text{and} \quad -1 < \mu < 1.$$

Hence by (4.1)

$$(H_\nu M_{\nu-1/2}H_+M_{-(\nu-1/2)}f)(x) = x^{-(\nu+1/2)} \frac{d}{dx} x^{\nu+1} \int_0^\infty t^{-1/2}(Y_{\nu+1}(xt) + \Gamma(\nu+1)(2/xt)^{\nu+1}/\pi)f(t)dt.$$

Now from [**1**; 7.2.8(52) et seq.],

$$\frac{d}{dz} z^{\nu+1} Y_{\nu+1}(z) = z^{\nu+1} Y_\nu(z),$$

whence

$$\frac{d}{dx} x^{\nu+1} Y_{\nu+1}(xt) = tx^{\nu+1} Y_\nu(xt),$$

and the differentiation may be taken under the integral sign since $f \in C_0$. Hence

$$(H_\nu M_{\nu-1/2}H_+M_{-(\nu-1/2)}f)(x) = \int_0^\infty (xt)^{1/2} Y_\nu(xt)f(t)dt = (\mathscr{Y}_\nu f)(x) \text{ a.e.},$$

and thus on C_0,

$$\mathscr{Y}_\nu = H_\nu M_{\nu-1/2}H_+M_{-(\nu-1/2)}.$$

THEOREM 4.2. *Suppose $1 < p < \infty$, $\gamma(p) \leqq \mu < 3/2 - |\nu|$. Then $\mathscr{Y}_\nu$ can be extended to $\mathscr{L}_{\mu,p}$ as an element of $[\mathscr{L}_{\mu,p}, \mathscr{L}_{1-\mu,q}]$ for any $q \geqq p$ such that $q' \geqq 1/\mu$, and except when $\mu = \frac{1}{2} - \nu$, $\mathscr{Y}_\nu$ is one-to-one and $\mathscr{Y}_\nu(\mathscr{L}_{\mu,p})$*

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$= H_\nu(\mathscr{L}_{\mu,p})$. *Further, on* $\mathscr{L}_{\mu,p}$

(4.2) $\quad \mathscr{Y}_\nu = H_\nu M_{\nu-1/2} H_+ M_{-(\nu-1/2)},$ *and*

(4.3) $\quad \mathscr{Y}_\nu = -M_{-(\nu-1/2)} H_- M_{\nu-1/2} H_\nu.$

Also, if $f \in \mathscr{L}_{\mu,p}$, $1 < p \leqq 2$, $\gamma(p) \leqq \mu < 3/2 - |\nu|$, *then with* Re $s = 1 - \mu$

(4.4) $$(\mathfrak{M}\mathscr{Y}_\nu f)(s) = -m_\nu(s) \cot \frac{\pi}{2}\left(s + \tfrac{1}{2} - \nu\right)(\mathfrak{M}f)(1-s).$$

Proof. Since $\gamma(p) \geqq \frac{1}{2}$, $|\nu| < 1$, and hence by Theorem 4.1, on C_0 (4.2) holds. But if $1 < p < \infty$, $\gamma(p) \leqq \mu < \nu + 3/2$, the transformation on the right of (4.2) is in $[\mathscr{L}_{\mu,p}, \mathscr{L}_{1-\mu,q}]$ for any $q \geqq p$ such that $q' \geqq 1/\mu$. For, $M_{-(\nu-1/2)}$ maps $\mathscr{L}_{\mu,p}$ boundedly onto $\mathscr{L}_{\mu+\nu-1/2,p}$; from Theorem 3.1, H_+ maps $\mathscr{L}_{\mu+\nu-1/2,p}$ into itself if $-1 < \mu + \nu - \frac{1}{2} < 1$; that is if $-(\nu + \frac{1}{2}) < \mu < 3/2 - \nu$, and this is so since for $|\nu| < 1$,

$$-(\nu + \tfrac{1}{2}) < \tfrac{1}{2} \leqq \gamma(p) \leqq \mu, \quad \text{and} \quad \mu < 3/2 - |\nu| \leqq 3/2 - \nu;$$

$M_{\nu-1/2}$ maps $\mathscr{L}_{\mu+\nu-1/2,p}$ boundedly onto $\mathscr{L}_{\mu,p}$; and since

$$\gamma(p) \leqq \mu < 3/2 - |\nu| \leqq 3/2 + \nu,$$

$H_\nu \in [\mathscr{L}_{\mu,p,}, \mathscr{L}_{1-\mu,q}]$ for any $q \geqq p$ such that $q' \geqq 1/\mu$.

Thus we can extend $\mathscr{Y}_\nu$ to $\mathscr{L}_{\mu,p}$ by defining it by (4.2) and then $\mathscr{Y}_\nu \in [\mathscr{L}_{\mu,p}, \mathscr{L}_{1-\mu,q}]$ for all $q \geqq p$ such that $q' \geqq 1/\mu$. Also, since $M_{\pm(\nu-1/2)}$ are isometric isomorphisms, and H_+ maps $\mathscr{L}_{\mu+\nu-1/2,p}$ one-to-one onto itself except when $\mu = \frac{1}{2} - \nu$, and since from Corollary 1 to Theorem 2.3, H_ν is one-to-one, then except when $\mu = \frac{1}{2} - \nu$, $\mathscr{Y}_\nu$ is one-to-one and $\mathscr{Y}_\nu(\mathscr{L}_{\mu,p}) = H_\nu(\mathscr{L}_{\mu,p})$.

From (2.3), (1.10) and (3.7), if $f \in \mathscr{L}_{\mu,p}$ where $1 < p \leqq 2$, $\gamma(p) \leqq \mu < 3/2 - |\nu|$, then with Re $s = 1 - \mu$

$$\begin{aligned}
(\mathfrak{M}\mathscr{Y}_\nu f)(s) &= (\mathfrak{M}H_\nu M_{\nu-1/2} H_+ M_{-(\nu-1/2)} f)(s) \\
&= m_\nu(s)(\mathfrak{M}M_{\nu-1/2} H_+ M_{-(\nu-1/2)} f)(1-s) \\
&= m_\nu(s)(\mathfrak{M}H_+ M_{-(\nu-1/2)} f)(\nu + \tfrac{1}{2} - s) \\
&= -m_\nu(s) \tan \frac{\pi}{2}\left(\nu + \tfrac{1}{2} - s\right)(\mathfrak{M}M_{-(\nu-1/2)} f)(\nu + \tfrac{1}{2} - s) \\
&= -m_\nu(s) \cot \frac{\pi}{2}\left(s + \tfrac{1}{2} - \nu\right)(\mathfrak{M}f)(1-s),
\end{aligned}$$

and (4.4) holds.

(4.2) holds by definition of $\mathscr{Y}_\nu$. For (4.3), we first note that the transformation on its right is in $[\mathscr{L}_{\mu,p}, \mathscr{L}_{1-\mu,q}]$ for the same parameter ranges as for $\mathscr{Y}_\nu$. For H_ν maps $\mathscr{L}_{\mu,p}$ boundedly into $\mathscr{L}_{1-\mu,q}$ since $\gamma(p) \leqq \mu < 3/2 - |\nu| \leqq 3/2 + \nu$; $M_{\nu-1/2}$ maps $\mathscr{L}_{1-\mu,q}$ boundedly onto $\mathscr{L}_{3/2-(\mu+\nu),q}$; H_- maps $\mathscr{L}_{3/2-(\mu+\nu),q}$ boundedly into itself if $0 < 3/2 -$

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$(\mu + \nu) < 2$, or $-\frac{1}{2} - \nu < \mu < 3/2 - \nu$, which we have seen is true; and $M_{-(\nu-1/2)}$ maps $\mathscr{L}_{3/2-(\mu+\nu),q}$ onto $\mathscr{L}_{1-\mu,q}$. Also, if $f \in \mathscr{L}_{1/2,2}$, from (1.10), (1.3), (3.8) and (4.4)

$$-(\mathfrak{M}M_{-(\nu-1/2)}H_-M_{\nu-1/2}H_\nu f)(s) = -(\mathfrak{M}H_-M_{\nu-1/2}H_\nu f)(s + \tfrac{1}{2} - \nu)$$

$$= -\cot\frac{\pi}{2}(s + \tfrac{1}{2} - \nu)(\mathfrak{M}M_{\nu-1/2}H_\nu f)(s + \tfrac{1}{2} - \nu)$$

$$= -\cot\frac{\pi}{2}(s + \tfrac{1}{2} - \nu)(\mathfrak{M}H_\nu f)(s)$$

$$= -m_\nu(s)\cot\frac{\pi}{2}(s + \tfrac{1}{2} - \nu)(\mathfrak{M}f)(1 - s) = (\mathfrak{M}\mathscr{Y}_\nu f)(s),$$

and thus (4.4) holds on $\mathscr{L}_{1/2,2}$, and hence on $\mathscr{L}_{\mu,p}$ since both sides of (4.4) are in $[\mathscr{L}_{\mu,p}, \mathscr{L}_{1-\mu,q}]$.

As a corollary of this result we obtain some information about the range of H_ν.

COROLLARY 1. *If* $1 < p < \infty$, $\gamma(p) \leqq \mu < 3/2 - |\nu|$, *then, except when* $\mu = \frac{1}{2} - \nu$, $H_\nu(\mathscr{L}_{\mu,p})$ *is invariant under the operator* $M_{-(\nu-1/2)}H_-M_{\nu-1/2}$.

Proof.

$$H_\nu(\mathscr{L}_{\mu,p}) = \mathscr{Y}_\nu(\mathscr{L}_{\mu,p}) = (M_{-(\nu-1/2)}H_-M_{\nu-1/2}H_\nu)(\mathscr{L}_{\mu,p})$$
$$= (M_{-(\nu-1/2)}H_-M_{\nu-1/2})(H_\nu(\mathscr{L}_{\mu,p})),$$

using (4.3).

Four comments seem to be in order about the results of Theorem 4.2 and Corollary 1. Firstly, the boundedness results seem to be maximal with respect to the spaces $\mathscr{L}_{\mu,p}$, except in the case $\nu = -\frac{1}{2}$, when they are not maximal. For it is easy to see that for $\mathscr{Y}_\nu$ to be bounded on $\mathscr{L}_{\mu,p}$, $m_\nu(s)\cot\pi(s + \frac{1}{2} - \nu)/2$ must be bounded on the line $\operatorname{Re} s = 1 - \mu$, and if $\nu \neq -\frac{1}{2}$, this requires $\frac{1}{2} \leqq \mu < 3/2 - |\nu|$, and easy examples show, using the integral representation of $\mathscr{Y}_\nu$ to be derived below, that we must have $\mu \geqq \gamma(p)$. If $\nu = -\frac{1}{2}$, since $Y_{-1/2}(x) = J_{1/2}(x)$, $\mathscr{Y}_{-1/2} = H_{1/2} = \mathscr{F}_s$, and $\mathscr{F}_s$ is bounded for $\gamma(p) \leqq \mu < 2$. Secondly, the exceptional value of μ for which

$$\mathscr{Y}_\nu(\mathscr{L}_{\mu,p}) \neq H_\nu(\mathscr{L}_{\mu,p})$$

and for which the result of Corollary 1 fails, namely $\mu = \frac{1}{2} - \nu$, is only possible if $-\frac{1}{2} < \nu \leqq 0$. For the condition $\gamma(p) \leqq \frac{1}{2} - \nu < 3/2 - |\nu|$ is equivalent to $-\frac{1}{2} < \nu \leqq \frac{1}{2} - \gamma(p)$ and $\gamma(p) \geqq \frac{1}{2}$. Further, if $\nu = 0$, $p = 2$ since $\gamma(p) = \frac{1}{2}$ only if $p = 2$, and thus

$$\mathscr{Y}_0(\mathscr{L}_{\mu,p}) = H_0(\mathscr{L}_{\mu,p}), \quad p \neq 2.$$

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Thirdly, since on $\mathscr{L}_{1/2,2}$, $H_\nu^2 = I$,

$$H_\nu(\mathscr{L}_{1/2,2}) = \mathscr{L}_{1/2,2},$$

and thus $\mathscr{Y}_\nu(\mathscr{L}_{1/2,2}) = \mathscr{L}_{1/2,2}$, $|\nu| < 1$.

Finally $H_\nu(\mathscr{L}_{\mu,p})$ has been characterized in [**5**], in terms of fractional integrals independent of ν and of $\mathscr{F}_c$ acting on $\mathscr{L}_{\gamma,p}$, and thus, except when $\mu = \frac{1}{2} - \nu$, $\mathscr{Y}_\nu(\mathscr{L}_{\mu,p})$ has the same characterization.

In order to obtain an integral representation for $\mathscr{Y}_\nu$, we need an analogue for $\mathscr{Y}_\nu$ of Theorem 2.1.

THEOREM 4.3. *If $f \in \mathscr{L}_{\mu,p}$, $g \in \mathscr{L}_{\mu,q}$ where $1 < p < \infty$, $1 < q < \infty$, $p^{-1} + q^{-1} \geqq 1$, and* max $(\gamma(p), \gamma(q)) \leqq \mu < 3/2 - |\nu|$, *then*

$$\int_0^\infty (\mathscr{Y}_\nu f)(x)g(x)dx = \int_0^\infty f(x)(\mathscr{Y}_\nu g)(x)dx. \tag{4.5}$$

Proof. This is practically the same as for Theorem 2.1.

THEOREM 4.4. *If $f \in \mathscr{L}_{\mu,p}$, where $1 < p < \infty$, $\gamma(p) \leqq \mu < 3/2 - |\nu|$, then for almost all $x > 0$,*

$$(\mathscr{Y}_\nu f)(x) = x^{-(\nu+1/2)} \frac{d}{dx} x^{\nu+1/2} \int_0^\infty (xt)^{1/2}(Y_{\nu+1}(xt) + \Gamma(\nu+1)(2/xt)^{\nu+1}/\pi)f(t)dt/t. \tag{4.6}$$

Proof. Since $q_{\nu,x} \in \mathscr{L}_{\mu,p'}$, from Theorem 4.3, for $x > 0$

$$\int_0^x t^{\nu+1/2}(\mathscr{Y}_\nu f)(t)dt = \int_0^\infty (q_{\nu,x})(t)(\mathscr{Y}_\nu f)(t)dt$$
$$= \int_0^\infty (\mathscr{Y}_\nu q_{\nu,x})(t)f(t)dt.$$

But from (4.3), (2.8), and (4.1),

$$\begin{aligned}(\mathscr{Y}_\nu q_{\nu,x})(t) &= -(M_{-(\nu-1/2)}H_-M_{\nu-1/2}H_\nu q_{\nu,x})(t)\\ &= -(M_{-(\nu-1/2)}H_-M_{\nu-1/2}r_{\nu,x})(t)\\ &= x^{\nu+1/2}(xt)^{1/2}(Y_{\nu+1}(xt) + \Gamma(\nu+1)(2/xt)^{\nu+1}/\pi)/t,\end{aligned}$$

and thus

$$\int_0^x t^{\nu+1/2}(\mathscr{Y}_\nu f)(t)dt = x^{\nu+1/2}\int_0^\infty (xt)^{1/2}(\mathscr{Y}_{\nu+1}(xt) + \Gamma(\nu+1)(2/xt)^{\nu+1}/\pi)f(t)dt/t,$$

and the result follows on differentiating.

5. The boundedness and range of $\mathscr{H}_\nu$. We shall determine the boundedness properties of $\mathscr{H}_\nu$ and find its range by showing that a

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relation exists between $\mathcal{H}_\nu$ and $H_{\nu+1}$. We shall also find an integral representation for $\mathcal{H}_\nu$. First we need the following lemma.

LEMMA 5.1. *Let*

$$(5.1)\qquad l_\nu(s) = (\Gamma(\tfrac{1}{2}(s+\nu+3/2))\Gamma(\tfrac{1}{2}(s-\nu-\tfrac{1}{2})))/ (\Gamma(\tfrac{1}{2}(s+\nu+\tfrac{1}{2}))\Gamma(\tfrac{1}{2}(s-\nu+\tfrac{1}{2}))),\ \nu > -2.$$

Then: (a) *there is a transformation* $S_\nu \in [\mathcal{L}_{\mu,p}]$ *for* $1 < p < \infty$, $\mu > \max(\nu + \tfrac{1}{2}, -(\nu+3/2))$ *such that if* $f \in \mathcal{L}_{\mu,p}$, $1 < p \leqq 2$, $\mu > \max(\nu+\tfrac{1}{2}, -(\nu+3/2))$, *then for* $\operatorname{Re} s = \mu$,

$$(5.2)\qquad (\mathfrak{M}S_\nu f)(s) = l_\nu(s)(\mathfrak{M}f)(s).$$

S_ν *maps* $\mathcal{L}_{\mu,p}$ *one-to-one onto itself if* $1 < p < \infty$, $\mu > \max(\nu+\tfrac{1}{2}, -(\nu+3/2))$, $\mu \neq -(\nu+\tfrac{1}{2})$. *Also:* (b) *there is a transformation* $T_\nu \in [\mathcal{L}_{\mu,p}]$ *for* $1 < p < \infty$, $\mu < \min(\tfrac{1}{2}-\nu, \nu+5/2)$ *such that if* $f \in \mathcal{L}_{\mu,p}$, $1 < p \leqq 2$, $\mu < \min(\tfrac{1}{2}-\nu, \nu+5/2)$, *then for* $\operatorname{Re} s = \mu$

$$(5.3)\qquad (\mathfrak{M}T_\nu f)(s) = l_\nu(1-s)(\mathfrak{M}f)(s).$$

T_ν *maps* $\mathcal{L}_{\mu,p}$ *one-to-one onto itself if* $1 < p < \infty$, $\mu < \min(\tfrac{1}{2}-\nu, \nu+5/2)$, $\mu \neq \nu + 3/2$. *Further:* (c) *if* $f \in \mathcal{L}_{\mu,p}$, $g \in \mathcal{L}_{1-\mu,p'}$, *where* $1 < p < \infty$, $\mu > \max(\nu+\tfrac{1}{2}, -(\nu+3/2))$, *then*

$$(5.4)\qquad \int_0^\infty (S_\nu f)(x)g(x)dx = \int_0^\infty f(x)(T_\nu g)(x)dx.$$

In addition: (d) *if* $\nu > -2$,

$$(5.5)\qquad (T_\nu r_{\nu+1,x})(t) = x^{\nu+3}t^{-(\nu+5/2)}\int_0^t v^{\nu+2}\mathbf{H}_\nu(xv)dv, \text{ a.e.}$$

Proof. Clearly l_ν is holomorphic in $\alpha(l_\nu) < \operatorname{Re} s < \beta(l_\nu)$, where $\alpha(l_\nu) = \max(\nu+\tfrac{1}{2}, -(\nu+3/2))$ and $\beta(l_\nu) = \infty$. Also, from [**1**; 1.18(6)], if $\sigma > \alpha(l_\nu)$, then as $|t| \to \infty$

$$|l_\nu(\sigma+it)| \sim (|t|^{(\sigma+\nu+3/2)/2}|t|^{(\sigma-\nu-1/2)/2})/ (|t|^{(\sigma+\nu+1/2)/2}|t|^{(\sigma-\nu+1/2)/2}) = 1$$

uniformly in $\sigma_1 \leqq \sigma \leqq \sigma_2$, where $\alpha(l_\nu) < \sigma_1 \leqq \sigma_2 < \beta(l_\nu)$. Hence in the strip $\sigma_1 \leqq \operatorname{Re} s \leqq \sigma_2$, $|l_\nu|$ is bounded. Further,

$$l_\nu'(\sigma+it) = \tfrac{1}{2}l_\nu(\sigma+it)(\psi(\tfrac{1}{2}(\sigma+\nu+3/2+it)) + \psi(\tfrac{1}{2}(\sigma-\nu-\tfrac{1}{2}+it)) - \psi(\tfrac{1}{2}(\sigma+\nu+\tfrac{1}{2}+it)) - \psi(\tfrac{1}{2}(\sigma-\nu+\tfrac{1}{2}+it)),$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. But from [**1**; 1.18(7)] as $|z| \to \infty$ in $|\arg z| \leqq \pi - \delta$,

$$\psi(z) = \log z - (2z)^{-1} + O(|z|^{-2}).$$

Also, if a and t are real as $|t| \to \infty$,

$$\log (a + it) = \log it + \log (1 - ia/t) = \log (it) - ia/t + O(t^{-2}),$$

while $(a + it)^{-1} = -i/t + O(t^{-2})$, so that

$$\psi(a + it) = \log it - i(a - \tfrac{1}{2})/t + O(t^{-2}).$$

Hence as $|t| \to \infty$

$$|l_\nu'(\sigma + it)| = |l_\nu(\sigma + it)|\, |-i((\sigma + \nu + 1) + (\sigma - \nu - 1)$$
$$- (\sigma + \nu) - (\sigma - \nu))/2t + O(t^{-2})| = O(t^{-2})$$

since $|l_\nu(\sigma + it)| = O(1)$. Thus $l_\nu \in \mathscr{A}$; see [**4**; Definition 3.1].

Hence by [**4**; Theorem 1], there is a transformation $S_\nu \in [\mathscr{L}_{\mu,p}]$ for $1 < p < \infty$, $\alpha(l_\nu) < \mu < \beta(l_\nu)$ such that if $f \in \mathscr{L}_{\mu,p}$, $1 < p \leqq 2$, $\alpha(l_\nu) < \mu < \beta(l_\nu)$, then (5.2) holds.

To prove the remainder of (a), we notice that $1/l_\nu$ is holomorphic in either of the strips $\alpha_1 < \operatorname{Re} s < \beta_1$ or $\alpha_2 < \operatorname{Re} s < \beta_2$, where $\alpha_1 = \max(\nu - \frac{1}{2}, -(\nu + \frac{1}{2}))$, $\beta_1 = \infty$, $\alpha_2 = \min(\nu - \frac{1}{2}, -(\nu + \frac{1}{2}))$, $\beta_2 = \alpha_1$, and calculations almost identical to those performed above for l_ν show that $1/l_\nu \in \mathscr{A}$ with either $\alpha(1/l_\nu) = \alpha_1$, $\beta(1/l_\nu) = \beta_1$ or $\alpha(1/l_\nu) = \alpha_2$, $\beta(1/l_\nu) = \beta_2$. Hence by [**4**; Theorem 1], S_ν maps $\mathscr{L}_{\mu,p}$ one-to-one onto itself for $1 < p < \infty$, $\max(\alpha(l_\nu), \alpha_1) < \mu < \min(\beta(l_\nu), \beta_1)$ or $\max(\alpha(l_\nu), \alpha_2) < \mu < \min(\beta(l_\nu), \beta_2)$. Putting the various values of the α's and β's into these inequalities we obtain that S_ν maps $\mathscr{L}_{\mu,p}$ one-to-one onto itself for $1 < p < \infty$, $\mu > \max(\nu + \frac{1}{2}, -(\nu + 3/2))$, $\mu \neq -(\nu + \frac{1}{2})$.

(b) follows from (a); for if $k_\nu(s) = l_\nu(1 - s)$, then $l_\nu \in \mathscr{A}$ implies $k_\nu \in \mathscr{A}$ with $\alpha(k_\nu) = 1 - \beta(l_\nu)$, $\beta(k_\nu) = 1 - \alpha(l_\nu)$, etc., and all results about S_ν are true for T_ν with μ replaced by $1 - \mu$.

We first prove (c) for $p = 2$. For then, from [**6**; Theorem 72],

$$\int_0^\infty (S_\nu f)(x) g(x) dx = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} (\mathfrak{M} S_\nu f)(s) (\mathfrak{M} g)(1 - s) ds$$
$$= \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} l_\nu(s) (\mathfrak{M} f)(s) (\mathfrak{M} g)(1 - s) ds$$
$$= \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} (\mathfrak{M} f)(s) l_\nu(1 - (1 - s)) (\mathfrak{M} g)(1 - s) ds$$
$$= \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} (\mathfrak{M} f)(s) (\mathfrak{M} T_\nu g)(1 - s) ds = \int_0^\infty f(x) (T_\nu g)(x) dx.$$

But both sides of (5.4) are bounded bilinear functionals on $\mathscr{L}_{\mu,p} \times \mathscr{L}_{1-\mu,p'}$, and thus (5.4) is true if $f \in \mathscr{L}_{\mu,p}$, $g \in \mathscr{L}_{1-\mu,p'}$, $1 < p < \infty$, $\mu > \max(\nu + \frac{1}{2}, -(\nu + 3/2))$.

To prove (5.5), we notice that since $\nu > -2$, $-(\nu + 3/2) < 1$, and $-(\nu + 3/2) < \nu + 5/2$. Thus we can choose μ, $-(\nu + 3/2) < \mu < 1$

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so that $\mu < \min(\frac{1}{2} - \nu, \nu + 5/2)$ and then $r_{\nu+1,x} \in \mathscr{L}_{\mu,2}$, and $r_{\nu+1,x} = H_{\nu+1}q_{\nu+1,x}$, and $T_\nu r_{\nu+1,x} \in \mathscr{L}_{\mu,2}$. Hence from (2.3) and (5.3), if Re $s = \mu$,

$$(\mathfrak{M}T_\nu r_{\nu+1,x})(s) = (\mathfrak{M}T_\nu H_{\nu+1}q_{\nu+1,x})(s)$$

$$= l_\nu(1-s)m_{\nu+1}(s)(\mathfrak{M}q_{\nu+1,x})\,(1-s)$$

$$= \frac{2^{s-1/2}x^{\nu+5/2-s}}{\nu+5/2-s}\cdot\frac{\Gamma(\frac{1}{2}(\nu+3/2+s))\Gamma(\frac{1}{2}(\frac{1}{2}-\nu-s))}{\Gamma(\frac{1}{2}(\nu+3/2-s))\Gamma(\frac{1}{2}(3/2-\nu-s))}$$

$$= \frac{2^{s-1/2}x^{\nu+5/2-s}}{\nu+5/2-s}\,\frac{\Gamma\,(\frac{1}{2}(\nu+s+\frac{1}{2}))\sin\frac{\pi}{2}\,(3/2-\nu-s)}{\Gamma\,(\frac{1}{2}(\nu-s+3/2))\sin\frac{\pi}{2}\,(\frac{1}{2}-\nu-s)}$$

$$= \frac{x^{\nu+5/2-s}}{\nu+5/2-s}\,m_\nu(s)\tan\frac{\pi}{2}\,(s+\nu+\tfrac{1}{2}),$$

where we have used [**1**; 1.2(6)]. But then by (1.9)

$$(T_\nu r_{\nu+1,x})(t)$$

$$= x^{\nu+5/2}\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\mu-iR}^{\mu+iR}(tx)^{-s}m_\nu(s)\tan\frac{\pi}{2}\,(s+\nu+\tfrac{1}{2})ds/(\nu+5/2-s),$$

where the limit is in the topology of $L_{1/2,2}$. However, closing the contour to the left, a long but straightforward residue calculus calculation yields that pointwise a.e.

$$x^{\nu+5/2}\lim_{R\to\infty}\int_{\mu-iR}^{\mu+iR}(tx)^{-s}m_\nu(s)\tan\frac{\pi}{2}\,(s+\nu+\tfrac{1}{2})ds/(\nu+5/2-s)$$
$$= x^{\nu+3}t^{-(\nu+5/2)}\int_0^t v^{\nu+2}\mathbf{H}_\nu(xv)dv,$$

and thus this must be $(T_\nu r_{\nu+1,x})(t)$ a.e., and (5.5) holds.

THEOREM 5.1. *If $\nu > -2$, then, on C_0, $\mathscr{H}_\nu = H_{\nu+1}S_\nu$.*

Proof. Suppose $f \in C_0$. Then for all $\mu > \max(\nu+\frac{1}{2}, -(\nu+3/2))$, $S_\nu f \in \mathscr{L}_{\mu,2}$. Since $\nu > -2$, $-(\nu+3/2) < \nu+5/2$, and thus μ exists such that

$$\tfrac{1}{2} \leqq \mu < \nu+5/2 \quad\text{and}\quad \mu > \max(\nu+\tfrac{1}{2}, -(\nu+3/2)).$$

Hence $H_{\nu+1}S_\nu f$ is defined, and by Theorem 2.2 and (5.4), for almost all $x > 0$

$$(H_{\nu+1}S_\nu f)(x) = x^{-(\nu+3/2)}\frac{d}{dx}\int_0^\infty r_{\nu+1,x}(t)(S_\nu f)(t)dt$$
$$= x^{-(\nu+3/2)}\frac{d}{dx}\int_0^\infty (T_\nu r_{\nu+1,x})(t)f(t)dt$$

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provided $r_{\nu+1,x} \in \mathscr{L}_{1-\mu,2}$, which is so from Lemma 2.1 since $\frac{1}{2} \leqq \mu < \nu + 5/2$ and hence $-(\nu + 3/2) < 1 - \mu \leqq \frac{1}{2} < 1$. Thus, for almost all $x > 0$, by (5.5),

$$(H_{\nu+1}S_\nu f)(x) = x^{-(\nu+3/2)} \frac{d}{dx} x^{\nu+3} \int_0^\infty t^{-(\nu+5/2)} f(t) dt \int_0^t v^{\nu+2} \mathbf{H}(xv) dv$$
$$= x^{-(\nu+3/2)} \frac{d}{dx} \int_0^\infty t^{1/2} f(t) dt \int_0^x v^{\nu+2} \mathbf{H}_\nu(tv) dv$$
$$= \int_0^\infty (xt)^{1/2} \mathbf{H}_\nu(xt) f(t) dt = (\mathscr{H}_\nu f)(x),$$

the differentiation under the integral sign being allowed since $f \in C_0$, and the result is proved.

THEOREM 5.2. *Suppose* $1 < p < \infty$, $\nu + \frac{1}{2} < \mu < \nu + 5/2$, $\mu \geqq \gamma(p)$. *Then* $\mathscr{H}_\nu$ *can be extended to* $\mathscr{L}_{\mu,p}$ *as an element of* $[\mathscr{L}_{\mu,p}, \mathscr{L}_{1-\mu,q}]$ *for any* $q \geqq p$ *such that* $q' \geqq 1/\mu$, *and except when* $\mu = -(\nu + \frac{1}{2})$, $\mathscr{H}_\nu$ *is one-to-one and*

$$\mathscr{H}_\nu(\mathscr{L}_{\mu,p}) = H_{\nu+1}(\mathscr{L}_{\mu,p}).$$

Further, if $f \in \mathscr{L}_{\mu,p}$, $1 < p \leqq 2$, $\nu + \frac{1}{2} < \mu < \nu + 5/2$, $\mu \geqq \gamma(p)$, *for* $\operatorname{Re} s = 1 - \mu$

$$(5.6) \qquad (\mathfrak{M}\mathscr{H}_\nu f)(s) = m_\nu(s) \tan \frac{\pi}{2} (s + \nu + \tfrac{1}{2}) (\mathfrak{M} f)(1 - s).$$

In addition, if $\nu > -1$, $1 < p < \infty$, $\nu + \frac{1}{2} < \mu < \nu + 3/2$, $\mu \geqq \gamma(p)$, *then*

$$(5.7) \qquad \mathscr{H}_\nu = H_\nu M_{-(\nu+1/2)} H_- M_{\nu+1/2},$$

and

$$(5.8) \qquad \mathscr{H}_\nu = -M_{\nu+1/2} H_+ M_{-(\nu+1/2)} H_\nu.$$

Proof. Since $\frac{1}{2} \leqq \gamma(p) \leqq \mu < \nu + 5/2$, $\nu > -2$, and thus by Theorem 5.1, on C_0, $\mathscr{H}_\nu = H_{\nu+1} S_\nu$. But by Lemma 5.1, $S_\nu \in [\mathscr{L}_{\mu,p}]$ for $\mu > \max(\nu + \frac{1}{2}, -(\nu + 3/2))$. Since $\max(\nu + \frac{1}{2}, -(\nu + 3/2)) = -(\nu + 3/2)$ only if $-2 < \nu < -1$, and for $-2 < \nu < -1$, $-(\nu + 3/2) < \frac{1}{2} \leqq \gamma(p) \leqq \mu$, so that for the values of μ under consideration in this theorem, $S_\nu \in [\mathscr{L}_{\mu,p}]$, and S_ν maps $\mathscr{L}_{\mu,p}$ one-to-one onto itself except when $\mu = -(\nu + \frac{1}{2})$. Also, since $\gamma(p) \leqq \mu < \nu + 5/2$, $H_{\nu+1} \in [\mathscr{L}_{\mu,p}, \mathscr{L}_{1-\mu,q}]$ for all $q \geqq p$ such that $q' \geqq 1/\mu$. Hence $H_{\nu+1} S_\nu \in [\mathscr{L}_{\mu,p}, \mathscr{L}_{1-\mu,q}]$ for all such q.

Thus we can extend $\mathscr{H}_\nu$ to $\mathscr{L}_{\mu,p}$ for $1 < p < \infty$, $\nu + \frac{1}{2} < \mu < \nu + 5/2$, $\mu \geqq \gamma(p)$, by defining it to be $H_{\nu+1} S_\nu$, and then $\mathscr{H}_\nu \in [\mathscr{L}_{\mu,p}, \mathscr{L}_{1-\mu,q}]$ for all $q \geqq p$ such that $q' \geqq 1/\mu$. Also, since S_ν is one-to-one except when $\mu = -(\nu + \frac{1}{2})$, and by Theorem 2.4 $H_{\nu+1}$ is one-to-one, $\mathscr{H}_\nu$ is one-to-one except when $\mu = -(\nu + \frac{1}{2})$. Since also $S_\nu(\mathscr{L}_{\mu,p}) = \mathscr{L}_{\mu,p}$ except when $\mu = -(\nu+\frac{1}{2})$, $\mathscr{H}_\nu(\mathscr{L}_{\mu,p}) = H_{\nu+1}(\mathscr{L}_{\mu,p})$ except when $\mu = -(\nu+\frac{1}{2})$.

From (2.3) and (5.2), if $f \in \mathscr{L}_{\mu,p}$, $1 < p \leqq 2$, $\nu + \frac{1}{2} < \mu < \nu + 5/2$, $\mu \geqq \gamma(p)$, then for Re $s = 1 - \mu$,

$$(\mathfrak{M}\mathscr{H}_\nu f)(s) = (\mathfrak{M}H_{\nu+1}S_\nu f)(s) = m_{\nu+1}(s)l_\nu(1-s)(\mathfrak{M}f)(1-s).$$

But we saw in the proof of Lemma 5.1 that

$$m_{\nu+1}(s)l_\nu(1-s) = m_\nu(s) \tan\frac{\pi}{2}(s+\nu+\tfrac{1}{2}),$$

and (5.6) follows.

For (5.7), note that if $\nu > -1$, $H_\nu M_{-(\nu+1/2)}H_- M_{\nu+1/2} \in [\mathscr{L}_{\mu,p}, \mathscr{L}_{1-\mu,q}]$ for all $q > p$ such that $q' \geqq 1/\mu$. For, $M_{\nu+1/2}$ maps $\mathscr{L}_{\mu,p}$ boundedly onto $\mathscr{L}_{\mu-\nu-1/2,p}$; from Theorem 3.1, H_- maps $\mathscr{L}_{\mu-\nu-1/2,p}$ boundedly into itself since $\nu + \frac{1}{2} < \mu < \nu + 3/2$ and thus $0 < \mu - \nu - \frac{1}{2} < 1$; $M_{-(\nu+1/2)}$ maps $\mathscr{L}_{\mu-\nu-1/2,p}$ boundedly onto $\mathscr{L}_{\mu,p}$; and H_ν maps $\mathscr{L}_{\mu,p}$ boundedly into $\mathscr{L}_{1-\mu,q}$ since $\gamma(p) \leqq \mu < \nu + 3/2$. But if $f \in \mathscr{L}_{\mu,2}$, $\frac{1}{2} \leqq \mu < \nu + 3/2$, then by (2.3), (3.8), and (1.10), if Re $s = 1 - \mu$,

$$\begin{aligned}(\mathfrak{M}H_\nu M_{-(\nu+1/2)}H_- M_{\nu+1/2}f)(s) &= m_\nu(s)(\mathfrak{M}M_{-(\nu+1/2)}H_-M_{\nu+1/2}f)(1-s)\\ &= m_\nu(s)(\mathfrak{M}H_-M_{\nu+1/2}f)(\tfrac{1}{2}-\nu-s)\\ &= m_\nu(s)\cot\frac{\pi}{2}(\tfrac{1}{2}-\nu-s)(\mathfrak{M}M_{\nu+1/2}f)(\tfrac{1}{2}-\nu-s)\\ &= m_\nu(s)\tan\frac{\pi}{2}(s+\nu+\tfrac{1}{2})(\mathfrak{M}f)(1-s) = (\mathfrak{M}\mathscr{H}_\nu f)(s),\end{aligned}$$

so that on $\mathscr{L}_{\mu,2}$, (5.7) holds. But both sides of (5.7) are in $[\mathscr{L}_{\mu,p}, \mathscr{L}_{1-\mu,q}]$ if $1 < p < \infty$, $\nu + \frac{1}{2} < \mu < \nu + 3/2$, $\mu \geqq \gamma(p)$, and hence since $\gamma(p) \geqq \frac{1}{2}$, (5.7) must hold on such $\mathscr{L}_{\mu,p}$. (5.8) follows similarly.

As a corollary of this result, we obtain further information about the range of H_ν.

COROLLARY 1. *If* $1 < p < \infty$, $\nu + \frac{1}{2} < \mu < \nu + 3/2$, $\mu \geqq \gamma(p)$, *then* $H_\nu(\mathscr{L}_{\mu,p})$ *is invariant under the operator* $M_{\nu+1/2}H_+M_{-(\nu+1/2)}$.

Proof. Since $\nu + 3/2 > \gamma(p) \geqq \frac{1}{2}$, $\nu > -1$. Also $\mu \neq -(\nu+\frac{1}{2})$, since if it were, $-(\nu+\frac{1}{2}) \geqq \gamma(p) \geqq \frac{1}{2}$, and $\nu \leqq -1$. But then, using [**5**; Theorem 1] and (5.8),

$$\begin{aligned}H_\nu(\mathscr{L}_{\mu,p}) = H_{\nu+1}(\mathscr{L}_{\mu,p}) &= \mathscr{H}_\nu(\mathscr{L}_{\mu,p})\\ &= (M_{\nu+1/2}H_+M_{-(\nu+1/2)}H_\nu)(\mathscr{L}_{\mu,p})\\ &= (M_{\nu+1/2}H_+M_{-(\nu+1/2)})(H_\nu(\mathscr{L}_{\mu,p})).\end{aligned}$$

Three comments may be made here. Firstly, the boundedness results again appear to be maximal with respect to the spaces $L_{\mu,p}$, for the same reasons as for $\mathscr{Y}_\nu$. Secondly, the exceptional value of μ for which $\mathscr{H}_\nu(L_{\mu,p}) \neq H_{\nu+1}(L_{\mu,p})$, namely $\mu = -(\nu + \frac{1}{2})$, can only occur for $-3/2 < \nu \leqq -1$ since $\frac{1}{2} \leqq \gamma(p) \leqq \mu < \nu + 3/2$; if $\nu = -1$, $p = 2$. Thirdly, $\mathscr{H}_\nu(\mathscr{L}_{1/2,2}) = \mathscr{L}_{1/2,2}$, $-2 < \nu < 0$, $\nu \neq -1$.

In order to develop an integral representation for $\mathscr{H}_\nu$, we need an analogue of Theorem 2.1.

THEOREM 5.3. *If $f \in \mathscr{L}_{\mu,p}$, $g \in \mathscr{L}_{\mu,q}$, $1 < p < \infty$, $1 < q < \infty$, $1/p + 1/q \geqq 1$, $\nu + \frac{1}{2} < \mu < \nu + 5/2$ and $\mu > \max(\gamma(p), \gamma(q))$, then*

$$(5.9) \qquad \int_0^\infty (\mathscr{H}_\nu f)(x) g(x) dx = \int_0^\infty f(x) (\mathscr{H}_\nu g)(x) dx.$$

Proof. This is practically the same as that for Theorem 2.1.

THEOREM 5.4. *If $f \in \mathscr{L}_{\mu,p}$, where $1 < p < \infty$, $\nu + \frac{1}{2} < \mu < \nu + 5/2$, $\mu \geqq \gamma(p)$, then for almost all $x > 0$,*

$$(5.10) \qquad (\mathscr{H}_\nu f)(x) = x^{-(\nu+1/2)} \frac{d}{dx} x^{\nu+1/2} \int_0^\infty (xt)^{1/2} \mathbf{H}_{\nu+1}(xt) f(t) dt/t, \quad \nu > -1$$

and

$$(5.11) \qquad (\mathscr{H}_\nu f)(x) = -x^{\nu-1/2} \frac{d}{dx} x^{-(\nu-1/2)} \int_0^\infty (xt)^{1/2} (\mathbf{H}_{\nu-1}(xt) - (xt)^\nu / (2^{\nu-1} \pi^{1/2} \Gamma(\nu + \tfrac{1}{2}))) f(t) dt/t, \quad -2 < \nu < 1.$$

Proof. If $\nu > -1$, $-(\nu + \frac{1}{2}) < \frac{1}{2} \leqq \gamma(p) \leqq \mu$, and by Lemma 2.1 $q_{\nu,x} \in \mathscr{L}_{\mu,p'}$. Hence from Theorem 5.3, if $x > 0$

$$\int_0^x t^{\nu+1/2} (\mathscr{H}_\nu f)(t) dt = \int_0^\infty q_{\nu,x}(t) (\mathscr{H}_\nu f)(t) dt = \int_0^\infty (\mathscr{H}_\nu q_{\nu,x})(t) f(t) dt.$$

Now also $q_{\nu,x} \in \mathscr{L}_{\mu,2}$, and hence from (5.6), with $\operatorname{Re} s = 1 - \mu$

$$\begin{aligned}(\mathfrak{M} \mathscr{H}_\nu q_{\nu,x})(s) &= m_\nu(s) \tan \frac{\pi}{2} (s + \nu + \tfrac{1}{2}) (\mathfrak{M} q_{\nu,x})(1 - s) \\ &= \frac{x^{\nu+3/2-s}}{\nu + 3/2 - s} m_\nu(s) \tan \frac{\pi}{2} (s + \nu + \tfrac{1}{2}).\end{aligned}$$

Hence from (1.9)

$$(\mathscr{H}_\nu q_{\nu,x})(t) = x^{\nu+3/2} \lim_{R \to \infty} \frac{1}{2\pi i} \int_{1-\mu-iR}^{1-\mu+iR} (xt)^{-s} m_\nu(s) \tan \frac{\pi}{2} (s + \nu + \tfrac{1}{2}) ds / (\nu + 3/2 - s),$$

where the limit is in the topology of $\mathscr{L}_{\mu,2}$. But, closing the contour to the left, by a residue calculus calculation similar to that mentioned in the proof of Lemma 5.1(d), pointwise a.e.

$$x^{\nu+3/2} \lim_{R\to\infty} \frac{1}{2\pi i} \int_{1-\mu-iR}^{1-\mu+iR} (xt)^{-s} m_\nu(s) \tan \frac{\pi}{2} (s+\nu+\tfrac{1}{2}) ds/(\nu+3/2-s)$$

$$= (xt)^{1/2} \mathbf{H}_{\nu+1}(xt)/t,$$

and (5.10) follows.

(5.11) follows in a similar manner, using $q_{-\nu,x}$, since $q_{-\nu,x} \in L_{\mu,p'}$ if $\nu < 1$.

6. Inverses. In this section we shall investigate to what extent $\mathscr{Y}_\nu$ and $\mathscr{H}_\nu$ are inverse to each other. We note firstly that in order that $\mathscr{H}_\nu \mathscr{Y}_\nu$ or $\mathscr{Y}_\nu \mathscr{H}_\nu$ be defined on $\mathscr{L}_{\mu,p}$, it is necessary that $\mu = \frac{1}{2}$, $p = 2$, and $-1 < \nu < 0$. For, in order that $\mathscr{Y}_\nu$ be defined on $\mathscr{L}_{\mu,p}$, we need $\gamma(p) \leqq \mu < 3/2 - |\nu|$, and thus since $\gamma(p) \geqq \frac{1}{2}$, $|\nu| < 1$, and $\mu \geqq \frac{1}{2}$. But $\mathscr{Y}_\nu$ maps $\mathscr{L}_{\mu,p}$ into $\mathscr{L}_{1-\mu,q}$ and thus for $\mathscr{H}_\nu \mathscr{Y}_\nu$ to be defined we need $\frac{1}{2} \leqq \gamma(q) \leqq 1 - \mu$ and $\nu + \frac{1}{2} < 1 - \mu < \nu + 5/2$. Thus since $\mu \geqq \frac{1}{2}$ and $1 - \mu \geqq \frac{1}{2}$, $\mu = \frac{1}{2}$, $\gamma(p) = \frac{1}{2}$, and since then $\nu + \frac{1}{2} < \frac{1}{2}$, and $|\nu| < 1$, $\nu < 0$. However, if $-1 < \nu < 0$, then on $\mathscr{L}_{1/2,2}$, $\mathscr{Y}_\nu$ and $\mathscr{H}_\nu$ are inverses, as the following theorem shows.

THEOREM 6.1. *If $-1 < \nu < 0$, then on $\mathscr{L}_{1/2,2}$*

$$\mathscr{H}_\nu \mathscr{Y}_\nu = \mathscr{Y}_\nu \mathscr{H}_\nu = I.$$

Proof. Since $-1 < \nu < 0$, $\frac{1}{2} < 3/2 + \nu = 3/2 - |\nu|$, and hence from (4.2),

$$\mathscr{Y}_\nu = H_\nu M_{\nu-1/2} H_+ M_{-(\nu-1/2)}.$$

Also, since $-1 < \nu < 0$, $\nu + \frac{1}{2} < \frac{1}{2} < \nu + 3/2$, and hence from (5.8),

$$\mathscr{H}_\nu = -M_{\nu+1/2} H_+ M_{-(\nu+1/2)} H_\nu.$$

Note that $M_{\nu-1/2} H_+ M_{-(\nu-1/2)}$ maps $\mathscr{L}_{1/2,2}$ onto itself, as shown in the proof of Theorem 4.1, and hence, since on $\mathscr{L}_{1/2,2}$, $H_\nu^2 = I$, using (1.5), (3.10), and (3.3),

$$\begin{aligned} \mathscr{H}_\nu \mathscr{Y}_\nu &= -M_{\nu+1/2} H_+ M_{-(\nu+1/2)} H_\nu H_\nu M_{\nu-1/2} H_+ M_{-(\nu-1/2)} \\ &= -M_{\nu+1/2} H_+ M_{-(\nu+1/2)} M_{\nu-1/2} H_+ M_{-(\nu-1/2)} \\ &= -M_{\nu+1/2} H_+ M_{-1} H_+ M_1 M_{-(\nu+1/2)} \\ &= -M_{\nu+1/2} H_+ H_- M_{-(\nu+1/2)} = M_{\nu+1/2} M_{-(\nu+1/2)} = I, \end{aligned}$$

and similarly, using (4.3) and (5.7), $\mathscr{Y}_\nu \mathscr{H}_\nu = I$.

However we can also consider $\mathscr{H}_\nu$ in the product $\mathscr{H}_\nu \mathscr{Y}_\nu$ to be given by (5.10), and $\mathscr{Y}_\nu$ in the product $\mathscr{Y}_\nu \mathscr{H}_\nu$ to be given by (4.6), and doing this we obtain a considerable extension of the results of Theorem 6.1, as the following two theorems show.

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THEOREM 6.2. *If* $f \in \mathscr{L}_{\mu,p}$, *where* $1 < p < \infty$, $\gamma(p) \leqq \mu < \min(\frac{1}{2} - \nu, \nu + 3/2)$, *then for almost all* $x > 0$,

$$(6.1) \quad f(x) = x^{-(\nu+1/2)} \frac{d}{dx} x^{\nu+1/2} \int_0^\infty (xt)^{1/2} \mathbf{H}_{\nu+1}(xt) (\mathscr{Y}_\nu f)(t) dt/t.$$

Proof. Since $\gamma(p) \geqq \frac{1}{2}$, $\frac{1}{2} < \min(\frac{1}{2} - \nu, \nu + 3/2)$, and it follows that $-1 < \nu < 0$, so that $\gamma(p) \leqq \mu < 3/2 + \nu = 3/2 - |\nu|$, and thus from Theorem 4.2, $\mathscr{Y}_\nu f$ exists and is in $\mathscr{L}_{1-\mu,p}$. Hence $M_{\nu+1/2}\mathscr{Y}_\nu f \in \mathscr{L}_{1/2-\mu-\nu,p}$. Since $-1 < \nu < 0$,

$$-(\nu + \tfrac{1}{2}) < \tfrac{1}{2} \leqq \gamma(p) \leqq \mu < \min(\tfrac{1}{2} - \nu, \nu + 3/2) \leqq 1,$$

and hence, from Lemma 2.1, $r_{\nu,x} \in \mathscr{L}_{\mu,p'}$, and thus $M_{-(\nu+1/2)} r_{\nu,x} \in \mathscr{L}_{1/2+\mu+\nu,p'}$; note also that $-1 < \frac{1}{2} + \mu + \nu < 1$, since for $-1 < \nu < 0$,

$$-(\nu + 3/2) < -\tfrac{1}{2} < \mu < \tfrac{1}{2} - \nu,$$

and that

$$1 - (\tfrac{1}{2} + \mu + \nu) = \tfrac{1}{2} - \mu - \nu.$$

Hence from Theorem 3.1 and (3.4),

$$(6.2) \quad \int_0^\infty (H_+ M_{-(\nu+1/2)} r_{\nu,x})(t) (M_{\nu+1/2} \mathscr{Y}_\nu f)(t) dt$$
$$= -\int_0^\infty (M_{-(\nu+1/2)} r_{\nu,x})(t) (H_- M_{\nu+1/2} \mathscr{Y}_\nu f)(t) dt.$$

But from [**2**; 15.3(15)], remembering that H_+ is the restriction to $(0, \infty)$ of the Hilbert transformation of even functions,

$$(H_+ M_{-(\nu+1/2)} r_{\nu,x})(t) = -x^{\nu+1} t^{-(\nu+1)} \mathbf{H}_{\nu+1}(xt).$$

Also, from (4.3),

$$\mathscr{Y}_\nu = -M_{-(\nu-1/2)} H_- M_{\nu-1/2} H_\nu,$$

so that, using (1.5), (3.9), and (3.3),

$$\begin{aligned} H_- M_{\nu+1/2} \mathscr{Y}_\nu f &= -H_- M_{\nu+1/2} M_{-(\nu-1/2)} H_- M_{\nu-1/2} H_\nu f \\ &= -H_- M_1 H_- M_{-1} M_{\nu+1/2} H_\nu f \\ &= -H_- H_+ M_{\nu+1/2} H_\nu f = M_{\nu+1/2} H_\nu f, \end{aligned}$$

and substituting in (6.2), using (2.7), it follows that

$$(6.3) \quad x^{\nu+1/2} \int_0^\infty (xt)^{1/2} \mathbf{H}_{\nu+1}(xt) (\mathscr{Y}_\nu f)(t) dt/t$$
$$= x^{\nu+1/2} \int_0^\infty (xt)^{1/2} J_{\nu+1}(xt) (H_\nu f)(t) dt/t,$$

https://doi.org/10.4153/CJM-1980-079-4 Published online by Cambridge University Press

and the result follows from Theorem 2.3, since, as noted, $\gamma(p) \leqq \mu < \nu + 3/2$, and $\mu < 1$.

THEOREM 6.3. *If $f \in \mathscr{L}_{\mu,p}$, where $1 < p < \infty$, $\nu + \frac{1}{2} < \mu < \min(1, \nu + 3/2)$, $\mu \geqq \gamma(p)$, then for almost all $x > 0$,*

$$(6.4)\quad f(x) = x^{-(\nu+1/2)} \frac{d}{dx} x^{\nu+1/2} \int_0^\infty (xt)^{1/2} (Y_{\nu+1}(xt) + \Gamma(\nu+1)(2/xt)^{\nu+1}/\pi)(\mathscr{H}_\nu f)(t)dt/t.$$

Proof. Note that since $\nu + \frac{1}{2} < 1$ and $\nu + 3/2 \geqq \gamma(p) \geqq \frac{1}{2}$, $-1 < \nu < \frac{1}{2}$. By Theorem 5.2, $\mathscr{H}_\nu f \in \mathscr{L}_{1-\mu,p}$, and hence $M_{-(\nu-1/2)}\mathscr{H}_\nu f \in \mathscr{L}_{1/2+\nu-\mu,p}$. Clearly $-(\nu + \frac{1}{2}) < \frac{1}{2} \leqq \gamma(p) \leqq \mu < 1$, and hence by Lemma 2.1 $r_{\nu,x} \in \mathscr{L}_{\mu,p'}$, and hence $M_{\nu-1/2}r_{\nu,x} \in \mathscr{L}_{1/2+\mu-\nu,p'}$. Further, $-1 < \frac{1}{2} + \nu - \mu < 1$, since $\nu - \frac{1}{2} < \nu + \frac{1}{2} < \mu < \nu + 3/2$. Hence from Theorem 3.1 and (3.4)

$$(6.5)\quad \int_0^\infty (H_- M_{\nu-1/2} r_{\nu,x})(t)(M_{-(\nu-1/2)}\mathscr{H}_\nu f)(t)dt = -\int_0^\infty (M_{\nu-1/2} r_{\nu,x})(t)(H_+ M_{-(\nu-1/2)}\mathscr{H}_\nu f)(t)dt.$$

$(H_- M_{\nu-1/2} r_{\nu,x})(t)$ is given in Lemma 4.1. Also, since $\nu > -1$ and $v + \frac{1}{2} < \mu < \nu + 3/2$, from (5.8),

$$\mathscr{H}_\nu f = -M_{\nu+1/2} H_+ M_{-(\nu+1/2)} H_\nu f,$$

so that using (1.5), (3.8), and (3.3),

$$\begin{aligned} H_+ M_{-(\nu-1/2)}\mathscr{H}_\nu f &= -H_+ M_{-(\nu-1/2)} M_{\nu+1/2} H_+ M_{-(\nu+1/2)} H_\nu f \\ &= -M_1 M_{-1} H_+ M_1 H_+ M_{-(\nu+1/2)} H_\nu f = -M_1 H_- H_+ M_{-(\nu-1/2)} H_\nu f \\ &= M_{-(\nu-1/2)} H_\nu f, \end{aligned}$$

and substituting in (6.5), using (2.7), and multiplying both sides by -1, we obtain

$$(6.6)\quad x^{\nu+1/2} \int_0^\infty (xt)^{1/2}(Y_{\nu+1}(xt) + \Gamma(\nu+1)(2/xt)^{\nu+1}/\pi)(\mathscr{H}_\nu f)(t)dt/t = x^{\nu+1/2} \int_0^\infty (xt)^{1/2} J_{\nu+1}(xt)(H_\nu f)(t)dt/t,$$

and the result now follows from Theorem 2.3.

Inverses for $\mathscr{Y}_\nu$ and $\mathscr{H}_\nu$ for other ranges of the parameters involved can also be determined. For $\mathscr{Y}_\nu$, it follows from (4.2) that

$$\mathscr{Y}_\nu^{-1} = -M_{\nu-1/2} H_- M_{-(\nu-1/2)} H_\nu^{-1},$$

and H_ν^{-1} can be determined using the remarks at the end of Section 2.

For $\mathscr{H}_\nu$, it follows that since $\mathscr{H}_\nu = H_{\nu+1}S_\nu$, $\mathscr{H}_\nu^{-1} = S_\nu^{-1}H_{\nu+1}^{-1}$. $H_{\nu+1}^{-1}$ can be determined using the remarks at the end of Section 2, while it is easy to see that

$$S_\nu^{-1} = (J_{2,,1/2(\nu-1/2)/2})^{-1}J_{2,1/2,(\nu+1/2)/2},$$

where $J_{\nu,\beta,\eta}$ is given in [**3**; (1.3)], and $(J_{\nu,\beta,\eta})^{-1}$ can be determined in much the same way as $(I_{\alpha,\xi})^{-1}$ in Section 2.

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