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THE CLASS $A_\infty^+(g)$ AND THE ONE-SIDED REVERSE HÖLDER INEQUALITY

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ABSTRACT. We give a direct proof that w is an $A_\infty^+(g)$ weight if and only if w satisfies a one-sided, weighted reverse Hölder inequality.

1. **Introduction.** Given a function f and a non-negative, locally integrable weight g on $\mathbb{R}$, define the one-sided, weighted maximal function of f, $M_g^+ f$, to be

$$M_g^+ f(x) = \sup_{t>0} \frac{1}{g(I_t)} \int_{I_t} |f| g\,dx,$$

where $I_t = [x, x+t]$. Similarly, we can define the backwards, one-sided maximal operator M_g^-. If $g = 1$, this is the maximal operator as originally defined by Hardy and Littlewood [4]. Weighted norm inequalities for M_g^+ were first studied by Sawyer [8] (in the case $g = 1$) and by Martín-Reyes, Ortega Salvador and de la Torre [6]. They showed that for $1 < p < \infty$, M_g^+ is a bounded operator from $L^p(w)$ into itself if and only if w is in $A_p^+(g)$: there exists a constant C such that

$$\Big(\int_{I^-} w\,dx\Big)\bigg(\int_{I^+} \Big(\frac{w}{g}\Big)^{1-p'} g\,dx\bigg)^{p-1} \le C\Big(\int_I g\,dx\Big)^p,$$

where $I = [a, b]$ is any interval, $I^- = [a, c]$, and $I^+ = [c, b]$ for any $a < c < b$. These classes are analogous to the (A_p) classes which govern the weighted norm inequalities for the (two-sided) Hardy-Littlewood maximal operator.

More recently Martín-Reyes [5] gave simpler proofs of the weighted norm inequalities for M_g^+; and Martín-Reyes, Pick and de la Torre [7] showed that $A_\infty^+(g)$, the union of all the $A_p^+(g)$ classes, has many properties similar to those of (A_∞). In both papers a central step is to show that functions in $A_\infty^+(g)$ satisfy what they called a weak reverse Hölder inequality: there exists $\delta > 0$ such that for any interval $I = [a, b]$,

$$\int_I \Big(\frac{w}{g}\Big)^{1+\delta} g\,dx \le C \int_I w\,dx \cdot \bigg(M_g^-\Big(\frac{w}{g}\chi_I\Big)(b)\bigg)^\delta. \tag{1}$$

This inequality is less versatile than a reverse Hölder inequality, and the proofs which use it are correspondingly more difficult. In particular, the proof given by Martín-Reyes [5]

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that $w \in A_p^+(g)$ implies $w \in A_{p-\epsilon}^+(g)$ for some $\epsilon > 0$ uses the weighted norm inequalities for M_g^+. Martín-Reyes posed the problem of finding a proof of this result which only used the intrinsic properties of the class $A_p^+(g)$.

In [3] Cruz-Uribe, Neugebauer and Olesen showed that in the case $g = 1$, inequality (1) is equivalent to a one-sided reverse Hölder inequality:

$$\frac{1}{|I^-|}\int_{I^-} w^{1+\delta}\,dx \le C\left(\frac{1}{|I|}\int_I w\,dx\right)^{1+\delta},$$

where $I = [a,b]$ is any interval and $I^- = [a,c]$, where $2|I^-| = |I|$. Using this they gave a direct proof that $w \in A_p^+$ implies that $w \in A_{p-\epsilon}^+$. The purpose of this paper is to generalize their result to arbitrary g and to give a proof which avoids inequality (1). To be precise, we will prove the following theorem.

THEOREM 1.1. *Given a weight g, the following are equivalent:*

(1) $w \in A_\infty^+(g)$;

(2) For some $s > 1$, $w \in RH_s^+(g)$: there exists a constant $C > 0$ such that

$$\frac{1}{g(I^-)}\int_{I^-}\left(\frac{w}{g}\right)^s g\,dx \le C\left(\frac{1}{g(I)}\int_I w\,dx\right)^s,$$

where $I = [a,b]$ is any interval and $I^- = [a,c]$ is such that $2g(I^-) = g(I)$.

To prove Theorem 1.1 it will suffice to show that if $w \in A_\infty^+(g)$ then $w \in RH_s^+(g)$ for some $s > 1$. The converse is straightforward: if $w \in RH_s^+(g)$ then $g \in A_{s'}^-(w)$, and if $g \in A_\infty^-(w)$ then $w \in A_\infty^+(g)$. The first implication follows from the definitions if I^- and I^+ are such that $g(I^-) = g(I^+)$. (I want to thank A. de la Torre for this observation.) That this is true for arbitrary I^- and I^+ follows for $g = 1$ from Lemma 6.4 in [3], and the proof of this lemma extends with slight modification to arbitrary g. The second implication is from [7].

The proof that w is in $RH_s^+(g)$ is similar to the proof of inequality (1) in [6] or [5], each of which in turn follows the proof of the reverse Hölder inequality given by Coifman and C. Fefferman [2]. It depends on a sharp covering lemma for intervals on the real line. The proof itself is in Section 3 below; in Section 2 we gather some preliminary results.

Finally, note that the one-sided reverse Hölder inequality and the proof that if $w \in A_\infty^+(g)$ then $w \in RH_s^+(g)$ simplifies the proof of the main result in [7] (by eliminating the weak reverse Hölder inequality), and the proof that if $w \in A_p^+(g)$ then $w \in A_{p-\epsilon}^+(g)$ given in [3] extends to arbitrary g without change.

2. **Preliminary Results.**

Throughout this paper all functions are assumed to be locally integrable and the weight g is assumed to be positive. The letter C denotes a positive constant whose value may change at each appearance, and for $p > 1$, $p' = p/(p-1)$ is the conjugate exponent of p. Given a Borel set E and a function f, let $|E|$ denote the Lebesgue measure of E and $f(E) = \int_E f\,dx$.

We will need the following property of $A_\infty^+(g)$ weights proved by Martín-Reyes, Pick and de la Torre [7].

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LEMMA 2.1. *If $w \in A_\infty^+(g)$ then for every α, $0 < \alpha < 1$, there exists a $\beta > 0$ such that, given $t > 0$ and an interval $I = [a, b]$ on which $w(I_x) \geq tg(I_x)$ for all $I_x = [a, x]$, $x \in I$, then*

$$g\big(\{x \in I : w(x) > \beta t g(x)\}\big) > \alpha g(I).$$

We will also need the following covering lemma due to Jesus Aldaz; the proof is in Bliedtner and Loeb [1].

LEMMA 2.2. *If μ is a finite Borel measure on $\mathbb{R}$, and if $\mathcal{I}$ is an arbitrary collection of non-degenerate intervals, then for each $\delta > 0$ there exists a finite subcollection, $\mathcal{I}_\delta$, of disjoint intervals in $\mathcal{I}$ such that*

$$\mu\Big(\bigcup_{I \in \mathcal{I}} I\Big) \leq (2+\delta) \sum_{I_k \in \mathcal{I}_\delta} \mu(I_k).$$

Finally, we will need the following decomposition of finite intervals which can be thought of as a weighted Whitney decomposition. It was first used in a slightly different form in [5]; it appeared in this notation (for $g = 1$) in [3].

DEFINITION 2.3. Given a weight g and an interval $I = [a, b]$, form the "plus/minus" decomposition of I with respect to g as follows: let $x_0 = a$, and for $k > 0$ let x_k be the point such that $g([x_{k-1}, b]) = 2g([x_{k-1}, x_k])$. Then for $k \geq 1$ define the intervals $J_k = [x_{k-1}, x_{k+1}]$, $J_k^- = [x_{k-1}, x_k]$ and $J_k^+ = [x_k, x_{k+1}]$.

It is immediate from this definition that for all k, $g(J_k^-) = 2g(J_k^+)$, I is the union of the J_k^-'s, and the J_k's have finite overlap.

3. **Proof of Theorem 1.1.** We first prove that if $w \in A_\infty^+(g)$ then there exist positive constants β and C such that

$$w\big(\{x \in I^- : w(x) > tg(x)\}\big) \leq Ctg\big(\{x \in I : w(x) > \beta t g(x)\}\big), \tag{2}$$

for all $t > t_0 = 3w(I)/g(I)$, where $I = [a, b]$ is any interval and $I^- = [a, c]$ is such that $g(I^-) = \frac{2}{3} g(I)$. To show this, fix $I = [a, b]$ and $t > t_0$. Let $O(t) = \{x \in I^- : w(x) > tg(x)\}$. By the Lebesgue differentiation theorem, for almost every $x \in O(t)$, if $I_h = [x, x+h]$, $h > 0$, then

$$\frac{w(x)}{g(x)} = \lim_{h \to 0} \frac{1}{g(I_h)} \int_{I_h} \Big(\frac{w}{g}\Big) g\, dx.$$

Therefore, there exists $h_0 > 0$ such that if $0 < h \leq h_0$ then

$$\frac{w(I_h)}{g(I_h)} > t.$$

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On the other hand, fix h such that $x + h = b$. Then

$$\frac{w(I_h)}{g(I_h)} \leq 3\frac{w(I)}{g(I)} = t_0 < t.$$

Since this ratio is continuous in h, by the intermediate value theorem there exists $h_1 > h_0$ such that $I_{h_1} \subset I$, $w(I_{h_1})/g(I_{h_1}) = t$, and $w(I_h)/g(I_h) \geq t$ for all $0 < h < h_1$. Let $I_x = I_{h_1}$. Then, up to a set of measure zero, $O(t)$ is contained in the union of the I_x's. Therefore, by Lemma 2.2 there exists a finite, disjoint subcollection $\{I_j\}$ of the I_x's such that

$$w\big(O(t)\big) \leq w\Big(\bigcup I_x\Big) \leq 3\sum_j w(I_j).$$

By our construction of the I_x's and by Lemma 2.1, there exist positive constants α and β such that

$$\begin{aligned} 3\sum_j w(I_j) &= 3t\sum_j g(I_j) \\ &\leq \frac{3t}{\alpha}\sum_j g\Big(\big\{x \in I_j : w(x) > \beta t g(x)\big\}\Big) \\ &\leq Ctg\Big(\big\{x \in I : w(x) > \beta t g(x)\big\}\Big). \end{aligned}$$

Inequality (2) follows at once.

Now fix an interval I and form the plus/minus decomposition of I with respect to g described in Definition 2.3. For each k, since $g(J_k^-) = \frac{2}{3}g(J_k)$, inequality (2) holds for the interval $J_k = J_k^- \cup J_k^+$:

$$w\Big(\big\{x \in J_k^- : w(x) > tg(x)\big\}\Big) \leq Ctg\Big(\big\{x \in J_k : w(x) > \beta t g(x)\big\}\Big),$$

for $t > t_k = 3w(J_k)/g(J_k)$. Multiply this inequality by $t^{\delta-1}$ ($\delta > 0$ to be fixed below) and integrate from t_k to infinity. This gives

$$\begin{aligned} \int_{t_k}^{\infty} t^{\delta-1} w\Big(\big\{x \in J_k^- : w(x) > tg(x)\big\}\Big)\,dt &\leq C\int_0^{\infty} t^{\delta} g\Big(\big\{x \in J_k : w(x) > \beta t g(x)\big\}\Big)\,dt \\ &\leq \frac{D}{1+\delta}\int_{J_k}\Big(\frac{w}{g}\Big)^{1+\delta} g\,dx. \end{aligned}$$

The constant D depends only on the constants from Lemma 2.1. By Fubini's theorem, the left-hand side is equal to

$$\begin{aligned} &\int_{\{x \in J_k^- : w(x) > t_k g(x)\}} \int_{t_k}^{w(x)/g(x)} t^{\delta-1}\,dt\,w(x)\,dx \\ &\qquad = \int_{\{x \in J_k^- : w(x) > t_k g(x)\}} w(x)\cdot\frac{1}{\delta}\left[\left(\frac{w(x)}{g(x)}\right)^{\delta} - t_k^{\delta}\right] dx \\ &\qquad \geq \frac{1}{\delta}\int_{J_k^-}\Big(\frac{w}{g}\Big)^{1+\delta} g\,dx - \frac{t_k^{\delta}}{\delta}\int_{J_k^-} w\,dx. \end{aligned}$$

Therefore, for all k we have the inequality

$$\frac{1}{\delta}\int_{J_k^-}\Big(\frac{w}{g}\Big)^{1+\delta} g\,dx - \frac{D}{1+\delta}\int_{J_k}\Big(\frac{w}{g}\Big)^{1+\delta} g\,dx \leq \frac{t_k^{\delta}}{\delta}\int_{J_k^-} w\,dx,$$

which in turn implies that

$$g(J_k)^\delta \int_{J_k^-} \left(\frac{w}{g}\right)^{1+\delta} g\,dx - \frac{\delta D g(J_k)^\delta}{1+\delta} \int_{J_k} \left(\frac{w}{g}\right)^{1+\delta} g\,dx \le 3^\delta \left(\int_{J_k} w\,dx\right)^{1+\delta}.$$

Now take the sum of these inequalities over all $k > 0$. Since the J_k's have finite overlap, the right-hand side becomes

$$3^\delta \sum_k \left(\int_{J_k} w\,dx\right)^{1+\delta} \le 3^\delta \left(\sum_k \int_{J_k} w\,dx\right)^{1+\delta} \le C\left(\int_I w\,dx\right)^{1+\delta}.$$

Since $J_k = J_k^- \cup J_k^+$, the left-hand side becomes

$$\sum_k \left[\left(1 - \frac{\delta D}{1+\delta}\right) g(J_k)^\delta \int_{J_k^-} \left(\frac{w}{g}\right)^{1+\delta} g\,dx - \frac{\delta D}{1+\delta} g(J_k)^\delta \int_{J_k^+} \left(\frac{w}{g}\right)^{1+\delta} g\,dx\right].$$

Since $J_k^+ = J_{k+1}^-$, this will be a telescoping series in which all terms but the first cancel one another if there exists $\delta > 0$ such that

$$\left(1 - \frac{\delta D}{1+\delta}\right) g(J_{k+1})^\delta = \frac{\delta D}{1+\delta} g(J_k)^\delta.$$

Since $g(J_k) = 2g(J_{k+1})$, this is equivalent to

$$\frac{1}{2^\delta}\left(1 - \frac{\delta D}{1+\delta}\right) = \frac{\delta D}{1+\delta}.$$

This is clearly true for some $\delta > 0$. Therefore, for this value of δ the series is equal to

$$\left(1 - \frac{\delta D}{1+\delta}\right) g(J_1)^\delta \int_{J_1^-} \left(\frac{w}{g}\right)^{1+\delta} g\,dx.$$

Since $J_1^- = I^-$ and $g(J_1) = \frac{3}{4} g(I)$, it follows that $w \in RH_s^+(g)$ for $s = 1 + \delta$.

References

1. J. Bliedtner and P. Loeb, *A reduction technique for limit theorems in analysis and probability theory*, Ark. Mat. **30**(1992), 25–43.

2. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51**(1974), 241–250.

3. D. Cruz-Uribe, SFO, C. J. Neugebauer and V. Olesen, *The one-sided minimal operator and the one-sided reverse Hölder inequality*, Studia Math. **116**(1995), 255–270.

4. G. H. Hardy and J. E. Littlewood, *A maximal theorem with function-theoretic applications*, Acta Math. **54**(1930), 81–116.

5. F. J. Martín-Reyes, *New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Proc. Amer. Math. Soc. (3)**117**(1993), 691–698.

6. F. J. Martín-Reyes, P. Ortega Salvador, and A. de la Torre, *Weighted inequalities for one-sided maximal functions*, Trans. Amer. Math. Soc. (2)**319**(1990), 517–534.

7. F. J. Martín-Reyes, L. Pick and A. de la Torre, *A^+_∞ condition*, Can. J. Math. **45**(1993), 1231–1244.

8. E. Sawyer, *Weighted inequalities for the one sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc. **297**(1986), 53–61.

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