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A SIMPLE C^*-ALGEBRA GENERATED BY TWO FINITE-ORDER UNITARIES

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§ 1. Introduction. We present an example which illustrates several peculiar phenomena that may occur in the theory of C^*-algebras. In particular, we show that a C^*-subalgebra of a nuclear (amenable) C^*-algebra need not be nuclear (amenable).

The central object of this paper is a pair of abstract unitary matrices,

$$u = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right] \quad \text{and} \quad v = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right],$$

acting on a *common* Hilbert space. For an explicit construction, we may decompose an infinite-dimensional Hilbert space H into $H = H_0 \oplus H_1$, $H_1 = H_\alpha \oplus H_\beta$ with $\dim H_0 = \dim H_1 = \dim H_\alpha = \dim H_\beta$, letting u, $v \in B(H)$ be any two unitary operators such that

$$u: \begin{cases} H_0 \to H_1 \\ H_1 \to H_0 \end{cases}, \quad v: \begin{cases} H_0 \to H_\alpha \\ H_\alpha \to H_\beta \\ H_\beta \to H_0 \end{cases}$$

and $u^2 = 1$, $v^3 = 1$. Whereas many choices of u, v are possible, it might be surprising to see that $C^*(u, v)$, the C^*-algebra generated by u and v, is algebraically unique; namely, if (u_1, v_1) is another pair of such unitaries, then $C^*(u, v)$ is canonically $*$-isomorphic with $C^*(u_1, v_1)$ (Theorem 2.6). In fact, we deduce further that $C^*(u, v)$ is a simple C^*-algebra with a unique (faithful) trace (Theorem 2.8).

In spite of its very elegant structure, $C^*(u, v)$ does possess several "pathological" properties. Foremost, we show directly that $C^*(u, v)$ is non-amenable in the sense of Johnson [**16**]. This appears to be the first explicit example in literature that a C^*-algebra may have a non-vanishing cohomological coefficient with respect to a dual Banach bimodule. In answering a question of Thayer [**22**], we show that $C^*(u, v)$, as a tracial C^*-algebra, is not quasi-diagonal. Another peculiar fact lies in the non-nuclearity of $C^*(u, v)$; actually, we prove that $C^*(u, v)$ fails to enjoy a completely positive metric approximation property. The last result, having an interesting counterpart in Banach space theory, leads to a highly plausible conjecture: $C^*(u, v)$ might not have a Schauder basis.

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Recently, Cuntz has given a systematic investigation of $\mathcal{O}_2$, the simple C^*-algebra generated by 2 isometries s, t satisfying $ss^* + tt^* = 1$. It has been shown that $\mathcal{O}_2$ is nuclear [**14**] and even amenable [**19**]. By a straightforward verification, we see that $C^*(u, v)$ is indeed a subalgebra of $\mathcal{O}_2$. This enables us to assert that a C^*-subalgebra of a nuclear (amenable) C^*-algebra need not be nuclear (amenable).

Notably, the proof of Theorem 2.8 is a variant of Powers' treatment [**18**] on $C_{\text{reg}}^*(\mathbf{F}_2)$, the left regular representation of the free group on two generators. While some results of this paper are applicable to $C_{\text{reg}}^*(\mathbf{F}_2)$ as well, we remark, however, that $C^*(u, v)$ is feasible for a space-free description and possesses a much more tractable structure. Typically, u, v being finite-order unitaries leads to immediate consequences: $C^*(u, v)$ is singly generated and contains non-trivial projections. It appears unlikely (in fact, it remains open) that $C_{\text{reg}}^*(\mathbf{F}_2)$ could have similar properties.

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§ 2. The basic structure. We begin with simple observations on words in u and v subject to the equality $u^2 = v^3 = 1$. By a *reduced word* w, we mean a formal product $x_1x_2 \ldots x_l$, with $x_i \in \{u, v, v^{-1}\}$, satisfying the condition: whenever $j < l$ we have

$$\begin{cases} x_j = u \Rightarrow x_{j+1} = v^{\pm 1} \\ x_j = v^{\pm 1} \Rightarrow x_{j+1} = u. \end{cases}$$

The subscript l is called the *length* of w. Obviously, each formal product, subject to $u^2 = v^3 = 1$, can be simplified to a unique reduced word.

LEMMA 2.1. *Let w be a reduced word in u, v $(u^2 = v^3 = 1)$ of length $l > 0$, and $z_n = (vu)^n v^{-1}$. Then the reduced form for $z_n^{-1} w z_n$ begins with $v^{\pm 1}$ and ends with $v^{\pm 1}$ whenever $2n \geqq l$.*

Proof. We note that the last entry v^{-1} in z_n continues to be the last entry in the reduced form for wz_n, because length(w) < length(z_n). Suppose the reduced form for $z_n^{-1}wz_n$ were ending with u. Then the last entry v^{-1} in wz_n would be cancelled out by the entry v in the left multiplication of $z_n^{-1} = v(uv^{-1})^n$; i.e., $z_n^{-1}wz_n = 1$, $w = 1$, which were impossible from the assumption of length(w) > 0. Therefore, the reduced form for $z_n^{-1}wz_n$ ends with $v^{\pm 1}$, and similarly begins with $v^{\pm 1}$. Thus the lemma is proved.

We will also need some manipulations on 2×2 matrix-operators. With respect to a fixed orthogonal decomposition of a Hilbert space $H = H_0 \oplus H_1$, we can write each operator $a \in B(H)$ in the form

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

In particular, $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is the projection onto H_0, and

$$\begin{cases} a = \begin{bmatrix} 0 & \# \\ \# & \# \end{bmatrix} \text{ if and only if } eae = 0, \\ a = \begin{bmatrix} \# & \# \\ \# & 0 \end{bmatrix} \text{ if and only if } (1-e)a(1-e) = 0, \end{cases}$$

where #'s stand for entries we do not have to evaluate.

LEMMA 2.2. *Let $u_1, \ldots u_n$ be unitaries such that each $u_i u_j^*$ is of the form $\begin{bmatrix} \# & \# \\ \# & 0 \end{bmatrix}$ whenever $i \neq j$. Suppose b is an operator of the form $\begin{bmatrix} 0 & \# \\ \# & \# \end{bmatrix}$; then $\|(1/n) \sum_{i=1}^n u_i^* b u_i\| \leqq 2\|b\|/\sqrt{n}$.*

Proof. We first assume that $b = \begin{bmatrix} 0 & 0 \\ \# & \# \end{bmatrix}$. Then it is clear that whenever $c = \begin{bmatrix} \# & \# \\ 0 & 0 \end{bmatrix}$, we have $\|b + c\|^2 \leqq \|b\|^2 + \|c\|^2$. From the fact

$$(u_j u_i^*) b (u_i u_j^*) = \begin{bmatrix} \# & \# \\ \# & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \# & \# \end{bmatrix} \begin{bmatrix} \# & \# \\ \# & 0 \end{bmatrix} = \begin{bmatrix} \# & \# \\ 0 & 0 \end{bmatrix} \quad (i \neq j),$$

we derive that

$$\begin{aligned} \|\textstyle\sum_{i=1}^n u_i^* b u_i\|^2 &= \|u_1(\textstyle\sum_{i=1}^n u_i^* b u_i) u_1^*\|^2 = \|b + \textstyle\sum_{i=2}^n u_1 u_i^* b u_i u_1^*\|^2 \\ &\leqq \|b\|^2 + \|\textstyle\sum_{i=2}^n u_1 u_i^* b u_i u_1^*\|^2 = \|b\|^2 + \|\textstyle\sum_{i=2}^n u_i^* b u_i\|^2 \end{aligned}$$

and continuing this process,

$$\|\textstyle\sum_{i=1}^n u_i^* b u_i\|^2 \leqq n\|b\|^2.$$

Now given $b = \begin{bmatrix} 0 & \# \\ \# & \# \end{bmatrix}$, we write $b = b_1 + b_2^*$ where $b_1 = \begin{bmatrix} 0 & 0 \\ \# & \# \end{bmatrix}$, $b_2 = \begin{bmatrix} 0 & 0 \\ \# & 0 \end{bmatrix}$. Then

$$\begin{aligned} \|\textstyle\sum_{i=1}^n u_i^* b u_i\| &\leqq \|\textstyle\sum_{i=1}^n u_i^* b_1 u_i\| + \|\textstyle\sum_{i=1}^n u_i^* b_2 u_i\| \\ &\leqq \sqrt{n}\|b_1\| + \sqrt{n}\|b_2\| \leqq 2\sqrt{n}\|b\|. \end{aligned}$$

Therefore, the desired inequality follows.

PROPOSITION 2.3. *Let e be a projection, and u, v be two unitary operators satisfying $u^2 = v^3 = 1$ and $(1-e)u(1-e) = 0$, $eve = 0$. Then*

(i) *$C^*(u, v)$ has a unique tracial state τ.*

(ii) *For each $a \in C^*(u, v)$ and $\epsilon > 0$, there exist an integer n and unitary operators $u_1, \ldots u_n \in C^*(u, v)$ such that*

$$\|\tau(a)1 - (1/n)\textstyle\sum u_i^* a u_i\| \leqq \epsilon.$$

Proof. (0) Letting $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, we have $u = \begin{bmatrix} \# & \# \\ \# & 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 & \# \\ \# & \# \end{bmatrix}$. Hence $v^{-1} = v^* = \begin{bmatrix} 0 & \# \\ \# & \# \end{bmatrix}$, $uv^{\pm 1} = \begin{bmatrix} \# & \# \\ 0 & \# \end{bmatrix}$, and for $\delta_i = \pm 1$,

$$(uv^{\delta_1})(uv^{\delta_2}) \ldots (uv^{\delta_n}) = \begin{bmatrix} \# & \# \\ 0 & \# \end{bmatrix},$$

$$(uv^{\delta_1})(uv^{\delta_2}) \ldots (uv^{\delta_n})u = \begin{bmatrix} \# & \# \\ \# & 0 \end{bmatrix},$$

$$v^{\delta_0}(uv^{\delta_1})(uv^{\delta_2}) \ldots (uv^{\delta_n}) = \begin{bmatrix} 0 & \# \\ \# & \# \end{bmatrix}.$$

That is, a reduced word beginning and ending with u is of the form $\begin{bmatrix} \# & \# \\ \# & 0 \end{bmatrix}$, while a reduced word beginning and ending with $v^{\pm 1}$ is of the form $\begin{bmatrix} 0 & \# \\ \# & \# \end{bmatrix}$.

(1) On the algebra of all finite linear combinations of reduced words in u,v ($u^2 = v^3 = 1$), we define a linear functional τ such that $\tau(a) = \alpha$ whenever $a = \alpha 1 + \sum \alpha_i w_i$ (w_i are reduced words of length > 0). A routine computation leads to $\tau(ab) = \tau(ba)$. We claim that τ is contractive in operator norm. Thus τ is well defined on the pre-C^*-algebra generated by u, v, and the extension of τ by continuity is a trace on $C^*(u, v)$. (Note that the positivity of τ follows from $\|\tau\| = 1 = \tau(1)$.)

To prove the claim, we note that if $a = \alpha 1 + \sum_{i=1}^{k} \alpha_i w_i$, then from Lemma 2.1, there exists a unitary operator z ($= z_n$ for a sufficiently large n) such that the reduced form for $z^* w_i z$ begins with $v^{\pm 1}$ and ends with $v^{\pm 1}$ for each i. Therefore $z^* a z = \alpha 1 + \sum \alpha_i z^* w_i z$ admits a 2×2 matrix-operator expression

$$\begin{bmatrix} \alpha 1 & 0 \\ 0 & \alpha 1 \end{bmatrix} + \begin{bmatrix} 0 & \# \\ \# & \# \end{bmatrix} = \begin{bmatrix} \alpha 1 & \# \\ \# & \# \end{bmatrix},$$

and $|\tau(a)| = |\alpha| \leqq \|z^* a z\| = \|a\|$ as claimed.

(2) To prove (ii), it suffices to assume that a is a finite linear combination of words in u, v. Replacing a by its unitary equivalence, we may further assume from the preceding paragraph that $a = \alpha 1 + b$, where $\tau(a) = \alpha$, $b = \begin{bmatrix} 0 & \# \\ \# & \# \end{bmatrix}$. Letting $u_i = uv^{-1}(uv)^i$, we get that

$$u_i u_j^* = uv^{-1}(uv)^{i-j} vu = \begin{cases} uv^{-1}(uv)^{i-j-1} uv^{-1} u & (\text{if } i > j) \\ uvu(v^{-1}u)^{j-i-1} vu & (\text{if } i < j) \end{cases}$$

has a reduced form beginning and ending with u when $i \neq j$; hence $u_i u_j^* =$

$\begin{bmatrix} \# & \# \\ \# & 0 \end{bmatrix}$ $(i \neq j)$. Therefore, by Lemma 2.2, we derive that

$$\|\tau(a)1 - (1/n)\sum u_i^*au_i\| = \|(1/n)\sum u_i^*bu_i\| \leqq 2\|b\|/\sqrt{n} = 2\|a - \tau(a)1\|/\sqrt{n} \leqq 4\|a\|/\sqrt{n}$$

which is smaller than any prescribed ϵ when n is sufficiently large.

(3) It remains to prove that τ is the unique trace. Suppose $C^*(u, v)$ has another trace σ, then from (ii), we have

$$|\tau(a) - \sigma(a)| = |\sigma(\tau(a)1 - (1/n)\sum u_i^*au_i)| \leqq \|\tau(a)1 - (1/n)\sum u_i^*au_i\| \leqq \epsilon.$$

Since ϵ is arbitrary, we conclude that $\tau = \sigma$ as desired. Thus the theorem is proved.

We are going to deal with a pair of concrete unitary operators. Let H be an infinite-dimensional Hilbert space with orthogonal decompositions $H = H_0 \oplus H_1$, $H_1 = H_\alpha \oplus H_\beta$, where $\dim H_0 = \dim H_1 = \dim H_\alpha = \dim H_\beta$. With respect to $H = H_0 \oplus H_1$, we define $u \in B(H)$ by the expression $u = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (herein, H_0 and H_1 are identified via an arbitrary onto isometry). Similarly, with respect to $H = H_0 \oplus H_\alpha \oplus H_\beta$, we write

$$v = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \in B(H)$$

by identifying $H_0 = H_\alpha = H_\beta$. In other words, (u, v) is any pair of unitary operators in $B(H)$ such that $u^2 = v^3 = 1$ and

$$u: \begin{cases} H_0 \to H_1 \\ H_1 \to H_0 \end{cases}, \quad v: \begin{cases} H_0 \to H_\alpha \\ H_\alpha \to H_\beta \\ H_\beta \to H_0 \end{cases}.$$

To get an equational description, we let $e \in B(H)$ be the projection onto H_0; then $u, v \in B(H)$ are completely characterized by the equations

(2.1) $\quad u = u^{-1} = u^*, \; e + u^*eu = 1,$

(2.2) $\quad v^2 = v^{-1} = v^*, \; e + v^*ev + vev^* = 1.$

LEMMA 2.4. *Let e be a projection of the form $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then an operator v satisfies (2.2) if and only if v is the form $\begin{bmatrix} 0 & s_2^* \\ s_1 & s_2s_1^* \end{bmatrix}$ where s_1, s_2 are isometries satisfying $s_1s_1^* + s_2s_2^* = 1$.*

Proof. The "if" part follows from a straightforward computation. Conversely, suppose v satisfies (2.2) and $v = \begin{bmatrix} x & s_2^* \\ s_1 & y \end{bmatrix}$ with x, y, s_1, s_2 to be determined. Then the equation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e + v^*ev + vev^* = \begin{bmatrix} 1 + x^*x + xx^* & \# \\ \# & s_1s_1^* + s_2s_2^* \end{bmatrix}$$

leads to

$$1 = s_1s_1^* + s_2s_2^* \quad \text{(a)}$$

and $x = 0$, i.e., $v = \begin{bmatrix} 0 & s_2^* \\ s_1 & y \end{bmatrix}$. From v being an order-3 unitary, we derive that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = v^*v = \begin{bmatrix} s_1^*s_1 & \# \\ \# & \# \end{bmatrix}, \quad \text{i.e., } s_1 \text{ is an isometry.}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = vv^* = \begin{bmatrix} s_2^*s_2 & \# \\ ys_2 & \# \end{bmatrix}, \text{ i.e., } s_2 \text{ is an isometry, } 0 = ys_2 \quad \text{(b)},$$

and

$$\begin{bmatrix} 0 & s_1^* \\ s_2 & y^* \end{bmatrix} = v^* = v^2 = \begin{bmatrix} \# & \# \\ ys_1 & \# \end{bmatrix}, \text{ i.e., } s_2 = ys_1 \quad \text{(c)}.$$

Combining (a) (b) (c), we get

$$y = y(s_1s_1^* + s_2s_2^*) = ys_1s_1^* = s_2s_1^*,$$

i.e., $v = \begin{bmatrix} 0 & s_2^* \\ s_1 & s_2s_1^* \end{bmatrix}$, as desired.

We denote by $\mathcal{O}_2$, the C^*-algebra generated by two isometries s_1, s_2 satisfying $s_1s_1^* + s_2s_2^* = 1$, as analyzed in detail by Cuntz. An interesting feature of $\mathcal{O}_2$ lies in the fact that $\mathcal{O}_2$ is independent of the choice of s_1, s_2 (namely, if t_1, t_2 are isometries satisfying $t_1t_1^* + t_2t_2^* = 1$, then $C^*(s_1, s_2)$ is canonically *-isomorphic with $C^*(t_1, t_2)$) ([**14**, Theorem 1.12]). The following lemma is a simple consequence.

LEMMA 2.5. *$M_2(\mathcal{O}_2)$ is *-isomorphic with $\mathcal{O}_2$*

Proof. Letting $\mathcal{O}_2 = C^*(s_1, s_2)$ where s_1, s_2 are isometries satisfying $1 = s_1s_1^* + s_2s_2^*$, we note that

$$s_1^*s_2 = s_1^*(s_1s_1^* + s_2s_2^*)s_2 = s_1^*s_2 + s_1^*s_2 = 2s_1^*s_2,$$

i.e., $s_1^*s_2 = 0$ and, by taking adjoint, $s_2^*s_1 = 0$. Now we write

$$t_1 = \begin{bmatrix} s_1 & s_2 \\ 0 & 0 \end{bmatrix}, \quad t_2 = \begin{bmatrix} 0 & 0 \\ s_1 & s_2 \end{bmatrix}, \in M_2(\mathcal{O}_2);$$

then

$$t_i^*t_i = \begin{bmatrix} s_1^*s_1 & s_1^*s_2 \\ s_2^*s_1 & s_2^*s_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

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and

$$t_1t_1^* + t_2t_2^* = \begin{bmatrix} s_1s_1^* + s_2s_2^* & 0 \\ 0 & s_1s_1^* + s_2s_2^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence $C^*(t_1, t_2)$ is *-isomorphic with $C^*(s_1, s_2)$ from the fact mentioned before this lemma. On the other hand, a direct computation, through

$$t_1t_1^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad t_2t_2^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad t_1t_2^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

leads to an easy conclusion that $C^*(t_1, t_2) = M_2(\mathcal{O}_2)$. Therefore, $M_2(\mathcal{O}_2)$ is *-isomorphic with $\mathcal{O}_2$ as desired.

THEOREM 2.6. *Suppose e is a projection and u, v are operators satisfying* (2.1)–(2.2). *Then $C^*(e, u, v)$ is *-isomorphic with $\mathcal{O}_2$. Consequently, $C^*(e, u, v)$ is algebraically unique; i.e., if e_1, e_2 are projections and (e_i, u_i, v_i), $i = 1, 2$, are triples satisfying* (2.1)–(2.2), *then $C^*(e_1, u_1, v_1)$ is canonically *-isomorphic with $C^*(e_2, u_2, v_2)$.*

Proof. We write $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $u = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; then by Lemma 2.4,

$$v = \begin{bmatrix} 0 & s_2^* \\ s_1 & s_2s_1^* \end{bmatrix}$$

where s_1, s_2 are isometries satisfying $s_1s_1^* + s_2s_2^* = 1$. Hence it follows easily that

$$C^*(e, u, v) = M_2(C^*(s_1, s_2)) \simeq M_2(\mathcal{O}_2) \simeq \mathcal{O}_2.$$

Since all *-isomorphisms are canonically defined, $C^*(e, u, v)$ is thus algebraically unique.

COROLLARY 2.7 *Suppose e is a projection and u, v are operators satisfying* (2.1)–(2.2). *Then $C^*(u, v)$ is *-isomorphic with the left regular representation of a discrete group.*

Proof. We denote by G, the quotient group of $\mathbf{F}_2$ with a presentation $\{u, v\colon u^2 = v^3 = 1\}$. Then there exists S, a subset of G, satisfying the condition

(2.3) $\quad S \cap uS = \emptyset, S \cap vS = \emptyset,$ and $G = S \cup uS = S \cup vS \cup v^{-1}S.$

To construct such S, we may define the membership of S by induction on the length of reduced words as follows:

(i) The empty word $1 \notin S$.

(ii) Whenever z is a reduced word of length n $(\geqq 0)$, and $w = xz$ $(x \in \{u, v, v^{-1}\})$ is a reduced word of length $n + 1$,we have that

$$\begin{cases} w \in S \text{ if } x \neq v^{-1} \text{ and } z \notin S, \\ w \notin S \text{ if } x = v^{-1} \text{ or } z \in S. \end{cases}$$

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Now on the Hilbert space $l^2(G)$, we let $E \in B(l^2(G))$ be the projection onto $l^2(S)$, and let $L(u)$, $L(v) \in B(l^2(G))$ be the left translations induced by u, v respectively. It is straightforward to check that $L(u)$ and $L(v)$ are unitaries satisfying $L(u)^2 = L(v)^3 = 1$. From the condition (2.3), we get

$$E + L(u)^*EL(u) = 1, E + L(v)^*EL(v) + L(v)EL(v)^* = 1.$$

Therefore, by Theorem 2.6, we conclude that $C^*(e, u, v)$ is canonically *-isomorphic with $C^*(E, L(u), L(v))$. In particular, $C^*(u, v)$ is *-isomorphic with $C^*(L(u), L(v)) = C_{\text{reg}}^*(G)$ as desired.

The proof of the following theorem is an immitation of Powers' work on the simplicity of $C_{\text{reg}}^*(F_2)$ ([**18**], see also [**1**, Theorem *VD*]).

THEOREM 2.8. *Suppose e is a projection and u, v are operators satisfying* (2.1)-(2.2). *Then $C^*(u, v)$ is a simple C^*-algebra possessing a unique tracial state.*

Proof. Clearly, u and v satisfy the hypothesis of Proposition 2.3. Hence $C^*(u, v)$ has a unique tracial state τ. We claim that τ is faithful. Indeed, τ is associated with a representation π_τ of $C^*(u, v)$ on a Hilbert space H_τ and a cyclic vector ξ_τ, satisfying $\tau(a) = \langle \pi_\tau(a)\xi_\tau, \xi_\tau \rangle$ for all $a \in C^*(u, v)$. It is straightforward to check that H_τ is identical with $l^2(G)$, where G is the group with a presentation $\{u, v: u^2 = v^3 = 1\}$, under the correspondence

$$\sum_i^n \alpha_i w_i \in l^2(G) \mapsto \sum_i^n \alpha_i \pi_\tau(w_i)\xi_\tau \in H_\tau \ (w_i \in G).$$

Consequently, $\pi_\tau(C^*(u, v))$ is just $C_{\text{reg}}^*(G)$. From Corollary 2.7, we deduce that $C^*(u, v)$ is canonically *-isomorphic with $\pi_\tau(C^*(u, v))$; i.e., π_τ is faithful, thus τ is faithful as claimed.

Alternatively, we observe directly that $C_{\text{reg}}^*(G)$, as of the proof of Corollary 2.7, has a faithful trace, defined by $a \mapsto \langle a\xi, \xi \rangle$ where $\xi \in l^2(G)$ satisfying $\xi(1) = 1$ and $\xi(g) = 0$ for $g \neq 1$.

Now suppose J is a non-zero 2-sided ideal of $C^*(u, v)$; then J contains a positive element $a \neq 0$, whence $\tau(a) = \alpha \neq 0$ from the faithfulness of τ. By Proposition 2.3(*ii*), there exist unitaries $u_1, \ldots u_n \in C^*(u, v)$ such that $\|\alpha 1 - (1/n)\sum u_i^* a u_i\| \leqq \alpha/2$. Hence $\sum u_i^* a u_i$ is invertible, and the ideal generated by a is $C^*(u, v)$. Therefore $J = C^*(u, v)$ and we conclude that $C^*(u, v)$ is simple.

§ 3. Some peculiar properties. Throughout this section, we let e be a projection, and u, v be operators satisfying the conditions (2.1)-(2.2); i.e., u, v are unitary operators satisfying $u^2 = v^3 = 1$ and

(†) $\quad e + u^*eu = 1, e + v^*ev + vev^* = 1.$

From § 2, $C^*(u, v)$ is a tracial C^*-algebra that can be imbedded into $\mathcal{O}_2$ as a subalgebra. Whereupon, we will show further that $C^*(u, v)$ is an explicit example for several types of pathologies in C^*-algebraic structure theory. Notably, a condition similar to (†) has been employed in recent literature ([**8**, p. 173], [**12**, Lemma 4.2], and [**2**]) for exploring some striking but different features of $C_{\text{reg}}^*(\mathbf{F}_2)$. While partial results of this section are also applicable to $C_{\text{reg}}^*(\mathbf{F}_2)$, we will work on $C^*(u, v)$ exclusively since $C^*(u, v)$ possesses a much more flexible and tractable structure.

In regard to some algebraic topological aspects of Banach algebras, Johnson [**16**] has introduced the notion of amenability. Namely, a unital Banach algebra A is *amenable* if and only if for each Banach A-bimodule X, each bounded derivation $D: A \to X^*$ is inner (i.e., given any bounded linear map $D: A \to X^*$ satisfying

$$D(ab)(x) = D(a)(bx) + D(b)(xa) \text{ for all } a, b \in A, x \in X,$$

there exists $\theta \in X^*$ such that $D(a)(x) = \theta(ax - xa)$ for all $a \in A, x \in X$).

The existence of non-amenable C^*-algebras is revealed in a result of Bunce [**5**]; the C^*-algebra generated by the left regular representation of a discrete group is amenable if and only if the group is amenable. Due to the recent work of Connes [**13**, Corollary 2] (see also [**6**, Corollary 5] for a different proof), we know further that every non-nuclear C^*-algebra is non-amenable. However, all known proofs, involving deep structure theory, probably do not admit easy interpretations on concrete examples. The following explicit demonstration on $C^*(u, v)$ may provide a clearer illustration of the non-amenability.

Example A: A separable non-amenable C^*-algebra.

Demonstration. Let X be the quotient Banach space $B(H)/C^*(u, v)$; then X^* can be identified with

$$\mathcal{S} = \{\theta \in B(H)^*: \theta|_{C^*(u,v)} = 0\}$$

and $\mathcal{S}$ is thus a dual $C^*(u, v)$-bimodule under the action

$$(a.\theta)(t) = \theta(ta), \ (\theta.a)(t) = \theta(at)$$

for $a \in C^*(u, v)$, $t \in B(H)$, $\theta \in \mathcal{S}$. We construct a linear map $D: C^*(u, v) \to \mathcal{S}$ by $D(a)(t) = \rho(at - ta)$, where ρ is any state on $B(H)$ satisfying $\rho|_{C^*(u,v)} = \text{trace } \tau$. It is straightforward to check that $D(a)|_{C^*(u,v)} = 0$ and D is a bounded derivation. Now suppose that D is inner; i.e., there exists $\theta \in \mathcal{S}$ such that for all $a \in C^*(u, v)$, $t \in B(H)$,

$$\theta(at - ta) = D(a)(t) = \rho(at - ta).$$

Letting $\sigma = \rho - \theta$, we have that $\sigma(at - ta) = 0$, and

$$\sigma(1) = \rho(1) - \theta(1) = \rho(1) = 1.$$

Thus

$$\sigma(u^*eu - e) = \sigma(u^*eu - uu^*e) = 0,$$

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and similarly,

$$\sigma(v^*ev - e) = 0, \sigma(vev^* - e) = 0.$$

Hence the condition (†) leads to

$$2\sigma(e) = \sigma(1) = 1, 3\sigma(e) = \sigma(1) = 1,$$

which is impossible. Therefore, D is not inner, and $C^*(u, v)$ is not amenable.

Remark. Bunce ([**4**, Proposition 2]: See also [**5**, Proposition 1]) has proved that a unital C^*-algebra A is amenable if and only if for any A-bimodules $X \subseteq Y$, each $\theta \in X^*$ satisfying $\theta(w^*xw) = \theta(x)$ for all $x \in X$, unitary $w \in A$ has an extension $\theta_1 \in Y^*$ satisfying $\theta_1(w^*yw) = \theta_1(y)$ for all $y \in Y$, unitary $w \in A$. Using this result, we can also deduce immediately that $C^*(u, v)$ is non-amenable. Namely, by letting $A = X = C^*(u, v)$, $Y = C^*(e, u, v)$ or $B(H)$, and $\theta =$ trace τ, it is transparent that the condition (†) does not admit θ to have the above extension.

We say that a C^*-algebra $A \in B(H)$ is *quasi-diagonal* if and only if there is an increasing net of finite-rank projections $p_\nu \in B(H)$ converging to 1 strongly and $\|ap_\nu - p_\nu a\| \to 0$ for all $a \in A$. In answering a question of Thayer [**22**, p. 56], we show that $C^*(u, v)$ serves as

Example B. A tracial separable C^*-algebra that is not quasi-diagonal.

Demonstration. Suppose there is a rank-n projection $p \in B(H)$ such that $\|up - pu\| \leqq \epsilon$ and $\|vp - pv\| \leqq \epsilon$. Then

$$p(e - vev^*)p = pev^*(vp - pv)p + (pev^*p)(pvp) - (pvp)(pev^*p) + p(vp - pv)ev^*p.$$

Letting τ_n be the unital trace on $pB(H)p$ ($\simeq M_n$), we have

$$|\tau_n(p(e - vev^*)p)| = |\tau_n(pev^*(vp - pv)p) + \tau_n(p(vp - pv)ev^*p)| \leqq 2\|vp - pv\| \leqq 2\epsilon.$$

Similarly,

$$|\tau_n(p(e - v^*ev)p)| \leqq 2\epsilon, |\tau_n(p(e - u^*eu)p)| \leqq 2\epsilon.$$

But from (†), we have

$$\tau_n(p(e + u^*eu)p) = 1, \tau_n(p(e + v^*ev + vev^*)p) = 1.$$

Hence,

$$2\tau_n(pep) \geqq 1 - 2\epsilon, 3\tau_n(pep) \leqq 1 + 4\epsilon,$$

which are impossible, if ϵ is sufficiently small. Therefore, $C^*(u, v)$ is not quasi-diagonal.

Remark. The meaning of a *quasi-diagonal C^*-algebra* is equivalent to that of a *quasi-triangular C^*-algebra*. On the other hand, by the spectral characterization theorem for quasi-triangular operators [**3**], we can deduce that every operator in $C^*(u, v)$ is quasi-triangular. Hence, $C^*(u, v)$ serves as an example for another peculiar feature: a C^*-algebra such that every element is quasi-triangular, but globally the C^*-algebra, as a whole, is not quasi-triangular.

We proceed to establish that $C^*(u, v)$ fails to have certain approximation properties. For this purpose, we will estimate how far a general unital completely positive linear map is away from being multiplicative.

We note that each unital completely positive linear map φ on a unital C^*-algebra A has the property:

$$\varphi(a^*a) \geqq \varphi(a^*)\varphi(a) \geqq 0 \text{ for all } a \in A.$$

In case $\varphi(a_0^*a_0) = \varphi(a_0^*)\varphi(a_0)$, we have then $\varphi(aa_0) = \varphi(a)\varphi(a_0)$ for all $a \in A$ (see e.g., [**7**, Theorem 3.1]). Hence the value of $\|\varphi(a_0^*a_0) - \varphi(a_0^*)\varphi(a_0)\|$ can be regarded as the "amount of non-multiplicativity" of φ at a_0. This notion can be further justified in the following proposition and its corollary.

PROPOSITION 3.1. *Let A be a unital C^*-algebra and φ be a unital completely positive linear map: $A \to B(H)$. If*

$$\|\varphi(a_i^*a_i) - \varphi(a_i^*)\varphi(a_i)\| \leqq \delta_i^2 \ (i = 0, 1), \textit{ then}$$
$$\|\varphi(a_1^*a_0) - \varphi(a_1^*)\varphi(a_0)\| \leqq \delta_1\delta_0.$$

Proof. From Stinespring's decomposition theorem [**20**], there exist a *-representation π of A on a Hilbert space K, and an into isometry $s: H \to K$ such that $\varphi(a) = s^*\pi(a)s$ for all $a \in A$. Letting $x_i = (1 - ss^*)\pi(a_i)s$, we get

$$\begin{aligned} x_i^*x_j = s^*\pi(a_i^*)(1 - ss^*)\pi(a_j)s &= s^*\pi(a_i^*a_j)s - s^*\pi(a_i^*)ss^*\pi(a_j)s \\ &= \varphi(a_i^*a_j) - \varphi(a_i^*)\varphi(a_j). \end{aligned}$$

Therefore,

$$\begin{aligned} \|\varphi(a_1^*a_0) - \varphi(a_1^*)\varphi(a_0)\| &= \|x_1^*x_0\| \leqq \|x_1^*x_1\|^{1/2}\|x_0^*x_0\|^{1/2} \\ &= \|\varphi(a_1^*a_1) - \varphi(a_1^*)\varphi(a_1)\|^{1/2}.\|\varphi(a_0^*a_0) - \varphi(a_0^*)\varphi(a_0)\|^{1/2} \leqq \delta_1\delta_0 \end{aligned}$$

as desired.

COROLLARY 3.2. *Let A be a unital C^*-algebra and φ be a unital completely positive linear map: $A \to B(H)$. If*

$$\|\varphi(a_0^*a_0) - \varphi(a_0^*)\varphi(a_0)\| \leqq \delta^2,$$

then

$$\|\varphi(aa_0) - \varphi(a)\varphi(a_0)\| \leqq \delta\|a\| \textit{ for all } a \in A.$$

Proof. Note that

$$0 \leqq \varphi(aa^*) - \varphi(a)\varphi(a^*) \leqq \varphi(aa^*) \leqq \|aa^*\| = \|a\|^2.$$

So Proposition 3.1 applies.

We say that a unital C^*-algebra A is *nuclear* if and only if it satisfies one of the following four equivalent conditions (see [**9**, Theorem 3.1] [**10**, Corollary 3.2] [**11**, Theorem 3]):

(I) There exist finite-rank unital completely positive linear maps $\varphi_\nu: A \to A$ such that φ_ν converge to the identity map in point-norm topology.

(II) There exist unital completely positive linear maps

$$A \overset{\sigma_\nu}{\to} M_{n_\nu} \overset{\theta_\nu}{\to} A$$

(certainly, n_ν changes with ν) such that $\theta_\nu \circ \sigma_\nu$ converge to the identity map in point-norm topology.

(III) For each C^*-algebra B, the algebraic tensor product $A \otimes B$ has a unique C^*-norm.

(IV) The second dual of A is injective.

It is well known that there exist (non-nuclear) C^*-algebras failing to enjoy the condition (III) or (IV) (see e.g., [**21**, pp. 119–121] [**23**, Prop. 7.4] [**17**, p. 175] [**8**, pp. 172–173] [**24**, Corollary 1.9]). By virtue of abstract theorems, these C^*-algebras do not satisfy the condition (I) or (II) either. However, the abstract proofs involve deep W^*-algebra theory, throwing not much light on explicit examples. Thus, it might be desirable to see directly (without using any W^*algebra theory) that $C^*(u, v)$ fails to have completely positive metric approximation property. In fact, $C^*(u, v)$ serves as

Example C. A singly generated non-nuclear C^*-algebra.

Demonstration. Since v is a finite-order unitary, there exists a hermitian operator a such that $C^*(v) = C^*(a)$. Therefore, from u and a being hermitian, $C^*(u, v) = C^*(u, a) = C^*(u + \sqrt{-1}a)$ is a singly generated C^*-algebra.

Now suppose $C^*(u, v)$ is nuclear; then from condition (II), given $\delta > 0$, there exist unital completely positive linear maps $C^*(u, v) \overset{\sigma}{\to} M_n \overset{\theta}{\to} C^*(u, v)$ such that

$$\|\theta(\sigma(u)) - u\| \leqq \tfrac{1}{2}\delta^2, \text{ and } \|\theta(\sigma(v)) - v\| \leqq \tfrac{1}{2}\delta^2.$$

Extending σ to a completely positive linear map $\sigma_1: C^*(e, u, v) \to M_n$ and letting

$$\varphi = \theta \circ \sigma_1: C^*(e, u, v) \to C^*(u, v),$$

we still have $\|\varphi(v) - v\| \leqq \frac{1}{2}\delta^2$ and

$$\|\varphi(v^*v) - \varphi(v^*)\varphi(v)\| = \|1 - \varphi(v^*)\varphi(v)\| \leqq \|v^*[v - \varphi(v)]\| + \|[v - \varphi(v)]^*\varphi(v)\| \leqq \tfrac{1}{2}\delta^2 + \tfrac{1}{2}\delta^2 = \delta^2.$$

From Corollary 3.2, we get

$$\|\varphi(v^*ev) - \varphi(v^*)\varphi(e)\varphi(v)\| \leqq \|\varphi(v^*ev) - \varphi(v^*e)\varphi(v)\| + \|[\varphi(ev) - \varphi(e)\varphi(v)]^*\varphi(v)\| \leqq 2\delta.$$

Hence,

$$\|\varphi(v^*ev) - v^*\varphi(e)v\| \leqq \|\varphi(v^*ev) - \varphi(v^*)\varphi(e)\varphi(v)\| + \|\varphi(v^*)\varphi(e)[\varphi(v) - v]\| + \|[\varphi(v) - v]^*\varphi(e)v\| \leqq 2\delta + \tfrac{1}{2}\delta^2 + \tfrac{1}{2}\delta^2 = 2\delta + \delta^2, = \epsilon, \text{ say.}$$

Applying the trace τ, we derive

$$|(\tau \circ \varphi)(v^*ev - e)| = |\tau(\varphi(v^*ev) - v^*\varphi(e)v)| \leqq \|\varphi(v^*ev) - v^*\varphi(e)v\| \leqq \epsilon,$$

and similarly,

$$|(\tau \circ \varphi)(u^*eu - e)| \leqq \epsilon, \ |(\tau \circ \varphi)(vev^* - e)| \leqq \epsilon.$$

Thus the equations $e + u^*eu = 1$, $e + v^*ev + vev^* = 1$ lead to

$$2\tau(\varphi(e)) \geqq 1 - \epsilon, \ 3\tau(\varphi(e)) \leqq 1 + 2\epsilon,$$

which are impossible for a sufficiently small ϵ. Therefore, $C^*(u, v)$ is non-nuclear as asserted.

Finally, we conclude with an example that gives a negative answer to a widely held conjecture.

Example D. A non-nuclear (non-amenable) C^*-subalgebra of a nuclear (amenable) C^*-algebra.

Demonstration. From the demonstrations of Examples A and C, we know that $C^*(u, v)$ is neither nuclear nor amenable. From [**14**] and [**19**], we know that $\mathcal{O}_2$ is nuclear and even amenable. As established in § 2, $C^*(u, v) \subseteq C^*(e, u, v) \simeq \mathcal{O}_2$. Thus we are done.

References

1. C. A. Akemann and P. A. Ostrand, *Computing norms in group C*-algebras*, Amer. J. Math. *98* (1976), 1015–1047.

2. J. Anderson, *A C*-algebra A for which* Ext *A is not a group*, Ann. of Math. *107* (1978), 455–458.

3. C. Apostal, C. Foias, and D. Voiculescu, *Some results on non-quasi-triangular operators*, IV, Rev. Roumaine Math. Pures Appl. *18* (1973), 487–514.

4. J. Bunce, *Characterizations of amenable and strongly amenable C*-algebras*, Pacific J. Math. *43* (1972), 563–572.

5. ——— *Finite operators and amenable C*-algebras*, Proc. Amer. Math. Soc. *56* (1976), 145–151.

6. J. Bunce and W. L. Paschke, *Quasi-expectations and amenable von Neumann algebras*, Proc. Amer. Math. Soc. *71* (1978), 232–236.

7. M. D. Choi, *A Schwarz inequality for positive linear maps on C*-algebras*, Ill. J. Math. *18* (1974), 565–574.

8. M. D. Choi and E. G. Effros, *Injectivity and operator spaces*, J. Func. Anal. *24* (1977), 156–209.

9. ——— *Nuclear C*-algebras and the approximation property*, Amer. J. Math. *100* (1978), 61–79.

10. ——— *Separable nuclear C*-algebras and injectivity*, Duke Math. J. *43* (1976), 309–322.

11. ——— *Nuclear C^*-algebras and injectivity: the general case*, Indiana U. Math. J. *26* (1977), 443–446.

12. ——— *Lifting problems and the cohomology of C^*-algebras*, Can. J. Math. *29* (1977), 1092–1111.

13. A. Connes, *On the cohomology of operator algebras*, J. Func. Anal. *28* (1978), 248–253.

14. J. Cuntz: *Simple C^*-algebras generated by isometries*, Commun. Math. Phys. *57* (1977), 173–185.

15. E. G. Effros, *Aspects of non-commutative order*, Notes for a lecture given at the Second U.S.-Japan Seminar on C^*-algebras and Applications to Physics, 1977.

16. B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. *127* (1972).

17. E. C. Lance, *On nuclear C^*-algebras*, J. Func. Anal. *12* (1973) 157–176.

18. R. T. Powers, *Simplicity of the C^*-algebra associated with the free group on two generators*, Duke Math. J. *42* (1975), 151–156.

19. J. Rosenberg, *Amenability of cross products of C^*-algebras*, Comm. Math. Phys. *57* (1977), 187–191.

20. W. F. Stinespring, *Positive functions on C^*-algebras*, Proc. Amer. Math. Soc. *6* (1955), 211–216.

21. M. Takesaki, *On the crossnorm of the direct product of C^*-algebras*, Tôhoku Math. J. *16* (1964), 111–122.

22. F. J. Thayer, *Quasi-diagonal C^*-algebras*, J. Func. Anal. *25* (1977), 50–57.

23. J. Tomiyama, *Tensor products and projections of norm one in von Neumann algebras*, Lecture Notes, Univ. of Copenhagen, 1970.

24. S. Wassermann, *On tensor products of certain group C^*-algebras*, J. Func. Anal. *23* (1976), 239–254.

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