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ROUQUIER'S CONJECTURE AND DIAGRAMMATIC ALGEBRA

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Abstract

We prove a conjecture of Rouquier relating the decomposition numbers in category $\mathcal{O}$ for a cyclotomic rational Cherednik algebra to Uglov's canonical basis of a higher level Fock space. Independent proofs of this conjecture have also recently been given by Rouquier, Shan, Varagnolo and Vasserot and by Losev, using different methods. Our approach is to develop two diagrammatic models for this category $\mathcal{O}$; while inspired by geometry, these are purely diagrammatic algebras, which we believe are of some intrinsic interest. In particular, we can quite explicitly describe the representations of the Hecke algebra that are hit by projectives under the KZ-functor from the Cherednik category $\mathcal{O}$ in this case, with an explicit basis. This algebra has a number of beautiful structures including categorifications of many aspects of Fock space. It can be understood quite explicitly using a homogeneous cellular basis which generalizes such a basis given by Hu and Mathas for cyclotomic KLR algebras. Thus, we can transfer results proven in this diagrammatic formalism to category $\mathcal{O}$ for a cyclotomic rational Cherednik algebra, including the connection of decomposition numbers to canonical bases mentioned above, and an action of the affine braid group by derived equivalences between different blocks.

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1. Introduction

One of the most powerful tools in the theory of category $\mathcal{O}$ for a semisimple Lie algebra is to consider it not just as a lonely category but as a module over the monoidal category of projective functors. This perspective was essential for a number of significant advances in our understanding of category $\mathcal{O}$; one example is the theory of Soergel bimodules [Soe90, Soe92]. In category $\mathcal{O}$ for cyclotomic

Cherednik algebras, defined in [GGOR03], the rôle of projective functors is played by the induction functors of Bezrukavnikov and Etingof [BE09].

In this paper, we exploit the fact that these functors essentially control the entire structure of the category, just as is the case for category $\mathcal{O}$. In category $\mathcal{O}$, all projectives were obtained by acting on a single projective with translation functors. In the Cherednik case, this method of control is a bit more indirect. In brief, category $\mathcal{O}$ for a cyclotomic Cherednik algebra is the unique collection of highest weight categories with a deformation which are tied together by induction functors, and a particular partial order on simples. Theorem 2.3, based on ideas from [RSVV16], makes this statement precise. These induction functors can also be repackaged into a highest weight categorical action of $\widehat{\mathfrak{sl}}_e$ (a notion defined by Losev [Los13]). Similar uniqueness theorems with other applications in representation theory have been proven by the author jointly with Brundan and Losev [LW15, BLW].

This fact is mainly of interest because we can give two constructions of categories which also satisfy these properties, and thus are equivalent to the Cherednik category $\mathcal{O}$. As in Rouquier [Rou08], we can associate a choice of parameters for the Cherednik algebra of $\mathbb{Z}/\ell\mathbb{Z} \wr S_n$ to a charge $\underline{\mathbf{s}} = (s_1, \dots, s_\ell) \in \mathbb{Z}^\ell$; we let $\mathcal{O}^{\underline{\mathbf{s}}}$ denote the sum of category $\mathcal{O}$ for these parameters over all n. (In fact, we can work with arbitrary parameters. See Section 3.2 for details.)

We also associate a graded finite-dimensional algebra with two presentations to the same data, as introduced in [Webb] under the name *WF Hecke algebras*. (Here, 'WF' stands for 'weighted framed.' This is explained in greater detail in [Webb].) We first introduce a 'Hecke-like' presentation which makes the connection to the KZ functor straightforward but which is not homogeneous, and then a 'KLR-like' presentation, which has the considerable advantage of being graded. We let $T^{\underline{\mathbf{s}}}$ denote this algebra with its induced grading.

We describe these presentations in considerable detail in Sections 2–4. The graded presentation is a generalization of the Khovanov–Lauda–Rouquier algebras [KL09, Rou], and a special case of a construction described by the author in [Webc]. In the terminology of that paper, it is a *reduced steadied quotient* of a *weighted KLR algebra*. Both these presentations are purely combinatorial/diagrammatic in description, though the formalism from which they are constructed is heavily influenced by geometry.

THEOREM A. *There is an equivalence of categories between the category of finite-dimensional (ungraded) representations of $T^{\underline{\mathbf{s}}}$ and the category $\mathcal{O}^{\underline{\mathbf{s}}}$.*

In particular, the category of graded modules over $T^{\underline{\mathbf{s}}}$ is a graded lift of $\mathcal{O}^{\underline{\mathbf{s}}}$ compatible with the graded lifts of the Hecke algebra defined by Brundan and Kleshchev [BK09].

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We should emphasize to the reader: this is the first explicit description of the category $\mathcal{O}$ for Cherednik algebras we know in the literature. As part of its proof, we give an explicit description of the modules given by the image of the Knizhnik–Zamolodchikov functor, a question which has been unresolved since the original definition of this functor in [GGOR03].

Furthermore, this development is also of theoretical interest. The algebra $T^{\underline{\mathbf{s}}}$ has a large number of desirable properties which are not easily seen from the Cherednik perspective:

THEOREM B.

(1) *The algebra $T^{\underline{\mathbf{s}}}$ is graded cellular; its basis vectors are indexed by pairs of generalizations of standard Young tableaux of the same shape.*

(2) *This equivalence gives an explicit description, including a basis and graded lift, of the image of projectives from $\mathcal{O}^{\underline{\mathbf{s}}}$ under the* KZ *functor.*

(3) *If the charges $\underline{\mathbf{s}}$ and $\underline{\mathbf{s}}'$ are permutations of each other modulo e, then the derived categories $D^b(T^{\underline{\mathbf{s}}}\text{-mod})$ and $D^b(T^{\underline{\mathbf{s}}'}\text{-mod})$ are equivalent, and in fact there is a strong categorical action of the affine braid group lifting that of the affine Weyl group on charges.*

(4) *The graded Grothendieck group $K_q^0(T^{\underline{\mathbf{s}}})$ is canonically isomorphic to Uglov's q-Fock space attached to the same charges.*

(5) *Under this isomorphism, the standard modules correspond to pure wedges, the projectives to Uglov's canonical basis, and the simples to its dual.*

The first four points of this theorem have purely algebraic proofs. The last point requires some geometric input from a category of perverse sheaves considered in [Webc]; this also resolves a long-standing conjecture of Rouquier, that the multiplicities of standard modules in projectives (which coincide by BGG reciprocity with the multiplicities of simples in standards) are given by the coefficients of a canonical basis specialized at $q = 1$. Note that we have constructed a q-analogue of these multiplicities using a grading on the algebras in question, rather than using depth in the Jantzen filtration on standards as in [RT10, Sha12]. Theorem B(3) was proven using geometric techniques in [GL, 5.1], but we eventually intend to show that our functors match theirs in forthcoming work [Web17].

Independent proofs of Theorem B(5) have recently appeared in work of Rouquier *et al.* [RSVV16] and of Losev [Los16], using very different methods from those contained here; both proofs proceed by proving the 'categorical dimension conjecture' of Vasserot and Varagnolo [VV10, 8.8]. Of course, it would be very interesting in the future to unify these proofs.

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The 'categorical dimension conjecture' actually leads to a stronger result, since instead of relating $\mathcal{O}^{\underline{\mathbf{s}}}$ to a diagrammatic category, it relates it to a truncation of parabolic category $\mathcal{O}$ for an affine Lie algebra, which is known to be Koszul by [**SVV14**, 2.16]; its Koszul dual is again a Cherednik category $\mathcal{O}$, with data specified by level-rank duality, as conjectured of Chuang and Miyachi [**CM**] and proven by Shan *et al.* [**SVV14**, B.5].

In our context, the consequence of these results is that:

THEOREM C. *For each weight μ, the algebra $T^{\underline{\mathbf{s}}}_{\mu}$ is standard Koszul, and its Koszul dual is Morita equivalent to another such algebra $T^{\underline{\mathbf{s}}^!}_{\mu^!}$, with parameters related by rank-level duality.*

We give an independent geometric proof that these algebras are Koszul in [**Web17**]. Since the grading and radical filtrations on the standards of a standard Koszul algebra coincide, this shows on abstract grounds that q-analogues of decomposition numbers using the grading coincide with those using the Jantzen filtration. This observation is also a key piece of evidence for the 'symplectic duality' conjectures on the author, Braden, Licata and Proudfoot. We develop the consequences of this observation further in later works [**BLPW**, **Web17**].

2. WF Hecke algebras

2.1. Hecke and Cherednik algebras. Consider the rational Cherednik algebra H of $\mathbb{Z}/\ell\mathbb{Z} \wr S_d$ (ranging over all values of d) over the base field $\mathbb{C}$ for the parameters $k = m/e$ where $(m, e) = 1$ and $h_j = s_j k - j/\ell$. That is, let S_0 be the set of complex reflections in $\mathbb{Z}/\ell\mathbb{Z} \wr S_d$ that switch two coordinate subspaces and S_1 the set which fix the coordinate subspaces. For each such reflection, let α_s be a linear function vanishing on $\ker(s-1)$, and $\alpha_s^\vee$ a vector spanning $\mathrm{im}(s-1)$ such that $\langle \alpha_s^\vee, \alpha_s\rangle = 2$. Let

$$\omega_s(y, x) = \frac{\langle y, \alpha_s\rangle\langle \alpha_s^\vee, x\rangle}{\langle \alpha_s^\vee, \alpha_s\rangle} = \frac{\langle y, \alpha_s\rangle\langle \alpha_s^\vee, x\rangle}{2}.$$

The RCA is the quotient of the algebra $T(\mathbb{C}^d \oplus (\mathbb{C}^d)^*)\#(\mathbb{Z}/\ell\mathbb{Z}\wr S_d)$ by the relations for $y, y' \in \mathbb{C}^d$, $x, x' \in (\mathbb{C}^d)^*$:

$$[x, x'] = [y, y'] = 0$$

$$[y, x] = \langle y, x\rangle + \sum_{s\in S_0} 2k\omega_s(y, x)s + \sum_{s\in S_1} k\omega_s(y, x) \sum_{j=0}^{\ell-1} \det(s)^{-j}(s_j - s_{j-1} - 1/\ell + \delta_{j,0})s.$$

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DEFINITION 2.1. Category $\mathcal{O}$, which we denote $\mathcal{O}^{\mathbf{s}}_d$ (leaving k implicit), is the full subcategory of modules over H which are generated by a finite-dimensional subspace invariant under $\mathrm{Sym}(\mathbb{C}^d)\#\mathbb{C}[\mathbb{Z}/\ell\mathbb{Z} \wr S_d]$ on which $\mathrm{Sym}(\mathbb{C}^d)$ acts nilpotently. Let $\mathcal{O}^{\mathbf{s}} \cong \bigoplus_d \mathcal{O}^{\mathbf{s}}_d$.

This category is closely tied to the cyclotomic Hecke algebra $H_d(q, Q_\bullet)$ by a functor $\mathsf{KZ}\colon \mathcal{O}^{\mathbf{s}}_d \to H_d(q, Q_\bullet)$ -mod, which is fully faithful on projectives.

DEFINITION 2.2. The cyclotomic Hecke algebra $H_d(q, Q_\bullet)$ is the algebra over $\mathbb{C}$ generated by $X_1^{\pm 1}, \dots, X_d^{\pm 1}$ and $T_1, \dots, T_{d-1}$ with relations

$$(T_i + 1)(T_i - q) = 0 \quad T_iT_{i\pm 1}T_i = T_{i\pm 1}T_iT_{i\pm 1} \quad T_iT_j = T_jT_i \quad (i \neq j \pm 1)$$
$$X_iX_j = X_jX_i \quad T_iX_iT_i = qX_{i+1} \quad X_iT_j = T_iX_j \quad (i \neq j, j+1)$$
$$(X_1 - Q_1)(X_1 - Q_2)\cdots(X_1 - Q_\ell) = 0$$

where $q = \exp(2\pi i k)$ and $Q_i = \exp(2\pi i k s_i)$.

One fact we use extensively is that these categories and functors deform nicely when our parameters are valued not in $\mathbb{C}$ but a local ring with residue field $\mathbb{C}$. Let $\mathscr{R} = \mathbb{C}[[h, z_1, \dots, z_\ell]]$. We can consider the Cherednik algebra over $\mathscr{R}$ with parameters $\mathsf{k} = k + h/2\pi i$ and $\mathsf{s}_j = (ks_j - z_j/2\pi i)/\mathsf{k}$. Let $\mathbb{O}^{\mathbf{s}}_d$ be the deformed category $\mathcal{O}$ of the Cherednik algebra over $\mathscr{R}$ for the complex reflection group $\mathbb{Z}/\ell\mathbb{Z} \wr S_d$ with the parameters as above (the 1-parameter deformations inside this one are discussed by Losev [Los, Section 3.1]). Let $\mathbb{O}^{\mathbf{s}} \cong \bigoplus_d \mathbb{O}^{\mathbf{s}}_d$. This category is also equipped with a Knizhnik–Zamolodchikov functor, landing in modules over the Hecke algebra $H_d(\mathsf{q}, \mathsf{Q}_\bullet)$ for $\mathsf{q} = qe^h$ and $\mathsf{Q}_i = Q_ie^{-z_i}$. We let $H_d(\mathsf{q})$ denote the usual affine Hecke algebra of rank d with parameter q. Fix an integer D.

THEOREM 2.3. *Assume $\mathbb{N}^{\mathbf{s}}_d$ are categories for each $d \leqslant D$ which satisfy:*

(1) $\mathbb{N}^{\mathbf{s}}_0 \cong \mathscr{R}$ -mod;

(2) $\mathbb{N}^{\mathbf{s}}_d$ *is a highest weight category over $\mathscr{R}$ in the sense of* [Rou08];

(3) $\mathbb{N}^{\mathbf{s}}_d$ *is endowed with adjoint $\mathscr{R}$-linear induction and restriction functors*

$$\mathrm{ind}\colon \mathbb{N}^{\mathbf{s}}_{d-1} \to \mathbb{N}^{\mathbf{s}}_d \qquad \mathrm{res}\colon \mathbb{N}^{\mathbf{s}}_d \to \mathbb{N}^{\mathbf{s}}_{d-1}$$

which preserve the categories of projective modules for all $d \leqslant D$. Furthermore, the powers ind^c *have compatible actions of the affine Hecke algebra $H_c(\mathsf{q})$;*

(4) *the d-fold restriction functor* $K = \mathrm{res}^d : \mathbb{N}^{\mathbf{s}}_d \to H_d(\mathsf{q})$-mod *lands in the subcategory* $H_d(\mathsf{q}, \mathsf{Q}_\bullet)$-mod*, and is a quotient functor to this subcategory that becomes an equivalence of categories after base change to* $R = \mathbb{C}((h, z_1, \dots, z_\ell))$ *and is* -1*-faithful after base change to* $\mathbb{C}$*;*

(5) *the category of* $\mathscr{R}$*-flat objects in* $\mathbb{N}^{\mathbf{s}}_d$ *are endowed with a duality which interwines a duality on modules over the Hecke algebra under induction functors and* K*;*

(6) *the order induced on simple representations of* $H_d(\mathsf{q}, \mathsf{Q}_\bullet) \otimes_{\mathscr{R}} R$ *by the highest weight structure on* $\mathbb{N}^{\mathbf{s}}_d$ *has a common refinement with that induced by* $\mathbb{O}^{\mathbf{s}}_d$*;*

(7) *if* $q = -1$*, then the image of* $\mathbb{N}^{\mathbf{s}}_2$ *under* K *contains the permutation module* $H_2(T + 1)$.

In this case, there is an equivalence $\mathbb{N}^{\mathbf{s}}_d \cong \mathbb{O}^{\mathbf{s}}_d$ *for all* $d \leqslant D$ *which matches* K *with the usual Knizhnik–Zamolodchikov functor* KZ.

Proof. This is heavily based on [**RSVV16**, 2.20], which we apply in this case with $R = \mathscr{R}$, $B = H_d(\mathsf{q}, \mathsf{Q}_\bullet)$, $F = \mathsf{KZ}$, $F' = K$. There are four conditions required by this lemma, which we consider in the order given there.

- The order induced by the two covers must have a common refinement: This is one of our assumptions.
- The functor KZ is fully faithful on standard or costandard filtered objects in $\mathbb{O}^{\mathbb{S}}_d$: This is proven in [**RSVV16**, 5.37].
- The functor K is fully faithful on $(\mathbb{N}^{\mathbf{s}}_d)^\Delta$ and $(\mathbb{N}^{\mathbf{s}}_d)^\nabla$: Using the duality, these two statements are equivalent. Thus, we need only establish that K is 0-faithful (that is faithful on standard filtered objects). We already assume that $K \otimes_{\mathscr{R}} \mathbb{C}((h, z_1, \dots, z_\ell))$ is an equivalence and thus 0-faithful. The result then follows from [**RSVV16**, 2.18].
- The image $\mathsf{KZ}(P)$ of any projective P in $\mathbb{O}^{\mathbb{S}}$ whose simple quotient L has $\mathrm{Ext}^i(L, T) \neq 0$ for some tilting T and $i = 0$ or 1 also lies in the image of K: By [**RSVV16**, 6.3], these images are precisely the modules of the form $H_d \otimes_{H_1} M$ for M in the image of projectives under KZ, and if $q = -1$, also the modules $H_d(T_1 + 1)$.

  By compatibility with induction functors, we only need to show that $\mathsf{KZ}(P)$ of any projective object in $\mathbb{O}^{\mathbf{s}}_1$ and $H_2(T + 1)$ (if $q = -1$) lie in this image. The latter is an assumption, so we need only address the former. In H_1, we have ℓ different simple representations over the generic point which correspond to

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the eigenvalues Q_i. We denote the corresponding standard module Δ_i. Let H_1^u be the stable kernel of $X_1 - u$, and consider $P_1^u = \mathrm{ind}(\mathscr{R}, H_1^u)$. Let m_u be the number of indices i such that $Q_i = u$. The object P_1^u is indecomposable (since H_1^u is and K is fully faithful on projectives), and thus has a unique standard quotient Δ_i for some i, the largest standard such that $Q_i = u$. The kernel of this map is a module we call P_2^u; this has a standard filtration, and thus a map to some other standard Δ_j. If we let P_j be the projective cover of Δ_j, we have an induced map $P_j \to P_2^u$. The induced map $K(P_j) \to K(P_2^u) = (X - \mathsf{Q}_i)H_1^u$ must be surjective, since it induces a surjective map $K(P_j) \otimes \mathbb{C} \to K(\Delta_i) \otimes \mathbb{C}$.

Consider $K(P_j)\otimes\mathbb{C}$. This must be the kernel of $(X_1-u)^m$ for some $1 \leqslant m \leqslant m_u$. In fact, $m > m_u$ because otherwise, we would have $K(P_j) \otimes \mathbb{C} \cong K(P_i) \otimes \mathbb{C}$ (impossible since $K \otimes \mathbb{C}$ is fully faithful on projectives). Thus, $K(P_j) \otimes \mathbb{C}$ has dimension $\leqslant m_u - 1$. Since the dimension of $K(P_2^u) \otimes \mathbb{C}$ is $m_u - 1$, we must have $K(P_2^u) \otimes \mathbb{C} \cong K(P_j^u) \otimes \mathbb{C}$, so by full faithfulness, $P_j \cong P_2^u$.

Applying this argument inductively, we find that P_1^u has a filtration by the different indecomposable projectives on which X_1-u is topologically nilpotent, with successive quotients being the different standards. In particular, the images of these projectives are

$$\mathscr{R}[X_1] \Big/ \prod_{\substack{Q_i=u \\ \Delta_i \leqslant \Delta_j}} (X_1 - \mathsf{Q}_i)$$

which are the same as the images for KZ.

This establishes all the conditions and finishes the proof. □

Note that this theorem can be easily applied to show that whenever the order induced on multipartitions of numbers $\leqslant D$ induced by the Cherednik algebra (which uses the c-function) coincides with dominance order, then we can apply this theorem with $\mathbb{N}_d^{\mathbf{s}}$ the category of modules over the rank d cyclotomic q-Schur algebra for the parameters $(\mathsf{q}, \mathsf{Q}_\bullet)$. This will occur whenever $s_1 \ll s_2 \ll \cdots \ll s_\ell$, though there is no choice of parameters where this will work for all D if $k \in \mathbb{Q}$.

2.2. Combinatorial preliminaries. The combinatorics that underlie category $\mathcal{O}$ for a Cherednik algebra are those of higher level Fock spaces and multipartitions. We must introduce a small generalization of the combinatorics that appear in twisted Fock spaces (in the sense of Uglov [Ugl00]). As we see later, this is just rearranging deck chairs, but it is quite convenient for us. Fix scalars $(r_1, \ldots, r_\ell) \in (\mathbb{C}/\mathbb{Z})^\ell$, and $k \in \mathbb{C}$ with $\kappa = \mathrm{Re}(k)$. Consider the subset of

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$\mathbb{C}/\mathbb{Z}$ defined by

$$U = \{r_i + km \pmod{\mathbb{Z}} \mid i = 1, \dots, \ell \text{ and } m \in \mathbb{Z}\}.$$

This set is finite if and only if $k \in \mathbb{Q}$. We endow this set with a graph structure by connecting $u \to u + k$ for every $u \in U$. We let $\mathfrak{g}_U$ be the Lie algebra whose Dynkin diagram is given by U if $k \notin \mathbb{Z}$. If $k \in \mathbb{Z}$, then we let $\mathfrak{g}_U$ be the product over U of copies of $\widehat{\mathfrak{gl}}_1$, the Heisenberg algebra on infinitely many variables, with the grading element ∂ adjoined. This is a product of either finitely many copies of $\widehat{\mathfrak{sl}}_e$ if $k = a/e$ with $(a, e) = 1$, or of $\mathfrak{sl}_\infty$ if k is irrational. Throughout, we fix e to be the denominator of k if $k \in \mathbb{Q}$, or $e = 0$ if $k \notin \mathbb{Q}$.

REMARK 2.4. For purposes of the internal theory of WF Hecke algebras, we only care about the exponentials $\exp(2\pi i r_j) = Q_j$ and $\exp(2\pi i k) = q$. Thus, we could just as easily define U to be the subset of $\mathbb{C}^\times$ of the form $\{Q_i q^m\}$ for $m \in \mathbb{Z}$. This definition easily translates to other fields, and applies equally well there. However, it is only over $\mathbb{C}$ that we can make sense of the connection to Cherednik algebras, so we focus on this case.

2.2.1. Weightings. Usually in the theory of twisted Fock spaces, one has a basis indexed by ℓ-multipartitions, and the structure of this space (especially its $\mathfrak{g}_U$-module structure) depends on an ℓ-tuple of integers, called the *charge* of the different partitions. These charges both determine the physical position of partitions on a line, and determine a fundamental weight of $\widehat{\mathfrak{sl}}_e$ by taking reduction modulo e. We wish to separate these functions of the charge, and generalize to the case where $\widehat{\mathfrak{sl}}_e$ is replaced by $\mathfrak{g}_U$.

DEFINITION 2.5. A *weighting* of an ℓ-multipartition is an ordered ℓ-tuple $(r_1, \dots, r_\ell) \in (\mathbb{C}/\mathbb{Z})^\ell$ and an ordered ℓ-tuple $(\vartheta_1, \dots, \vartheta_\ell) \in \mathbb{R}^\ell$ with $\vartheta_i \neq \vartheta_j$ (with no assumption of congruence between the two).

Given an arbitrary weighting, we associate a *residue* in U to each box of the diagram of a multipartition: the box (a, b, m) receives $r_m + k(b - a)$; note that all elements of U occur for some multipartition. We often match these residues with their corresponding simple roots of $\mathfrak{g}_U$. We let $\mathrm{res}(\xi/\eta)$ for a skew multipartition ξ/η denote the sum of the roots corresponding to each box in its diagram.

In essence, if the residues r_i and r_j do not differ by an integer multiple of k, the corresponding partitions will not interact; this is analogous to a result of Dipper and Mathas [DM02, 1.1] for Ariki–Koike algebras. Thus, let us concentrate on the case where the graph U is connected.

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DEFINITION 2.6. The *Uglov weighting* $\vartheta^{\pm}_{\underline{\mathbf{s}}}$ attached to an ℓ-tuple $(s_1, \dots, s_\ell)$ of integers (its *charge*), is that where $k = \kappa = \pm 1/e$ if $e > 0$ and k is an arbitrary positive irrational real if $e = 0$.

- The residue r_m is given by the reduction of $ks_m \pmod{\mathbb{Z}}$.
- The weights of the partitions are given by $\vartheta_j = \kappa s_j - je\kappa/\ell$.

The choice of $k = \pm 1/e$ is less significant than it might first appear; nothing about the combinatorics we consider later will change if $k = \pm a/e$ for any positive integer a coprime to e. In general, our combinatorics will reduce to familiar notions for those who work with charged multipartitions and twisted higher level Fock spaces in the Uglov case.

There is a symmetry of this definition: sending $k \to -k$ and $s \mapsto s^\star = (-s_\ell, \dots, -s_1)$ results in the same weighting up to shift, if we reindex $i \mapsto \ell - i + 1$, and send $r_i \mapsto -r_{\ell-i+1}$.

Actually, for any weighting with U connected, there is an Uglov weighting which gives the same algebras T^ϑ. Thus, we lose no generality by only considering Uglov weightings.

DEFINITION 2.7. For an arbitrary weighting with U connected, we define its *Uglovation* as follows:

- By assumption $r_j - r_1$ is an integer multiple of k. If $e = 0$, we let h_j be the unique integer such that $r_j - r_1 = kh_j$, and if $e > 0$, let h_j be the smallest such nonnegative integer. In particular $h_1 = 0$.
- Reindexing values except for the first, we can assume $\vartheta_j/\kappa - h_j$ are cyclically ordered (mod e).
- We let $s_1 = 0$ by convention. We let s_j be the unique integer such that $s_j \equiv h_j \pmod{e}$ and $0 \leqslant \vartheta_j/\kappa - \vartheta_1/\kappa - s_j \leqslant e$. That is,

$$s_j = h_j + e\lfloor(\vartheta_j/e\kappa - \vartheta_1/e\kappa)\rfloor.$$

2.2.2. Dominance order. We imagine our multipartition diagram drawn in 'Russian notation' with rows tilted northeast, and columns northwest if $\kappa > 0$ (and *vice versa* if $\kappa < 0$), with the bottom corner placed at ϑ_m, and the boxes having diagonal of length 2κ; see Figure 1. For a box at (i, j, m) in the diagram of ν, its *x-coordinate* is $\vartheta_m + \kappa(j - i)$, that is, the x-coordinate of the center of the box when partitions are drawn as we have specified. This coincides with the $\underline{\mathbf{s}}$-shifted content as in [GL] if we choose $\kappa = 1$ and $s_i = \vartheta_m$.

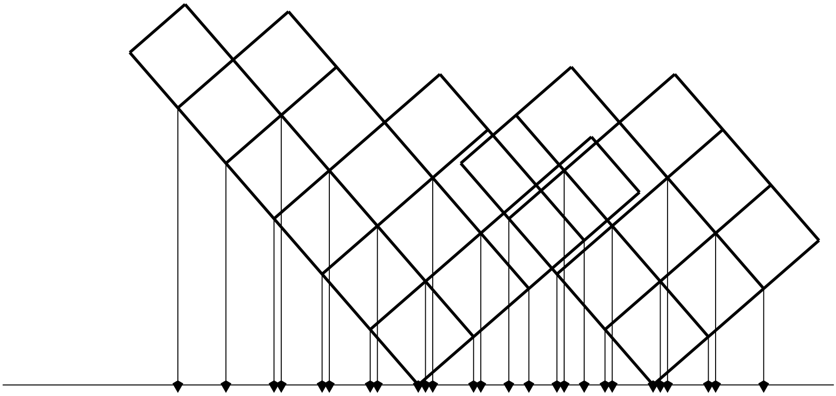

Figure 1. The set D_ξ attached to the multipartition $\xi = (6, 5, 3, 1); (4, 4, 3)$.

Definition 2.8. The *weighted dominance order* on multipartitions for a fixed weighting is the partial order where $\nu \geqslant \nu'$ if for each real number a, the number of boxes in ν with a fixed residue with x-coordinate less than a is greater than or equal to the same number in ν'.

On single partitions, this is a coarsening of the usual dominance order, but for multipartitions, it depends in a subtle way on the weighting. What is usually called dominance order on multipartitions arises when the partitions are far apart.

In order to clarify the relationship between our combinatorics and that for rational Cherednik algebras, it will be useful to refine this order using a numerical function; we let the *weighted c-function* be the function that assigns to a multipartition the sum of minus the x-coordinates of its boxes. The obvious order by c-function is thus a refinement of weighted dominance order.

Proposition 2.9. *If we let $\vartheta_i = s_i\kappa - i/\ell$ then our c-function agrees with the usual c-function (see* [GL, 2.3.8]*), up to addition and multiplication by constants. Note that when $\kappa = 1/e$ and the numbers s_i are integers, we recover our usual Uglov weighting for the charge $\underline{\mathbf{s}}$. Thus, in this case, the usual c-function order is a refinement of $\vartheta^+_{\underline{\mathbf{s}}}$-weighted dominance order.*

Proof. This follows instantly from the formula [GL, 2.3.8]. Note that we use a different convention from [GL]; their λ_i is our λ_{i-1}. Up to constants, we need

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only to show that our c-function agrees with

$$-\sum_{r=1}^{\ell}(r-1)|\xi^{(r)}| + \sum_{A\in\xi}\kappa\ell\,\mathrm{res}^{\mathbf{s}}(A) = \sum_{(i,j,r)\in\lambda} -r+1+(s_r+j-i)\cdot -\kappa$$
$$= \sum_{(i,j,r)\in\lambda} -(\kappa s_r + r - 1) - \kappa(j-i).$$

This is, indeed, the sum of minus the x-coordinates of the boxes under the Uglov weighting. □

2.2.3. Loadings for multipartitions and **i***-tableaux.* Fix a multipartition ξ, and give its diagram a very subtle tilt to the right. We create a subset by projecting the top corner of each box to the real number line, and weighting that point with the residue of the box. More precisely:

DEFINITION 2.10. We let

$$D_\xi := \{\vartheta_k + (i+j)\epsilon + \kappa(j-i) | (i,j,k) \text{ a box in the diagram of } \nu\}.$$

Obviously, this set depends on ϵ, but for $0 < \epsilon$ sufficiently small, its equivalence class will not change. This equivalence class will be independent of ϵ as long as $0 < \epsilon < |\vartheta_i - \vartheta_j + q\kappa|/|\xi|$ for integers q with $|q| \leqslant |\xi|$, so we exclude ϵ from the notation.

We can upgrade this set to a loading – that is, to a map $D_\xi \to U$. In [Webc], we would think of this as a map from $\mathbb{R} \to U \cap \{0\}$ that extends the map on D_ξ by 0 on all other points. The loading $\mathbf{i}_\xi$ sends $\vartheta_k + (i+j)\epsilon + \kappa(j-i)$ to the simple root α_m if there is a box (i, j, k) in the diagram of ν with residue $m = r_k + k(j-i)$, and 0 otherwise.

DEFINITION 2.11. Given a subset $D \subset \mathbb{R}$, let a *D-tableau* be a filling of the diagram of a multipartition with the elements of D such that:

- each $d \in D$ occurs exactly once; and
- $\mathbf{i}$ is the function that sends each real number to the residue of the box it occurs in if it occurs and 0 otherwise;
- the entry in $(1, 1, m)$ is greater than ϑ_m;
- the entry in (i, j, m) is greater than that in $(i-1, j, m)$ minus κ and greater than that in $(i, j-1, m)$ plus κ.

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If the differences between each pair of real numbers which occurs is greater than κ, this is just the notion of a standard tableau on a charged multipartition. Also, note that transposing each partition gives a tableau when κ is replaced by $-\kappa$.

If we upgrade the set D to a loading $\mathbf{i}: D \to U$, a **i**-*tableau* is a D-tableau such that $\mathbf{i}$ is the function that sends each element of D to the residue of the box it occurs in.

Note that this condition is the same as saying that if we add $\kappa(i - j)$ to the entry in box (i, j, k), we obtain a standard tableau on each component of the multipartition. Of course, this makes the dependence on the set D quite complicated, so we prefer not to think of them this way.

The *Russian reading word* of a **i**-tableau of shape η is the word obtained by reading the boxes of the tableau in order of the x-coordinate, reading up columns, that is, in the order of the loading $\mathbf{i}_\eta$, reading left to right.

For a usual standard tableau of shape η, the boxes where entries are below a fixed value form a new partition diagram. However, for a **i**-tableau, this is not the case; that said, one can make sense of a particular box being addable or removable relative to a value h.

DEFINITION 2.12. For a fixed box (i, j, m) whose entry is not h, we have a subdiagram of η given by the boxes (i', j', m) with entries $> h + (j' - i' - j + i)\kappa$. We say that $b = (i, j, m)$ is *addable* (respectively *removable*) *relative to h* if:

- it is addable (respectively removable) for this subdiagram; and
- if $b = (1, 1, m)$ then we have $\vartheta_m < h$.

That is, a box is addable (respectively removable) relative to h if it existing entry (if it is has one) is $>h$ (respectively $<h$) and making its entry h would not disturb the tableau conditions. These conditions are visually represented in Figure 2.

Note that the subdiagram we consider depends on h and $i - j$, and that it is only relevant whether the adjacent squares $(i, j \pm 1, m)$ and $(i \pm 1, j, m)$ are in this subdiagram. Note that:

- The box $(1, 1, m)$ is addable if $\vartheta_m < h$ and furthermore if it is in the diagram, then the entry is $>h$.
- The box (i, j, m) in the diagram of η is addable relative to h if h is less than the entry in (i, j, m), $h + \kappa$ is greater than the entry in $(i - 1, j, m)$ and $h - \kappa$ is greater than the entry in $(i, j - 1, m)$.

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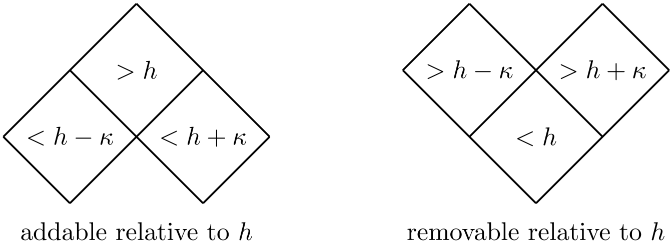


Figure 2. Relatively addable and removable boxes.

- If (i, j, m) is not in the diagram of η, it is addable relative to h if it is addable for the whole diagram and $h + \kappa$ is greater than the entry in $(i - 1, j, m)$ and $h - \kappa$ is greater than the entry in $(i, j - 1, m)$.
- A box (i, j, m) in the diagram of η is removable relative to h if h is greater than the entry in (i, j, m), $h + \kappa$ is less than the entry in $(i, j + 1, m)$ and $h - \kappa$ is greater than the entry in $(i + 1, j, m)$.

We say that a box (i', j', m') is right of (i, j, m) if the associated x coordinate in D_ξ is greater, that is if

$$\vartheta_{m'} + (i' + j')\epsilon + \kappa(j' - i') > \vartheta_m + (i + j)\epsilon + \kappa(j - i).$$

DEFINITION 2.13. The *degree* of a box b in a $\mathbf{i}$-tableau with entry h is the number of boxes of the same residue as and to the right of b which are addable relative to the entry h minus the number removable relative to h.

The *degree* of a $\mathbf{i}$-tableau is the sum of the degrees of the boxes.

Again, we wish to emphasize that this does not count elements which are addable or removable with respect to a fixed diagram; instead for each box (i', j', m') right of our fixed one, we compute a separate subdiagram with depends on $i' - j'$ and on h, and check whether it is addable or removable in this diagram.

2.3. WF Hecke algebras defined. We apply this combinatorics to define a diagrammatic version of the category $\mathcal{O}_m$. As in [Webb], let S be a local complete $\mathbb{k}$-algebra and let $\mathsf{q}, \mathsf{Q}_1, \ldots, \mathsf{Q}_\ell \in S$ be units with $q, Q_1, \ldots, Q_\ell$ their images in $\mathbb{k}$.

DEFINITION 2.14. We let a *type WF Hecke diagram* be a collection of curves in $\mathbb{R} \times [0, 1]$ with each curve mapping diffeomorphically to $[0, 1]$ via the projection to the y-axis. Each curve is allowed to carry any number of squares or the formal inverse of a square. We draw:

- a dashed line κ units to the right of each strand, which we call a *ghost*;
- red lines at $x = \vartheta_i$ each of which carries a label $\mathsf{Q}_i \in S$.

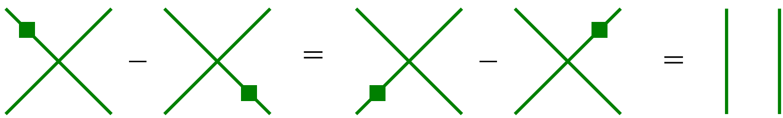
We now require that there are no triple points or tangencies involving any combination of strands, ghosts or red lines and no squares lie on crossings. We consider these diagrams equivalent if they are related by an isotopy that avoids these tangencies, double points and squares on crossings.

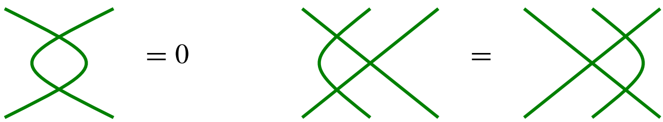
In examples, we usually draw these with the number Q_i written at the bottom of the strand, leaving the lift Q_i implicit.

Note that at any fixed value of y, the positions of the various strands in this horizontal slice give a finite subset D of $\mathbb{R}$. If this slice is chosen generically, in particular avoiding any crossings, we have that we have $\vartheta_i - d \neq m\kappa$ and $d' - d \neq m\kappa$ for all $d, d' \in D$, $m \in \mathbb{Z}$. We call such a subset *generic*.

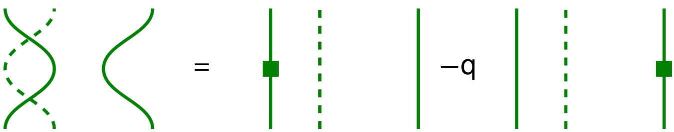
We can now define the object of primary interest in this section.

DEFINITION 2.15. The *type WF Hecke algebra* $\mathfrak{C}^\vartheta$ is the S-algebra generated by WF Hecke diagrams modulo the relations


(2.1a)



(2.1b)



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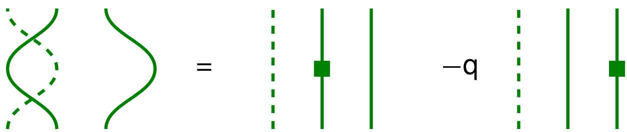

(2.1d)

(2.1e)

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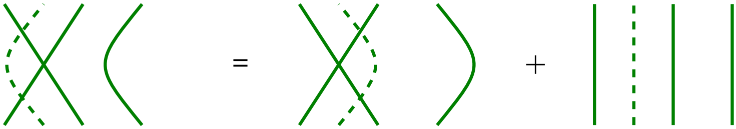

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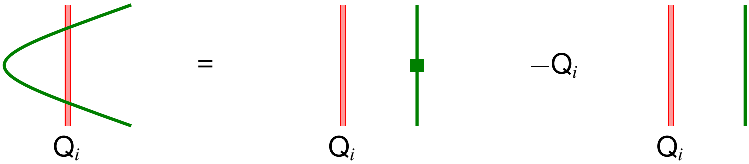
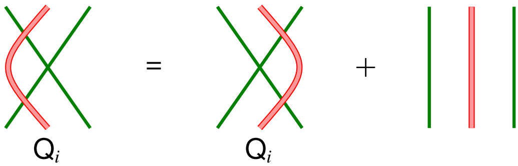
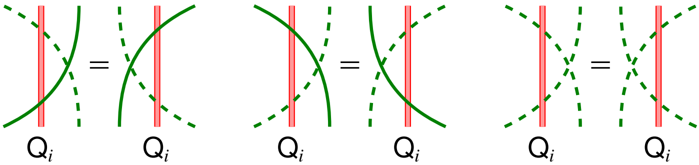


(2.1g)



(2.1h)



(2.1i)

and the nonlocal relation that a idempotent is 0 if the strands can be divided into two groups with a gap $>|\kappa|$ between them and all red strands in the right hand group.

Some care must be used when understanding what it means to apply these relations locally. In each case, the LHS and RHS have a dominant term which are related to each other via an isotopy through a disallowed diagram with a tangency, triple point or a square on a crossing. You can only apply the relations if this isotopy avoids tangencies, triple points and squares on crossings everywhere else in the diagram; one can always choose isotopy representatives sufficiently generic for this to hold.

2.4. A Morita equivalence. One must be slightly careful in the definition of these algebras, since as described they have $\aleph_1$ many idempotents. We usually fix a finite collection $\mathscr{D}$ of subsets of $\mathbb{R}$ and consider the subalgebra $\mathfrak{C}^{\vartheta}_{\mathscr{D}}$ where the green strands at the top and bottom of every diagram is equal to one of the sets in $\mathscr{D}$. This subalgebra is finite-dimensional. In fact, we describe a basis of it in Lemma 2.22. Recall that for each ℓ-multipartition ξ, we have a subset D_ξ defined as in Figure 1. Let $\mathscr{D}^{\circ}_{m}$ be the collection of these for all ℓ-multipartitions of size m.

LEMMA 2.16. *For all collections $\mathscr{D}$ of m-element subsets containing $\mathscr{D}_m^\circ$, the inclusion $\mathfrak{C}^\vartheta_{\mathscr{D}_m^\circ} \to \mathfrak{C}^\vartheta_{\mathscr{D}}$ induces a Morita equivalence.*

Proof. Let $e_\circ$ be the idempotent given by the sum of straight-line diagrams for the subsets D_ξ, then we already know that $e_\circ \mathfrak{C}^\vartheta_{\mathscr{D}} e_\circ = \mathfrak{C}^\vartheta_{\mathscr{D}_m^\circ}$, so in order to show that $e_\circ \mathfrak{C}^\vartheta_{\mathscr{D}}$ induces a Morita equivalence, we need only show that $\mathfrak{C}^\vartheta_{\mathscr{D}} e_\circ \mathfrak{C}^\vartheta_{\mathscr{D}} = \mathfrak{C}^\vartheta_{\mathscr{D}}$. We also simplify by only considering the case where $\kappa < 0$. The case where $\kappa > 0$ follows by similar arguments.

The underlying idea of the proof is that at $y = 1/2$, we push strands as far to the left as possible. Fix a real number ϵ. By applying an isotopy, we may assume that for any strand in the horizontal slice at $y = 1/2$, there is either a strand (red or black) or a ghost within ϵ to its left, or a strand within ϵ to the left of its ghost. Otherwise, we can simply move this strand to the left by ϵ. Eventually this process will terminate, or the slice at $y = 1/2$ will be unsteady and thus 0. We call such a diagram *left-justified*.

This defines an equivalence relation on strands generated by the relations that two strands are equivalent if one is with ϵ of the other or its ghost. Once we shrink ϵ to be much smaller than $\vartheta_i - \vartheta_j - p\kappa$ for all $i \neq j \in [1, \ell]$ and $p \in \mathbb{Z}$, we cannot have any pair of red strands which are equivalent, since the distance between two equivalent strands must within $m\epsilon$ of a multiple of κ. On the other hand, every equivalence class must contain a red strand, since otherwise, we can simply shift all its elements ϵ units to the left.

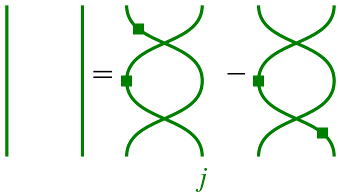
We now place a preorder on left-justified diagrams, given by the dominance order on the slice at $y = 1/2$ and then ordering by the distance of dots from the red line in its equivalence class. That is, for each equivalence class, we have a function $\delta(t)$ given by the number of dots on strands in the equivalence class within t units of the red strand. If we have two left-justified diagrams a, b with the same slice at $y = 1/2$, then $a \geqslant b$ if $\delta_a \geqslant \delta_b$ for every equivalence class.

First, we note that if we have a pair of black strands within 2ϵ of each other with a ghost between them, but no strand between their ghosts, we can apply (2.1f) to write this in terms of slices higher in this partial order. Similarly, if there is no ghost between the strands, but a strand between the ghosts, we can apply (2.1e). Thus, we need only consider the possibilities that two consecutive strands within 2ϵ of each other have both a strand between ghosts and a ghost between strands, or neither.

Consider the point of the closest dot in a fixed equivalence class to the red line. The strand that this dot sits on must be constrained from moving left by a ghost or a red strand. If it is a red strand, then we can apply the relation (2.1g) to write this in terms of a diagram with slice higher in dominance order and the diagram with the dot removed.

If the dot is to the left of the red line, then it cannot be constrained only by a strand left of its ghost, since in this case, we can just shift the strand and all to its left in the equivalence class ϵ units leftward. As usual, this process must terminate or the idempotent will be 0 in $\mathfrak{C}^{\vartheta}$. Thus, either a ghost or strand to its left is constraining it.

If the constraint is a ghost, we can apply (2.1d) to move this strand left. The correction term will have a dot closer to the red strands. If the constraint is a strand, then we can apply the relation



(2.2)

to move the dot left. Eventually, the dot will encounter a ghost and we can apply an earlier argument. Since the dot moved left by no more than $m\epsilon$ and then moved right by κ, over all it has moved right.

Symmetrically, if the dot is right of the red line, then we must have that it is constrained by a strand, either immediately to its left or left of its ghost. Otherwise, the original strand and all to its right in the equivalence class can be moved left by ϵ units. We can apply (2.2) for a strand immediately to the left or (2.1c) for one left of the ghost, to show that this factors through a slice higher in dominance order.

Thus, in all cases, if there is a dot anywhere, the diagram can be written as a sum of ones higher in our preorder. That is, we can assume that there are no dots. Furthermore, if we have a consecutive pair of strands with no ghost between them, we can apply the relation (2.2), and rewrite as a sum of diagrams higher in our order. Thus, every pair of strands within 2ϵ of each other must be separated by a ghost, and their ghost must be separated by a strand.

Each equivalence class breaks up into groups of strands within $m\epsilon$ of the points $m\kappa + \vartheta_p$ for $m \in \mathbb{Z}$. Left of the red strand, the leftmost element of each group of the equivalence class must be a ghost, there is a central group where the red strand itself is leftmost, and then right of the red strand leftmost element must be a strand.

This precisely means that the resulting slice has strands at the points in D_ξ for some ξ: the boxes of ξ are in bijection with strands; the equivalence classes correspond to the component partitions in the multipartition ξ, the box $(1, 1, p)$ corresponding to the strand which blocked by the red line p. Given the strand for the box (i, j, p), the box $(i+1, j, p)$ is the strand whose ghost is to the right of it,

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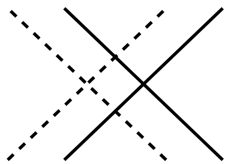
and the box $(i, j+1, p)$ is the strand caught on its ghost. The partition condition is precisely that two consecutive close black strands correspond to (i, j, p) and $(i+1, j+1, p)$, the ghost between them to $(i+1, j, p)$ and the strand between their ghosts to $(i, j+1, p)$.

This shows that the algebra is spanned by elements factoring through e_{D_ξ} for some multipartition ξ. □

2.5. Relationship to the Hecke algebra. For a real number $s > 0$, let $D_{s,m}$ be the set $\{s, 2s, \dots, ms\}$. For any affine Hecke diagram, we can embed it into the plane with top and bottom at $s, 2s, \dots, ms$, and if $s \gg |\kappa|$, we can assume that no strand passes between any crossing and its ghost. This will happen, for example, if we write the diagram as a composition of the diagrams of the type



If such a strand exists, we can just increase s by scaling the diagram horizontally; however, κ is left unchanged, so the strand will be pushed out from between the crossings.

PROPOSITION 2.17 [**Webb**, 5.5]. *For $s \gg |\kappa|$, the inclusion above induces an isomorphism of algebras $H_m(\mathsf{q}, \mathsf{Q}_\bullet) \cong \mathfrak{C}^{\vartheta}_{D_{s,m}}$, sending $T_i + 1$ to a crossing if $\kappa < 0$ and $T_i - q$ if $\kappa > 0$.*

This shows that the category $H_m(\mathsf{q}, \mathsf{Q}_\bullet)$ -mod is a quotient category of $\mathfrak{C}^{\vartheta}_{\mathscr{D}}$ for any collection $\mathscr{D}$ containing $D_{s,m}$.

We can extend this theorem a bit further to a 'relative setting.' Fix a collection $\mathscr{D}$, and fix $s > 0$ such that $s > |d| + |\kappa|$ for all elements $d \in D \in \mathscr{D}$. For $D \in \mathscr{D}$, let $D' = D \cup \{s, \dots, ms\}$, and $\mathscr{D}' = \{D'\}_{D \in \mathscr{D}}$.

LEMMA 2.18. *There is a natural map $\mathfrak{C}^{\vartheta}_{\mathscr{D}} \otimes H_m(\mathsf{q}) \to \mathfrak{C}^{\vartheta}_{\mathscr{D}'}$ given by horizontal composition. That is, the image of $a \otimes b$ is a diagram where $a \in \mathfrak{C}^{\vartheta}_{\mathscr{D}}$ is drawn attached to the points in D, and b is drawn attached to the points $\{s, \dots, ms\}$.*

Proof. We only need to check that horizontally composed diagrams in $\mathfrak{C}^{\vartheta}_{\mathscr{D}'}$ satisfy the correct relations. The relations of $H_m(\mathsf{q})$ are satisfied by the right-hand set of strands [**Webb**, 3.5], since all these relations are local in nature.

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For $\mathfrak{C}^{\vartheta}_{\mathscr{D}}$, we need only note that adding a diagram at the right will not change any of the relations. This is clear for the local relations, and unsteady idempotents remain unsteady, so the only nonlocal relation is preserved as well. □

Assume that $\mathscr{D}$ is a collection of sets of size m, and $\mathscr{E}$ a collection of sets of size $m+1$. We can consider the $\mathfrak{C}^{\vartheta}_{\mathscr{E}}$ - $\mathfrak{C}^{\vartheta}_{\mathscr{D}}$ module $e_{\mathscr{E}}\mathfrak{C}^{\vartheta}e_{\mathscr{D}'}$ where $\mathfrak{C}^{\vartheta}_{\mathscr{D}}$ acts on the right via the map above, and as before $\mathscr{D}'$ is the collection given by $\mathscr{D}$ with $\{s\}$ added to each set (where $\{s\}$ is assumed to be $|\kappa|$ larger than any element of $D \in \mathscr{D}$).

Tensor and Hom with this bimodule induces adjoint $\mathscr{R}$-linear induction and restriction functors

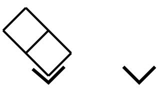
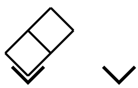
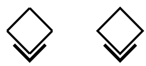
$$\text{ind}\colon \mathfrak{C}^{\vartheta}_{\mathscr{D}} \to \mathfrak{C}^{\vartheta}_{\mathscr{E}} \qquad \text{res}\colon \mathfrak{C}^{\vartheta}_{\mathscr{E}} \to \mathfrak{C}^{\vartheta}_{\mathscr{D}}.$$

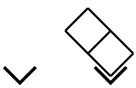
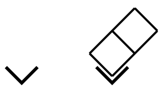
We consider some important properties of these functors later.

2.6. Cellular structure. In this section, we define a cellular structure on this algebra. Consider a generic subset $D \subset \mathbb{R}$, and a D-tableau S of shape ξ. We describe a WF Hecke diagram $\mathcal{B}_{\mathsf{S}} \in e_{D_\xi}\mathfrak{C}^{\vartheta}e_D$ which matches D_ξ at the top $y=1$ (that is its points are given by the projection of boxes in the diagram, as in Figure 1) and given by the set D at the bottom. The strands at the top are naturally in bijection with boxes in the diagram of ξ, and those at the bottom have a bijection given by the tableau S. The strands of the diagram $\mathcal{B}_{\mathsf{S}}$ connect the top and the bottom using this bijection, without creating any bigons between pairs of strands or strands and ghosts. This diagram is not unique up to isotopy (since we have not specified how to resolve triple points), but we can choose one such diagram arbitrarily.

EXAMPLE 2.19. Consider the example where $q=-1$ and $\kappa=-4$, with $Q_1=1$, $Q_2=-1$ and $d=2$. The resulting category, weighted order, and basis only depend on the difference of the weights $\vartheta_1-\vartheta_2$. In fact, there are only 3 different possibilities; the category changes when this value passes ± 4.

There are 5 multipartitions of size 2:

$p_1 =$  $p_2 =$  $p_3 =$ 

$p_4 =$  $p_5 =$ 

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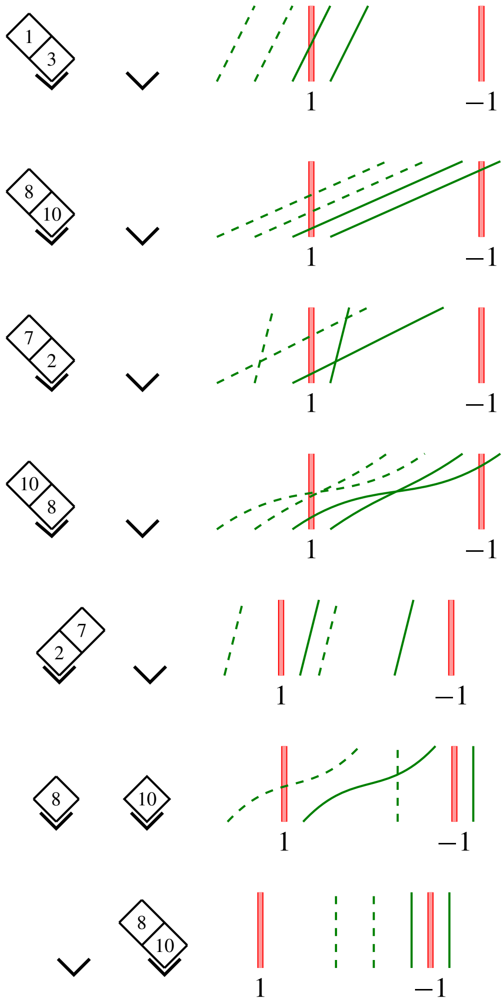
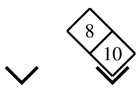
Case 1: $\vartheta_1 - \vartheta_2 < -4$. We exemplify this case with $\vartheta_1 = 0$, $\vartheta_2 = 9$. In this case, our order is $p_1 > p_2 > p_3 > p_4 > p_5$.

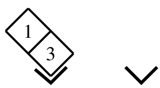
Consider the set $\mathscr{D} = \{\{1, 3\}, \{2, 7\}, \{8, 10\}\}$. For this collection, the tableaux with their corresponding B_S's are:



Note that



and



are not standard tableaux in the usual sense, but are D-tableaux as defined above.

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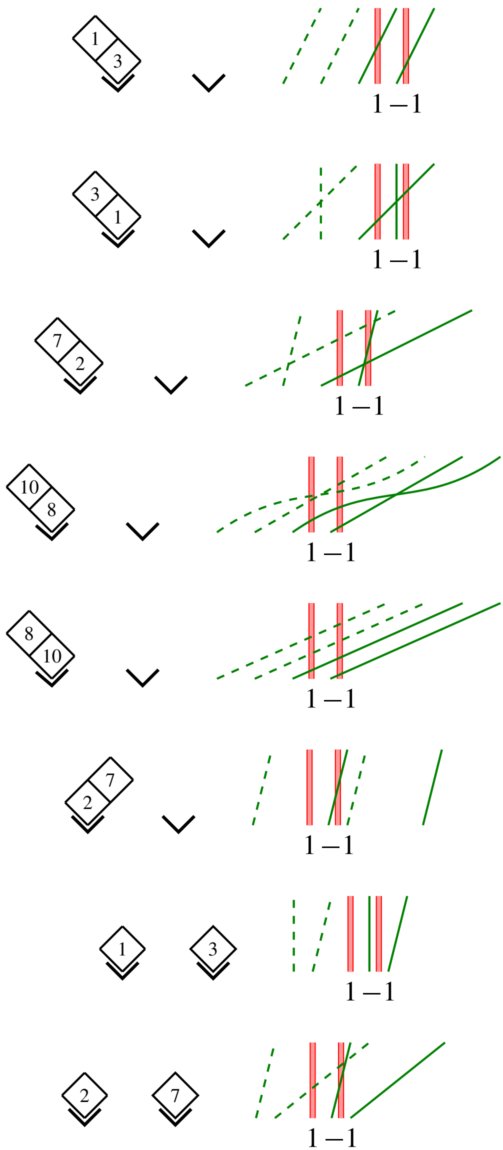
Case 2: $-4 < \vartheta_1 - \vartheta_2 < 4$. We exemplify this case with $\vartheta_1 = 0$, $\vartheta_2 = 1.5$. In this case, our partial order is $p_1, p_4 > p_3 > p_2, p_5$.

A loading in this case is given by specifying the position the point a labeled 1 and the point b labeled 2. We denote this loading $\mathbf{i}_{a,b}$. With $\mathscr{D}$ as before, the tableaux with their corresponding B_S's are:



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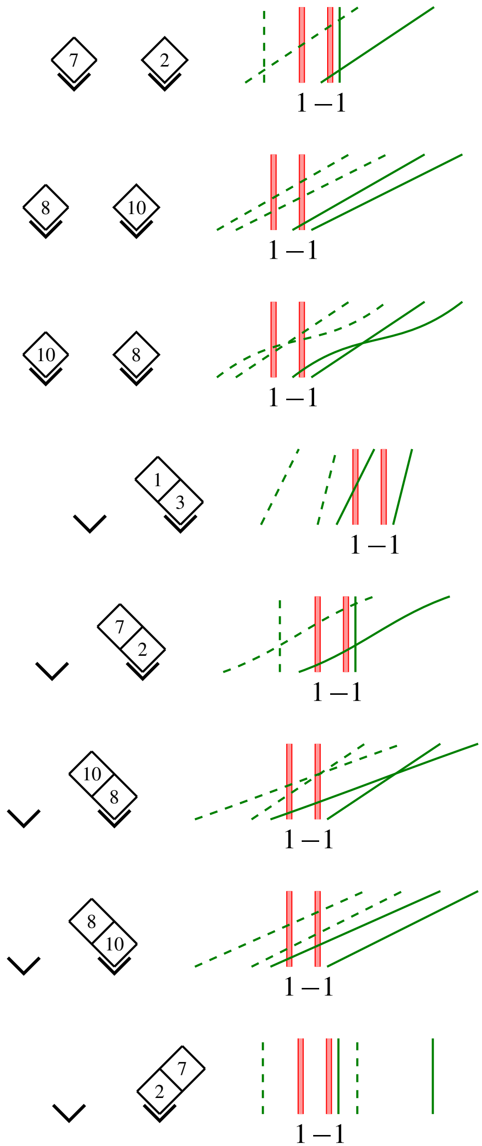

Case 3: $\vartheta_1 - \vartheta_2 > 4$. This is essentially the same as Case 1 with components reversed; in particular, the partial order is $p_4 > p_5 > p_3 > p_1 > p_2$.

DEFINITION 2.20. For each pair S, T of tableaux of the same shape, we let $\mathcal{C}_{\mathsf{S},\mathsf{T}} = \mathcal{B}_{\mathsf{S}}\mathcal{B}_{\mathsf{T}}^{*}$ where $*$ is the reflection of a diagram through a horizontal axis.

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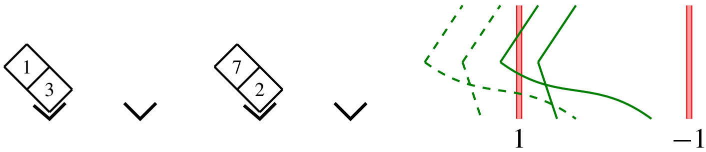
For example:



This yields $4^2 + 2^2 + 3^2 + 2^2 + 1^2 = 34$ basis vectors, which we will not write all of in the interest of saving trees.

LEMMA 2.21. *The space $e_D \mathfrak{C}^{\vartheta} e_{D'}$ is spanned by the elements $\mathcal{C}_{\mathsf{S},\mathsf{T}}$ for S a D-tableau and T a D'-tableau.*

Proof. By Lemma 2.16, every element can be written as a sum of elements of the form $ae_\xi b$ for different multipartitions ξ.

We need to show that $ae_\xi b$ can be written as a sum of the elements $B_{\mathsf{S},\mathsf{T}}$. Let us induct first on ξ according to weighted dominance order, and then on the number of crossings in the diagram below e_{D_ξ} plus the number above e_{D_ξ}. Note that we can assume that any bigon which appears is bisected by the line $y = 1/2$, and that all dots lie on this line. Thus, we can associate the top half and the bottom half to two fillings of the diagram of ξ, by filling each box with the top and bottom endpoint of each strand.

If a diagram has no crossings, it must be ordered in Russian reading order. There is only one way up to isotopy of drawing this diagram (since there are no crossings of two strands or two ghosts, and thus no triangles).

If there is a pair of entries which violate the partition condition, that means either a strand for the box (i, j, p) crosses a red strand to its left if $\mathsf{S}(1, 1, p) < \vartheta_i$, the ghost to its left if $\mathsf{S}(i, j, p) < \mathsf{S}(i, j-1, p) + \kappa$, or the ghost to its right if $\mathsf{S}(i, j, p) > \mathsf{S}(i+1, j, p)+\kappa$. In either case, doing just this crossing will result in a slice higher in dominance order, and we can isotope to assume that this crossing is the first thing we do. Thus, we can write this element using those corresponding to D-tableaux, and elements factoring through higher multipartitions.

Now, consider the general case. First of all, any pair of diagrams corresponding to the same tableau differ by shorter elements, which lie in the desired span by induction. Thus, we need only show that this is the span of some diagrams corresponding to tableaux (in our sense), not the fixed ones $\mathcal{C}_{\mathsf{S},\mathsf{T}}$.

However, if S is not a tableau, as we argued above, then either:

(i) $\mathsf{S}(1, 1, p) < \vartheta_p$, holds for some p;

(ii) or $\mathsf{S}(i, j, p) < \mathsf{S}(i, j-1, p) + \kappa$, holds for some i, j, p;

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(iii) or $\mathsf{S}(i, j, p) > \mathsf{S}(i+1, j, p) + \kappa$ holds for some i, j, p.

Each of these inequalities implies that there is a 'bad crossing':

(i) the green strand corresponding to the box $(1, 1, p)$ crosses the pth red strand;

(ii) or the green strand for the box (i, j, p) crosses the ghost of that for $(i, j-1, p)$;

(iii) or the green strand for (i, j, p) crosses the ghost of that for $(i+1, j)$.

If we choose a diagram for this filling where this 'bad crossing' is the first that occurs, then after isotopy, the slice after the 'bad crossing' is higher in dominance order than e_{D_ξ}. Thus, this diagram is in the span of diagrams factoring through a multipartition greater in weighted dominance order. Thus, these diagrams are in the span of $\mathcal{C}_{\mathsf{S}',\mathsf{T}'}$ for S', T' tableaux by induction; this completes the proof that these elements span. □

LEMMA 2.22. *The elements $\mathcal{C}_{\mathsf{S},\mathsf{T}}$ for S a D-tableau and T a D'-tableau of the same shape are a basis of $e_D\mathfrak{C}^\vartheta e_{D'}$ as a free S-module.*

Proof. Since we already know that these vectors span, we need only show that they are linearly independent. Note that if $D, D' = D_{s,m}$ for $s \gg 0$, then we know that $e_{D_{s,m}}\mathfrak{C}^\vartheta e_{D_{s,m}}$ is a free S-module of rank $m!\ell^m$. Thus, any spanning set of this size must be a basis. The vectors $\mathcal{C}_{\mathsf{S},\mathsf{T}}$ are thus a basis in this case, since $D_{s,m}$-tableaux for $s \gg 0$ are in canonical bijection with usual standard tableaux.

For a general choice of D, D', assume that we find a linear combination $\sum_{\mathsf{S},\mathsf{T}} c_{\mathsf{S},\mathsf{T}}\mathcal{C}_{\mathsf{S},\mathsf{T}} = 0$. Assume S has shape ξ which is minimal in dominance order among those with nonzero coefficients and that the number of crossings in $\mathcal{C}_{\mathsf{S},\mathsf{T}}$ is maximal among those corresponding to ξ with nonzero coefficients.

Given the tableau S, we can obtain a standard tableau of real numbers (in the usual sense) by considering the filling $\mathsf{S}^\circ(i, j, p) = \mathsf{S}(i, j, p) + \kappa(j-i)$. All entries of S° are distinct by the genericity of D.

Consider the diagram ϕ_{S} in $e_{D_{s,m}}\mathfrak{C}^\vartheta e_D$ which connects s at $y = 1$ to the strand corresponding to the smallest entry in S° at $y = 0$, connects $2s$ at $y = 1$ to the strand corresponding to the second smallest, and so on; we can define a similar diagram ϕ^*_{T} in $e_{D'}\mathfrak{C}^\vartheta e_{D_{s,m}}$. Consider the tableaux S', T' with the filling $s, \dots, ms$ which induce the same order on boxes as $\mathsf{S}^\circ, \mathsf{T}^\circ$. The product $\phi_{\mathsf{S}}\mathcal{C}_{\mathsf{S},\mathsf{T}}\phi^*_{\mathsf{T}}$ is the basis vector $\mathcal{C}_{\mathsf{S}',\mathsf{T}'}$. For every $\mathsf{S}'', \mathsf{T}''$ such that $c_{\mathsf{S}'',\mathsf{T}''} \neq 0$, we have that $\phi_{\mathsf{S}}\mathcal{C}_{\mathsf{S}'',\mathsf{T}''}\phi^*_{\mathsf{T}}$ is a sum of diagrams with no more crossings than $\mathcal{C}_{\mathsf{S}',\mathsf{T}'}$, and is thus a sum of basis vectors for higher multipartitions in dominance order, and ones for ξ with tableaux different from S and T.

Thus, if this linear combination is 0, it must be that the coefficient $c_{\mathsf{S},\mathsf{T}}$ is 0, since we have a basis of $\mathfrak{C}^{\vartheta}_{D_{s,m}}$. This is a contradiction and shows that the vectors $\mathcal{C}_{\mathsf{S},\mathsf{T}}$ are linearly independent. □

DEFINITION 2.23. A *cellular S-algebra* is an associative unital S-algebra A, free of finite rank, together with a *cell datum* $(\mathcal{P}, M, C, *)$ such that:

(1) $\mathcal{P}$ is a partially ordered set and $M(p)$ is a finite set for each $p \in \mathcal{P}$;

(2) $C : \dot{\bigsqcup}_{p\in\mathcal{P}} M(p) \times M(p) \to A$, $(\mathsf{T}, \mathsf{S}) \mapsto C^p_{\mathsf{T},\mathsf{S}}$ is an injective map whose image is a basis for A;

(3) the map $* : A \to A$ is an algebra anti-automorphism such that $(C^p_{\mathsf{T},\mathsf{S}})^* = C^p_{\mathsf{S},\mathsf{T}}$ for all $p \in \mathcal{P}$ and $\mathsf{S}, \mathsf{T} \in M(p)$;

(4) if $p \in \mathcal{P}$ and $\mathsf{S}, \mathsf{T} \in M(p)$ then for any $x \in A$ we have that

$$xC^p_{\mathsf{S},\mathsf{T}} \equiv \sum_{\mathsf{S}'\in M(p)} r_x(\mathsf{S}', \mathsf{S})C^p_{\mathsf{S}',\mathsf{T}} \pmod{A(> p)}$$

where the scalar $r_x(\mathsf{S}', \mathsf{S})$ is independent of T and $A(> \mu)$ denotes the subspace of A generated by $\{C^q_{\mathsf{S}'',\mathsf{T}''} \mid q > p, \mathsf{S}'', \mathsf{T}'' \in M(q)\}$.

The basis consisting of the $C^p_{\mathsf{S},\mathsf{T}}$ is then a *cellular basis* of A.

Recall that if A is an algebra with cellular basis, there is a natural *cell representation* S_ξ of A for each $\xi \in \mathcal{P}$ which is freely generated over S by symbols c_{T} for each $\mathsf{T} \in M(\xi)$, with the action rule $xc_{\mathsf{T}} = \sum_{\mathsf{S}\in M(\xi)} r_x(\mathsf{S}, \mathsf{T})c_{\mathsf{S}}$.

Fix a collection $\mathscr{D}$ of subsets of $\mathbb{R}$ such that each $D \in \mathscr{D}$ is generic. Let $M_{\mathscr{D}}(\xi)$ for a multipartition ξ be the set of tableaux whose entries form a set $D \in \mathscr{D}$. Let $\mathcal{P}^{\vartheta}_{\ell}$ be the set of ℓ-multipartitions with ϑ-weighted dominance order. Let $*: \mathfrak{C}^{\vartheta}_{\mathscr{D}} \to \mathfrak{C}^{\vartheta}_{\mathscr{D}}$ be the anti-automorphism given by reflection in a horizontal axis.

THEOREM 2.24. *The data $(\mathcal{P}^{\vartheta}_{\ell}, M_{\mathscr{D}}, \mathcal{C}, *)$ define a cellular S-algebra structure on $\mathfrak{C}^{\vartheta}_{\mathscr{D}}$.*

Proof. Consider the axioms of a cellular algebra, as given in Definition 2.23. Condition (1) is manifest.

Condition (2), that a basis is formed by the vectors $\mathcal{C}_{\mathsf{S},\mathsf{T}}$ where S and T range over tableaux for loadings from $\mathscr{D}$ of the same shape, follows from Lemma 2.22.

Condition (3) is clear from the calculation

$$\mathcal{C}^*_{\mathsf{S},\mathsf{T}} = (\mathcal{B}^*_{\mathsf{S}}\mathcal{B}_{\mathsf{T}})^* = \mathcal{B}^*_{\mathsf{T}}\mathcal{B}_{\mathsf{S}} = \mathcal{C}_{\mathsf{T},\mathsf{S}}.$$

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Thus, we need only check the final axiom, that for all x, we have an equality

$$x\mathcal{C}_{\mathsf{S},\mathsf{T}} \equiv \sum_{\mathsf{S}'\in M_B(\xi)} r_x(\mathsf{S}',\mathsf{S})\mathcal{C}_{\mathsf{S}',\mathsf{T}} \qquad (\star)$$

modulo the vectors associated to partitions higher in dominance order. The numbers $r_x(\mathsf{S}',\mathsf{S})$ are just the structure coefficients of x^* acting on the basis of S_ξ given by $\mathcal{B}_\mathsf{S}$. Since we have that $x\mathcal{B}^*_\mathsf{S} \equiv \sum_{\mathsf{S}'\in M_B(\xi)} r_x(\mathsf{S}',\mathsf{S})\mathcal{B}_{\mathsf{S}'}$ modulo diagrams factoring through loadings that are higher in weighted dominance order, the equation ($\star$) holds. This completes the proof. □

If A is an finite S-algebra with cellular basis $(\mathcal{P}, M, C, *)$, then for any basis vector $C^\xi_{\mathsf{S},\mathsf{T}}$, we have that $(C^\xi_{\mathsf{S},\mathsf{T}})^2 = a^\xi_{\mathsf{S},\mathsf{T}}C^\xi_{\mathsf{S},\mathsf{T}} + \cdots$ where other terms are in higher cells. A standard lemma (see [**KX99**, 2.1(3)] for the case of a field) shows that:

LEMMA 2.25. *The category* A-mod *is highest weight with standard modules given by the cell modules if for every* $\xi \in \mathcal{P}$, *there is some* S,T *with* $a^\xi_{\mathsf{S},\mathsf{T}}$ *a unit.*

COROLLARY 2.26. *The category* $\mathfrak{C}^\vartheta_{\mathscr{D}^\circ_m}$-mod *is highest weight.*

Proof. For any multipartition ξ, there is a tautological tableau T filling each box with the x-value of the corresponding point in D_ξ. Since $\mathcal{C}^\xi_{\mathsf{T},\mathsf{T}} = e_{D_\xi}$, this is an idempotent, and thus satisfies the conditions of Lemma 2.25. □

This cellular structure is also useful because it allows one to check that maps are isomorphisms by means of dimension counting. For example, this shows:

PROPOSITION 2.27. *For any multipartition* η, *the restriction of a cell module* $\operatorname{res}(S_\eta)$ *has a filtration* $N_n \subset N_{n-1} \subset \cdots \subset N_1$, *such that* $N_p/N_{p+1} \cong S_{\xi_p}$, *where* ξ_p *is the multipartition given by removing from* ξ *the pth removable box (read from left to right in Russian notation with weightings given by* ϑ_i*).*

Similarly, $\operatorname{ind}(S_\eta)$ *has a filtration* $M_1 \subset \cdots \subset M_q$ *such that* $M_p/M_{p-1} \cong S_{\xi^p}$, *where* ξ^p *is the multipartition given by adding to* ξ *the pth addable box.*

Proof. The module $e_{\mathscr{D}}\operatorname{res}(S_\eta)$ is spanned by a basis c_S indexed by tableaux S where the filling is given by a set in $\mathscr{D}$ with $\{s\}$ for $s \gg 0$ added. The entry s must be in a removable box, since it is more than $|\kappa|$ greater than any other entry. In terms of the diagram, this means we can factor it $c_\mathsf{S} = ab$ into two parts: in the bottom part b, we grab the strand corresponding to this removable box at $y = 0$, and pull it over to match with $x = s$; in the top part a, the strand at $x = s$ remains unchanged, and we act on the other strands by the tableau $\mathsf{S} \setminus \{s\}$, the tableau with the box labeled by s removed.

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By definition, N_p is the span of the basis vectors where this removable box is the pth, or one further leftward. The relations show that when multiplying by a diagram that does not touch the strand at s, this strand can only be shortened, not lengthened. After all, we cannot create new crossings with this strand, only break them with the correction terms in (2.1a)–(2.1i). Thus, N_p is a submodule.

Now, assume that T is a tableau with $\{s\}$ in the pth removable box. When we act on c_{T} by $x \in \mathfrak{C}^{\vartheta}_{\mathscr{D}}$ (so not acting on the strand at s), we have

$$xc_{\mathsf{T}} = x\mathcal{B}_{\mathsf{T}\setminus\{s\}}b = \sum_{\mathsf{S}\in M(\xi_p)} r_x(\mathsf{S}, \mathsf{T}\setminus\{s\})\mathcal{B}_{\mathsf{S}}b + \cdots$$

The remaining terms all lie in N_{p+1}, so we have seen that the map sending $c_{\mathsf{T}} \mapsto c_{\mathsf{T}\setminus\{s\}}$ is an isomorphism $N_p/N_{p+1} \cong S_{\xi_p}$.

The proof for $\operatorname{ind}(S_\eta)$ proceeds along similar lines; now we add a new strand at the bottom of the diagram, and O_p is the submodule spanned by all diagrams where the new strand goes no further left than the x-value for the pth addable box. □

Let us make a useful observation on the structure of the cell modules of this cell structure. Fix a set $D \subset \mathbb{R}$, and let X_d for $d \in D$ be the idempotent e_D with a square added on the strand at $x = d$. This defines an action $\Bbbk[X_{d_1}^{\pm 1}, \ldots, X_{d_m}^{\pm 1}]$ on any $\mathfrak{C}^{\vartheta}_{\{D\}}$-module. One can easily check that the symmetric polynomials in these variables are central. Summing over all $D \in \mathscr{D}$, we obtain a map $\zeta : \Bbbk[X_1^{\pm 1}, \ldots, X_m^{\pm 1}]^{S_m} \to Z(\mathfrak{C}^{\vartheta}_{\mathscr{D}})$ whenever all sets in $\mathscr{D}$ have size m.

LEMMA 2.28. *The map ζ is surjective and induces an isomorphism between $Z(\mathfrak{C}^{\vartheta}_{\mathscr{D}^\circ_m})$ and $Z(H_m(\mathsf{q}, \mathsf{Q}_\bullet))$.*

Proof. This follows immediately from the fact that -1-faithful covers induce an isomorphism between centers, and the fact that $Z(H_m(\mathsf{q}, \mathsf{Q}_\bullet))$ is generated by the symmetric polynomials in the $X_i^{\pm 1}$'s. □

Let σ be the sign of κ.

LEMMA 2.29. *The joint spectrum of $\Bbbk[X_{d_1}, \ldots, X_{d_m}]$ acting on $e_D S_\eta$ is the image of the map sending a D-tableau S of shape η to the point in $(\mathbb{C}^*)^D$ to the vector whose entry for $d \in D$ is $Q_p q^{\sigma(i-j)}$ where (i, j, p) in the diagram of η is the unique box with $\mathsf{S}(i, j, p) = d$. In particular, a symmetric Laurent polynomial $p(X_1, \ldots, X_m)$ acts as a unipotent transformation times this polynomial applied to the set $\{Q_p q^{\sigma(i-j)}\}$ for (i, j, p) ranges over the diagram of η.*

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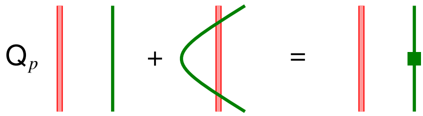
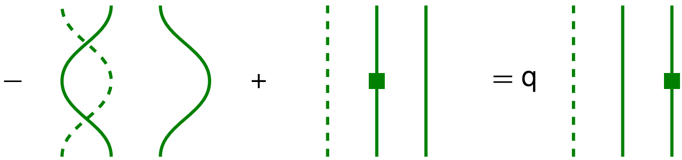
Proof. Assume that $\kappa < 0$. Now, filter S_η by all the span T_g of the basis vectors with $\geqslant g$ crossings of strands. The subspace T_g is invariant under $\Bbbk[X_{d_1}^{\pm 1}, \dots, X_{d_m}^{\pm 1}]$, and on the associated graded, we have that $X_d c_{\mathsf{S}} = \mathcal{B}_{\mathsf{S}} X_{d'} c_{\mathsf{T}}$ where T is the tautological tableau with filling D_ξ where $d' = b_p + \kappa(j-i) + \epsilon(j-i)$ and d fills the box (i, j, p) in S.

If $i = j = 0$ then by (2.1g), we have that



The second term of the LHS is 0 in S_η, so this is a X_d eigenvector with eigenvalue Q_p.

If $j \geqslant i$, then the strand corresponding to (i, j, p) is protected to the left by a ghost corresponding to $(i, j-1, p)$. Using the relation (2.1d):



Since the RHS is $\mathsf{q} \cdot \mathsf{Q}_p q^{j-1-i}$ times c_{S} (in the associated graded), and thus X_d has eigenvalue $\mathsf{Q}_p q^{j-i}$. Similarly, if $j < i$, then the strand is protected by a strand to the left of its ghost, and a similar argument using (2.1c) shows that the eigenvalue is the same in this case. □

3. Comparison of Cherednik and WF Hecke algebras

In this section, we prove a comparison theorem between the WF Hecke algebra and category $\mathcal{O}$ for a Cherednik algebra, using Theorem 2.3. Before moving to this proof, we need some preparatory lemmata.

3.1. Preparation.

If $\mathsf{q} - \zeta$ is a unit for every a root of unity ζ, and for every i, j, p, we have that $\mathsf{Q}_i - \mathsf{q}^p \mathsf{Q}_j$ is a unit, then the Hecke algebra $H_m(\mathsf{q}, \mathsf{Q}_\bullet)$ is semisimple by [**Ari02**]. In particular:

COROLLARY 3.1. *After base change to the fraction field* $R = \mathbb{C}((h, z_1, \dots, z_\ell))$, *the Hecke algebra* $H_m(\mathsf{q}, \mathsf{Q}_\bullet) \otimes_{\mathscr{R}} R$ *is semisimple.*

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LEMMA 3.2. *The isomorphism of Proposition 2.17 induces a Morita equivalence of* $\mathfrak{C}^{\vartheta}_{\mathscr{D}}$ *and* $H_m(\mathsf{q}, \mathsf{Q}_\bullet)$ *for every* $\mathscr{D}$ *of sets of size m containing* $D_{s,m}$ *if and only if the latter algebra is semisimple. In particular, it is an equivalence after the base change* $-\otimes_{\mathscr{R}} R$.

Proof. Since $\mathfrak{C}^{\vartheta}_{\mathscr{D}}$ is cellular with the number of cells given by the number of ℓ-multipartitions of m, this gives an upper bound on the number of simple modules this algebra can have. Corollary 2.26 shows that for at least one choice of $\mathscr{D}$, this bound is achieved. On the other hand, $H_m(\mathsf{q}, \mathsf{Q}_\bullet)$ has this number of nonisomorphic symbols if and only if it is semisimple.

We know that $H_m(\mathsf{q}, \mathsf{Q}_\bullet)$ -mod is a quotient category of $\mathfrak{C}^{\vartheta}_{\mathscr{D}}$ -mod. Since both categories are Noetherian, this quotient functor kills no module iff it kills no simple iff the number of simples over the two algebras coincide. This can only occur for all $\mathscr{D}$ if $H_m(\mathsf{q}, \mathsf{Q}_\bullet)$ is semisimple. ☐

LEMMA 3.3. *The functor* $K\colon \mathfrak{C}^{\vartheta}_{\mathscr{D}^\circ_m}$ -mod $\to H_m(\mathsf{q}, \mathsf{Q}_\bullet)$ -mod *is faithful on standard filtered objects, that is,* -1*-faithful.*

Proof. In the proof of Lemma 2.22, we showed that for any nonzero element $a \in e_D A_\xi$, we can choose $\phi_{\mathsf{S}} \in e_{D_{s,m}} \mathfrak{C}^{\vartheta}_{\mathscr{D}^\circ_m} e_D$ such that $\phi_{\mathsf{S}} a \neq 0$. That is, no submodule of a cell module is killed by e_D. Thus, the same is true of any module with a cell filtration. In particular, if $M \to N$ is a nonzero map between cell-filtered modules, then the image of this map is not killed by e_D, so we have a nonzero map $e_D M \to e_D N$. ☐

As noted in the proof of Theorem 2.3, [**RSVV16**, 2.18] now implies that:

COROLLARY 3.4. *The functor* K *is* 0*-faithful, and thus, in particular, fully faithful on projectives.*

A further corollary that will be quite useful for us regards the natural transformations of functors. For any monomials $F, F'\colon \mathfrak{C}^{\vartheta}_{\mathscr{D}^\circ_m} \to \mathfrak{C}^{\vartheta}_{\mathscr{D}^\circ_{m'}}$ in the functors ind, res, there are functors $F_{\mathsf{H}}, F'_{\mathsf{H}}\colon H_m(\mathsf{q}, \mathsf{Q}_\bullet)$ -mod $\to H_{m'}(\mathsf{q}, \mathsf{Q}_\bullet)$ -mod given by the same monomials.

LEMMA 3.5. $F_{\mathsf{H}} \circ K \cong K \circ F$.

Proof. It is enough to prove this when $F =$ ind, res itself.

The composition $\mathrm{res}_{\mathsf{H}} \circ K$ is given by the vector space $e_{D_{s,m}} M$, where $H_{m-1}(\mathsf{q}, \mathsf{Q}_\bullet)$ acts on the leftmost $m-1$ terminals. The functor $K \circ \mathrm{res}$ goes to the same

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vector space, but first separates the rightmost terminal, and then acts by $e_{D_{s,m-1}}$ on the remaining terminals. Thus, these functors are canonically isomorphic by the identity map.

The functor $K \circ \mathrm{ind}$ is given by tensor product with the bimodule $e_{D_{s,m+1}}\mathfrak{C}^{\vartheta}e_{\mathscr{D}'_m}$ where as before $\mathscr{D}'_m$ is $\mathscr{D}^{\circ}_m$ with one point added to each set, which we may as well take at $\{s(m+1)\}$. On the other hand, $\mathrm{ind}_{\mathsf{H}} \circ K$ is given by $H_{m+1}(\mathsf{q}, \mathsf{Q}_\bullet) \otimes_{H_m(\mathsf{q},\mathsf{Q}_\bullet)} e_{D_{s,m}}\mathfrak{C}^{\vartheta}e_{\mathscr{D}^{\circ}_m}$. The map

$$H_{m+1}(\mathsf{q}, \mathsf{Q}_\bullet) \otimes_{H_m(\mathsf{q},\mathsf{Q}_\bullet)} e_{D_{s,m}}\mathfrak{C}^{\vartheta}e_{\mathscr{D}^{\circ}_m} \to e_{D_{s,m+1}}\mathfrak{C}^{\vartheta}e_{\mathscr{D}'_m}$$

is given by considering an element of $H_{m+1}(\mathsf{q}, \mathsf{Q}_\bullet)$ as a diagram on between the slices $D_{s,m+1}$, attaching this to a diagram in $e_{D_{s,m}}\mathfrak{C}^{\vartheta}e_{\mathscr{D}^{\circ}_m}$ leaving the terminal at $s(m+1)$ free, and then attaching a segment to the strand at $s(m+1)$ to extend to the top of the diagram. This map is obviously surjective. On the other hand, the functors both for classical and WF Hecke algebras preserve cell-filtered modules with the same multiplicities by Proposition 2.27. Thus we can check that dimensions agree, and the map must be an isomorphism. □

COROLLARY 3.6. *We have a canonical isomorphism respecting composition between the natural transformations* $\mathrm{Hom}(F, F')$ *and* $\mathrm{Hom}(F_{\mathsf{H}}, F'_{\mathsf{H}})$.

Proof. We have natural maps

$$A\colon \mathrm{Hom}(F, F') \to \mathrm{Hom}(K \circ F, K \circ F')$$
$$B\colon \mathrm{Hom}(F_{\mathsf{H}}, F'_{\mathsf{H}}) \to \mathrm{Hom}(F_{\mathsf{H}} \circ K, F'_{\mathsf{H}} \circ K).$$

It suffices to prove that both these maps are isomorphisms. We can modify the argument of [Sha11, 2.4] to show this: we know from Proposition 2.27 that induction and restriction preserve the categories of standard filtered modules, and by 0-faithfulness, the functor K is fully faithful on the subcategory of standard filtereds. Thus any element of the kernel of A must kill all standard filtered modules and be 0; on the other hand, the surjectivity follows from fullness, since any object has a representation by projectives, which are standard filtered.

The map B is injective because K is a quotient functor. On other hand, 0-faithfulness implies that any projective has a copresentation by modules induced from $H_m(\mathsf{q}, \mathsf{Q}_\bullet)$. Thus, the action of any natural transformation a projective is determined by its action on an induction. This shows the surjectivity of B. □

Note that this shows that any property of ind, res that can be phrased in terms of natural transformations can be transfered from the analogous properties of the Hecke algebra. In particular:

COROLLARY 3.7. *The functors* ind, res *are biadjoint and commute with duality. If* $e \neq 1$, *then they induce a categorical* $\mathfrak{g}_U$*-action in the sense of Chuang and Rouquier* [CR08].

LEMMA 3.8. *If* $q = -1$, *then* $H_2(T+1)$ *is in the image of the functor* $K : \mathfrak{C}_2^{\mathbf{s}}\text{-mod} \to H_2(\mathsf{q}, \mathsf{Q}_\bullet)\text{-mod}$.

Proof. We claim that if D is the set $\{s, s + {}^{\kappa}/_{2}\}$ for $s \gg 0$ then $e_{d,s}\mathfrak{C}_d^{\mathbf{s}} e_D$ is isomorphic to 2 copies of this module. The module $e_{d,s}\mathfrak{C}_d^{\mathbf{s}} e_D$ is generated by the two elements

(3.1)

Both of these elements are killed by $T+1$, and thus give maps from $H_2(T+1) \cong H_2/H_2(T+1) \to e_{d,s}\mathfrak{C}_d^{\mathbf{s}} e_D$. The dimension of this module is ℓ^2. On the other hand, the dimension of $e_{d,s}\mathfrak{C}_d^{\mathbf{s}} e_D$ is the number of pairs of tableaux of the same shape on ℓ-multipartitions of 2, one with filling $s, 2s$ and the other with filling $s, s + {}^{\kappa}/_{2}$.

Each of $\ell(\ell-1)/2$ different ℓ-multipartitions consisting of 2 different 1 box diagrams give 4 basis vectors, so together they contribute $2\ell(\ell-1)$ basis vectors. For a multipartition with a single 2-box diagram, we can only have a tableau with filling $s, s + {}^{\kappa}/_{2}$ on (2) if $\kappa < 0$ or (1, 1) if $\kappa > 0$. In either case, the ℓ ways of placing this in different components contribute 2 basis vectors each, since either filling with $s, s + {}^{\kappa}/_{2}$ gives a tableau, but only one filling with $s, 2s$ does. Thus, we have dimension $2\ell(\ell-1) + 2\ell = 2\ell^2$. This shows that the map from $H_2(T+1)^{\oplus 2}$ is an isomorphism.

Thus, either of the elements shown in (3.1) generate a summand of $e_{d,s}\mathfrak{C}_d^{\mathbf{s}} e_D$ whose image under K is $H_2(T+1)$. □

3.2. A comparison theorem. Now, we consider the case where $\Bbbk = \mathbb{C}$, and S is one of $\mathbb{C}$, $\mathscr{R} = \mathbb{C}[[h, z_1, \ldots, z_\ell]]$ or $R = \mathbb{C}((h, z_1, \ldots, z_\ell))$. As before, we have parameters $\kappa, s_1, \ldots, s_\ell \in \mathbb{C}$ for the rational Cherednik algebra, and we consider

$$\mathsf{k} = k + \frac{h}{2\pi i} \qquad \mathsf{s}_j = \left(ks_j - \frac{z_j}{2\pi i}\right) \Big/ \mathsf{k}$$

$$q = \exp(2\pi i k) \quad Q_i = \exp(2\pi i k s_i) \quad \mathsf{q} = qe^h \quad \mathsf{Q}_i = Q_i e^{-z_i}.$$

We let $\kappa = \mathrm{Re}(k)$ and $\vartheta_i = \mathrm{Re}(ks_i) - i/\ell$, and let $\mathfrak{C}_d^{\mathbf{s}} := \mathfrak{C}^{\vartheta}_{\mathscr{D}_d^\circ}$ denote the WF

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Hecke algebra over $\mathscr{R}$ defined above attached to the collection $\mathscr{D} = \{D_\xi\}$ for ξ all ℓ-multipartitions of d.

THEOREM 3.9. *We have an equivalence of categories* $\mathbb{O}^{\mathbf{s}}_d \cong \mathfrak{C}^{\mathbf{s}}_d$ -mod *intertwining the functor* KZ *with the quotient functor* $M \mapsto e_{D_{s,d}} M$.

Proof. Of course, we use Theorem 2.3. Let us confirm the conditions of this theorem:

(1) we have an isomorphism $\mathfrak{C}^{\mathbf{s}}_0 \cong \mathscr{R}$;

(2) the highest weight structure follows from Lemma 2.25;

(3) the desired induction functors are induced by the map of Lemma 2.18; extension of scalars always preserves projectives.

(4) The image $\operatorname{ind}(\mathcal{R}, H_q(\mathsf{q}, \mathsf{Q}_\bullet))$ is the projective $\mathfrak{C}^{\mathbf{s}}_d e_{s,d}$. Thus, the functor K is just $M \mapsto e_{D_{s,d}} M$. This is clearly a quotient functor, and becomes an equivalence after base change by Lemma 3.2.

(5) The desired duality is just $M^\star := \operatorname{Hom}(M, \mathscr{R})$, which is naturally a $(\mathfrak{C}^{\mathbf{s}}_d)^{\mathrm{op}}$-module. We use the anti-automorphism $*$ to make this a $\mathfrak{C}^{\mathbf{s}}_d$-module again. We have $eM^\star \cong (e^* M)^\star$, so the commutation of this duality with the analogous one on the Hecke algebra follows from the fact that $e^*_{s,d} = e_{s,d}$. The duality on the Hecke algebra corresponds to the anti-automorphism sending $T_i \mapsto T_i$ and $X_i \mapsto X_i$.

(6) In both cases, the order induced on simples is a coarsening of c-function ordering. These match as calculated in Proposition 2.9.

(7) Finally, we need that if $q = -1$, then $H_2(T + 1)$ is in the image. This is precisely Lemma 3.8.

This confirms all the hypotheses, and thus shows that we have an equivalence. □

Let $\bar{\mathfrak{C}}^{\mathbf{s}}_d := \mathbb{C} \otimes_{\mathscr{R}} \mathfrak{C}^{\mathbf{s}}_d$:

COROLLARY 3.10. *The category* $\mathcal{O}^{\mathbf{s}}_d$ *over* H *is equivalent to the category* $\bar{\mathfrak{C}}^{\mathbf{s}}_d$ -mod.

While this equivalence is somewhat abstract, at least it gives us a concrete description of the image of projectives under the KZ functor. This image is generated as an additive category by the $H_d(q, Q_\bullet)$-modules $e_{s,d} \bar{\mathfrak{C}}^{\mathbf{s}}_d e_\xi$ for different partitions ξ. This is an explicit cell-filtered module over $H_d(q, Q_\bullet)$, with a basis we can compute with, though of course, not without some effort.

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3.3. Cyclotomic q-Schur algebras. This comparison theorem can also be applied to cyclotomic q-Schur algebras. The cyclotomic q-Schur algebra $\mathscr{S}_d(\mathsf{q}, \mathsf{Q}^\bullet)$ over the ring $\mathscr{R}$ attached to the data $(\mathsf{q}, \mathsf{Q}^\bullet)$ was defined by Dipper *et al.* [**DJM98**, 6.1] (for the set Λ, we use all multicompositions with d parts). One can easily confirm that the category of representations of this algebra satisfies all the properties of $\mathcal{Q}^{\underline{\mathbf{s}}}_d$ in Theorem 2.3, except that the order does not necessarily have a common refinement with the ordering on the simples of the Cherednik algebra. Thus, Theorem 2.3 shows that

COROLLARY 3.11. *If the c-function order for $k, \underline{\mathbf{s}}$ on charged ℓ partitions refines the usual dominance order on ℓ-multipartitions of $d \leqslant D$, then we have an equivalence of highest weight categories $\mathcal{O}^{\underline{\mathbf{s}}}_d \cong \mathscr{S}_d(\mathsf{q}, \mathsf{Q}^\bullet)$ -mod $\cong \bar{\mathfrak{C}}^{\mathbb{S}}_d$ -mod for all $d \leqslant D$.*

This condition will necessarily hold whenever $D|\kappa| < \vartheta_{j+1} - \vartheta_j$ for all i, j, but there is no uniform choice of $\underline{\mathbf{s}}$ where we have this Morita equivalence for all D; eventually, the orders will start to differ. Note that in [**Webb**, 5.6], we showed the latter Morita equivalence directly when the inequality above holds.

3.4. Change-of-charge functors: Hecke case. In the algebra $\mathfrak{C}^\vartheta$, we have required that the red lines are vertical, that is, the quantities ϑ_i, as well as κ are fixed. However, a natural and important question is how these algebras compare if these quantities are changed. We can relate them using natural bimodules between such pairs of algebras.

Given different choices ϑ_i, κ and ϑ'_i, κ' of these parameters, we can define a bimodule over $\mathfrak{C}^\vartheta$ and $\mathfrak{C}^{\vartheta'}$ (we leave the use of κ and κ' in the two algebras implicit).

DEFINITION 3.12. We let a *WF ϑ - ϑ' diagram* be a diagram like a WF Hecke diagram with:

- ℓ red line segments which go from $(\vartheta'_i, 0)$ to $(\vartheta_i, 1)$;
- green strands, which as usual project diffeomorphically to $[0, 1]$ on the y-axis and can carry squares. Each strand has a ghost whose distance from the strand now varies with the value of y: it is $y\kappa + (1 - y)\kappa'$ units to the right of the strand.

These diagrams must satisfy the genericity conditions from before, though these must be interpreted carefully: if two red strands cross, or a strand crosses its own ghost, this is not a 'true crossing' and it can be ignored for purposes of genericity. In particular, we can isotope another strand through it without issues.

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Let $\mathcal{K}^{\vartheta,\vartheta'}$ be the $\Bbbk$-span of the WF ϑ - ϑ' diagrams modulo the relations (2.1a)–(2.1i) and the steadying relation that a diagram is 0 if the strands can be divided into two groups with a gap $>|\kappa|$ between them and all red strands in the right hand group, as before.

PROPOSITION 3.13. *The space $\mathcal{K}^{\vartheta,\vartheta'}$ is naturally a $\mathfrak{C}^{\vartheta}$ - $\mathfrak{C}^{\vartheta'}$-bimodule.*

Proof. We wish to stack a diagram a from $\mathfrak{C}^{\vartheta}$ on top of one b from $\mathcal{K}^{\vartheta,\vartheta'}$. This will not literally be the case, since we require a diagram from $\mathcal{K}^{\vartheta,\vartheta'}$ to have its red lines to be straight, and the composition will have a kink where the diagrams join, and similarly a kink in each ghost at this point. However, we can apply a combination of isotopies and the relations to get rid of this kink. There is some ϵ such that replacing the red strands in a by ones going from $(\epsilon\vartheta_i' + (1-\epsilon)\vartheta_i, 0)$ to $(\vartheta_i, 1)$, and placing the ghosts $\kappa + (1-y)\epsilon(\kappa' - \kappa)$ units right of each strand results in an isotopic diagram. We can further choose this ϵ so that in the diagram b, replacing the red strands by ones going from $(\vartheta_i', 0)$ to $(\epsilon\vartheta_i' + (1-\epsilon)\vartheta_i, 1)$ and placing the ghosts $\kappa' + y(1-\epsilon)(\kappa - \kappa')$ units right of each strand results in an isotopic diagram as well. Now, we can stack these diagrams, with a scaled to fit between $y = 1-\epsilon$ and $y = 1$, and b to fit between $y = 0$ and $y = 1-\epsilon$. □

In this bimodule, we can construct analogues of the elements $\mathcal{C}_{\mathsf{S},\mathsf{T}}$, which we also denote $\mathcal{C}_{\mathsf{S},\mathsf{T}}$ by abuse of notation (the original elements $\mathcal{C}_{\mathsf{S},\mathsf{T}}$ will be a special case of these where $\vartheta = \vartheta'$). Unlike the algebra $\mathfrak{C}^{\vartheta}$, the construction of these requires breaking the symmetry between top and bottom of the diagram. Thus, we can make one choice to obtain a cellular basis of $\mathcal{K}^{\vartheta,\vartheta'}$ as a left module and another to obtain a cellular basis as a right module.

Let us first describe the basis which is cellular for the right module structure. Let $\mathcal{D}_{\mathsf{S}}$ be the element of the bimodule $\mathcal{K}^{\vartheta,\vartheta'}$ defined analogously with $\mathcal{B}_{\mathsf{S}}$. Its bottom is given by the set $\mathcal{D}_{\eta}$ (for the weighting ϑ'). Its top is given by the entries of S, with each entry giving the x-coordinate of a strand. The diagram proceeds by connecting the points in the loading associated to the same box in the top and bottom, while introducing the smallest number of crossings. As usual, this diagram is not unique; we choose any such diagram and fix it from now on.

DEFINITION 3.14. The right cellular basis for $e_{\mathbf{i}}\mathcal{K}^{\vartheta,\vartheta'}e_{\mathbf{j}}$ is given by $\mathcal{D}_{\mathsf{S}}\mathcal{B}_{\mathsf{T}}^*$ for S a $\mathbf{i}$-tableau for some loading $\mathbf{i}$ and the weighting ϑ (upon which the definition of $\mathbf{i}$-tableau depends), and T a $\mathbf{j}$-tableau for some loading $\mathbf{j}$ and the weighting ϑ'.

The left cellular basis for $e_{\mathbf{j}}\mathcal{K}^{\vartheta',\vartheta}e_{\mathbf{i}}$ is given by the reflections of these vectors, that is by $\mathcal{B}_{\mathsf{T}}\mathcal{D}_{\mathsf{S}}^*$.

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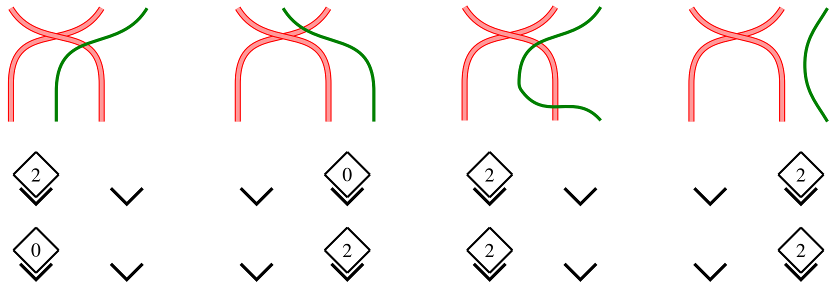
EXAMPLE 3.15. Let us illustrate with a small example. Consider $\mathfrak{C}^\vartheta$ with two red lines, both labeled with 1, and a single green line. Let $\vartheta = (1, -1)$ and $\vartheta' = (-1, 1)$. Thus, in each diagram, we have a red cross. A loading is determined by the position of its single dot. Let e_0 be the loading where it is at $y = 0$ and e_2 that where it is at $y = 2$. Each basis vector is attached to a pair of Young diagrams with one box total, so one is a single box and the other empty. A tableau on such a diagram is a single number, which is greater than the associated value of ϑ or ϑ'.

Thus, if the box is in the first component, its filling in S must be >1 and in T must be >-1; if the box is in the second component, the filling in S must be >-1 and in T must be >1. Thus, $e_0\mathscr{B}^{\vartheta',\vartheta}e_0$ is the 0 space, since 0 cannot give a tableau for both ϑ and ϑ' for either diagram. On the other hand, $e_2\mathscr{B}^{\vartheta',\vartheta}e_0$ and $e_0\mathscr{B}^{\vartheta',\vartheta}e_2$ are both 1-dimensional, with the only basis vector associated to $((1), \emptyset)$ in the first case, and to $(\emptyset, (1))$ in the second. Both these diagrams have a tableau with filling with all 2's, so $e_2\mathscr{B}^{\vartheta',\vartheta}e_2$ is 2-dimensional. For the right basis, these vectors are given by:



Note that we have drawn these in a way that the factorization into two diagrams is clear, but according the definition, we should really perform isotopies of these so that the red lines are straight.

LEMMA 3.16. *The vectors $\mathcal{D}_\mathsf{S}\mathcal{B}^*_\mathsf{T}$ are a basis for the bimodule $\mathcal{K}^{\vartheta,\vartheta'}$. Furthermore, the sum of vectors attached to partitions $\leqslant\xi$ in ϑ'-weighted order is a right submodule. In particular, as a right module, $\mathcal{K}^{\vartheta,\vartheta'}$ is standard filtered.*

Similarly, the left cellular basis shows that the bimodule $\mathcal{K}^{\vartheta,\vartheta'}$ is standard filtered as a left module.

Proof of Lemma 3.16. First, we wish to show that these elements span. By the Morita equivalence of Lemma 2.16, the bimodule $\mathcal{K}^{\vartheta,\vartheta'}$ is spanned by elements of the form $ae_\xi b$ where $a \in \mathcal{K}^{\vartheta,\vartheta'}$, ξ a multipartition and $b \in \mathfrak{C}^{\vartheta'}$. We prove by

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induction that $ae_\xi b$ lies in the span of the vectors $\mathcal{D}_\mathsf{S}\mathcal{B}^*_\mathsf{T}$ for S, T of shape $\geqslant \xi$ in ϑ'-weighted dominance order.

Without loss of generality, we can assume that b is one of the vectors of our cellular basis of Theorem 2.24. If the associated cell is not ξ, then b factors through e_ν for $\nu > \xi$, and the result follows by induction. If it is ξ, then we must have $b = \mathcal{B}^*_\mathsf{T}$ for some T.

We can also assume that a is a single diagram, with no bigons between pairs of strands or strands and ghosts. The slice at $y = 0$ of b is precisely D_ξ, and we can use this identification to match the strands with boxes of the diagram of this multipartition. Now, we can apply the argument of Lemma 2.22 to a: we can fill the diagram of ξ by the x-value at $y = 1$ of the strand corresponding to that box at $y = 0$. Let D be the set given by the slice at $y = 1$. If this filling is not a D-tableau for the weighting ϑ, then the corresponding diagram must have a 'bad crossing' in the same sense of the proof of Lemma 2.22, which we can slide to the bottom of the diagram, showing it factors through e_ν for $\nu > \xi$ in ϑ'-dominance order. Thus, we can assume that this filling is a D-tableau. As usual, any two diagrams for the same tableau differ by diagrams with fewer crossings, so by induction, choosing one diagram for each tableau suffices to span.

Thus, we need only show that these are linearly independent. As before, we can reduce to the case where $D = D' = D_{s,m}$ for $s \gg 0$ by Lemma 3.3; in this case, the bimodule $e_D\mathcal{K}^{\vartheta,\vartheta'}e_D$ is precisely the same as $e_D\mathfrak{C}^\vartheta e_D$. We can identify this space with the image of the corresponding idempotents acting on the cyclotomic Hecke algebra $\mathfrak{C}^\lambda$, so it has the correct dimension by Lemma 2.22. □

As with any cellularly filtered module, we can study the multiplicities of cell modules S_ξ for $\mathfrak{C}^\vartheta$ in $\mathcal{K}^{\vartheta,\vartheta'}e_{D'}$.

Corollary 3.17. *We have an equality of multiplicities*

$$[\mathcal{K}^{\vartheta,\vartheta'}e_D : S_\xi] = [\mathfrak{C}^{\vartheta'}e_D : S'_\xi].$$

We prove later (Lemma 5.12) that derived tensor product with this bimodule is an equivalence of derived categories, and in fact, that these can be organized into an action of the affine braid group.

One way to think about the significance of a weighting is that it induces a total order on the columns of the diagram of ξ (remember, we are always using Russian notation; in the usual notation for partitions, these would be diagonals). Let $>_\vartheta$ be this order.

DEFINITION 3.18. For total orders on a finite set, we say $>'$ is *between* $>$ and $>''$ if there is no pair of elements a, b such that $a > b$, $a >'' b$ and $b >' a$.

We say that a weighting ϑ' is *between* ϑ and ϑ'' if for any multipartition ξ, the induced order $>_{\vartheta'}$ on columns of ξ is between $>_\vartheta$ and $>_{\vartheta''}$.

LEMMA 3.19. *If ϑ' is between ϑ and ϑ'', then we have that*

$$\mathcal{K}^{\vartheta,\vartheta'} \overset{L}{\otimes} \mathcal{K}^{\vartheta',\vartheta''} \cong \mathcal{K}^{\vartheta,\vartheta''}.$$

Proof. There is an obvious map $\mathcal{K}^{\vartheta,\vartheta'} \overset{L}{\otimes} \mathcal{K}^{\vartheta',\vartheta''} \to \mathcal{K}^{\vartheta,\vartheta''}$ given by stacking the diagrams. First, we need to show that this map is surjective if ϑ' is between ϑ to ϑ''. This follows since after applying an isotopy, any diagram in $\mathcal{K}^{\vartheta,\vartheta''}$ can have its red strands meet with ϑ' at $y = 1/2$. Thus, slicing this diagram in half, we obtain diagrams from $\mathcal{K}^{\vartheta,\vartheta'}$ and $\mathcal{K}^{\vartheta',\vartheta''}$ which hit this one under the stacking map.

Note that

$$\dot{S}_\xi \overset{L}{\otimes} S_{\xi'} = \begin{cases} \Bbbk & \xi = \xi', \\ 0 & \xi \neq \xi'. \end{cases}$$

Furthermore, the multiplicity of $\dot{S}_\xi$ in $\mathcal{K}^{\vartheta,\vartheta'}$ as a right module is the number of D-tableau for ϑ of shape ξ and the multiplicity of S_ξ as a left module is the number of D'-tableau for ϑ'' of shape ξ. Thus, the dimension of $e_D\mathcal{K}^{\vartheta,\vartheta'} \overset{L}{\otimes} \mathcal{K}^{\vartheta',\vartheta''}e_{D'}$ is exactly the number of pairs of these with the same shape, which is the dimension of $e_D\mathcal{K}^{\vartheta,\vartheta''}e_{D'}$. Since a surjective map between finite-dimensional vector spaces of the same dimension is an isomorphism, we are done. □

4. Gradings and weighted KLR algebras

One great advantage of having a concrete presentation of the category $\mathcal{O}$ for a Cherednik algebra is that it allows us to think in a straightforward way about graded lifts of this category: they simply correspond to gradings on this algebra. The presentation we gave before is not homogeneous for an obvious grading, but we can give a different presentation which is, in the spirit of Brundan and Kleshchev's approach to gradings on Hecke algebras [BK09].

4.1. Weighted KLR algebras. As before, we choose $(r_1, \dots, r_\ell) \in (\mathbb{C}/\mathbb{Z})^\ell$ and a scalar $k \in \mathbb{C}$. Given this data, we have a graph $U \subset \mathbb{C}/\mathbb{Z}$ and associated Lie algebra $\mathfrak{g}_U$, as defined in Section 2.2. We have an associated highest weight

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$\lambda = \sum_i \omega_{r_i}$ of $\mathfrak{g}_U$ of level ℓ. Attached to this choice, we have a *Crawley-Boevey quiver* U_λ, as defined in [Webc, 3.1]. This adds a single vertex, which in this paper we index by ∞, and $\lambda^u = \alpha_u^\vee(\lambda)$ new edges from u to ∞. We let Ω_λ be the edge set of this new graph. We often refer to the edges of the original cycle as *old* and those we have added to the Crawley-Boevey vertex as *new*.

The data ϑ_i and $\kappa = \mathrm{Re}(k)$ gives a weighting on the Crawley-Boevey graph, that is, a function $\vartheta \colon \Omega_\lambda \to \mathbb{R}$, such that every edge of U has weight κ and ϑ_i giving the weights of the new edges.

As before, we let $\Bbbk$ be a field and we now assume that S is a $\Bbbk$-algebra with a choice of elements $h, z_1, \dots, z_\ell \in S$. The most interesting choices for us will be $\Bbbk$ itself with $h = z_1 = \cdots = z_\ell = 0$ or $\mathscr{R}$. For each edge, we set the polynomials $Q_e(u, v) = u - v + h \in \Bbbk[u, v]$ for old edges and $Q_{e_i}(u, v) = v - u - z_i \in \Bbbk[u, v]$ for new edges, and consider the *weighted KLR algebra* W^ϑ of the Crawley-Boevey quiver as defined in [Webc, Section 3.1]. As in that paper, we only consider dimension vectors with $d_\infty = 1$.

Let us briefly recall the definition of this algebra.

DEFINITION 4.1. We let a *weighted KLR diagram* be a collection of curves in $\mathbb{R} \times [0, 1]$ with each curve mapping diffeomorphically to $[0, 1]$ via the projection to the y-axis. Each curve is allowed to carry any number of dots, and has a label that lies in U. We draw:

- a dashed line κ units to the right of each strand, which we call a *ghost*;
- red lines at $x = \vartheta_i$ each of which carries a label ω_{r_j}.

We now require that there are no triple points or tangencies involving any combination of strands, ghosts or red lines and no dots lie on crossings. We consider these diagrams equivalent if they are related by an isotopy that avoids these tangencies, double points and dots on crossings.

Note that this is a bit different from the description in [Webc]; we have specialized to the case of Crawley-Boevey quiver with one vertical strand at $x = 0$ labeled with the vertex ∞. The red lines are the ghosts of this single vertical stand with label ∞.

This definition is quite similar to the conditions we considered in Section 2.3; the only difference is that we use black in place of green, label each of these strands with an element of U and denote the polynomial generators with a dot instead of a square (and do not allow negative powers of them).

For example, consider the case where $k = 3/4$ and $r_1 = r_2 = 0$, $r_3 = 3/4$, $r_4 = 1/2$ and $\vartheta_1 = 4$, $\vartheta_2 = 1$, $\vartheta_3 = 6$, $\vartheta_4 = -4$. Thus, the diagram with no black strands

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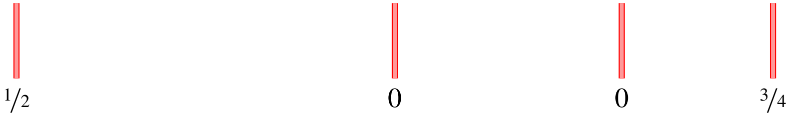
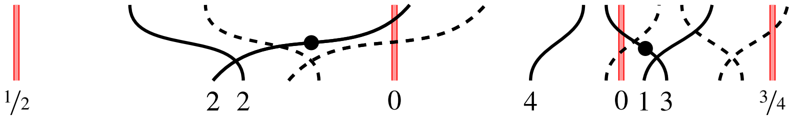
for this choice of weighting looks like:



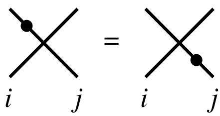
Adding in black strands will result in a diagram which looks (for example) like:



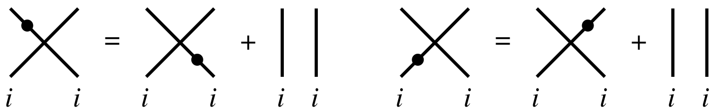
In W^ϑ, we have idempotents $e_{\mathbf{i}}$ indexed not just by sequences of nodes in the Dynkin diagram, but by combinatorial objects we call *loadings*, discussed earlier. A loading is a function from the real line to $U \cup \{\emptyset\}$ which is $\emptyset$ at all but finitely many points. Diagrammatically, we think of this as encoding the positions of the black strands on a horizontal line. Thus, a loading will arise from a generic horizontal slice of a weighted KLR diagram, and the idempotent corresponding to a loading has exactly that slice at every value of y.

Of course, there are infinitely many such loadings. Typically, we only consider these loadings up to *equivalence*, as defined in [Webc, 2.9]. There only finitely many equivalence classes, so the resulting algebra is more tractable.

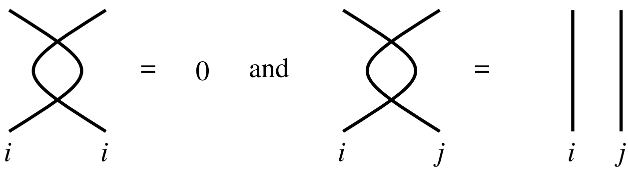
DEFINITION 4.2. The weighted KLR algebra $\tilde{T}^\vartheta$ is the quotient of the span of weighted KLR diagrams by the local relations:



for $i \neq j$ (4.1a)



(4.1b)



(4.1c)

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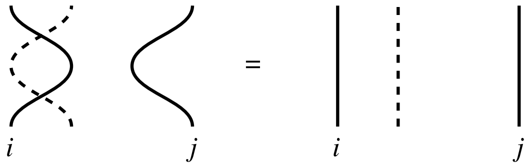
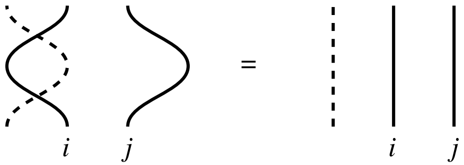
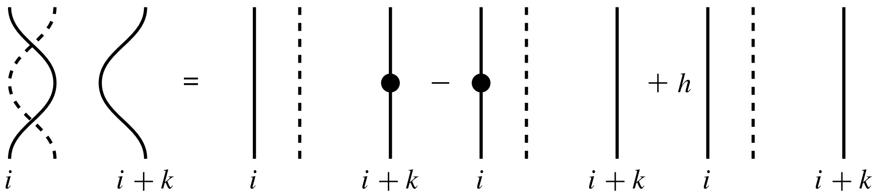


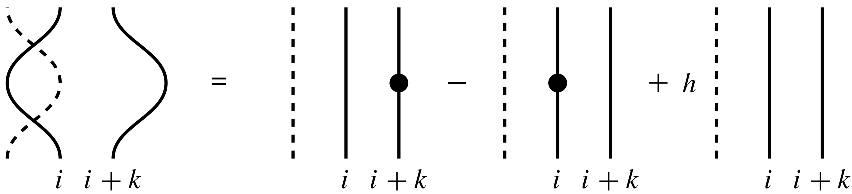
for $i + k \neq j$ (4.1d)



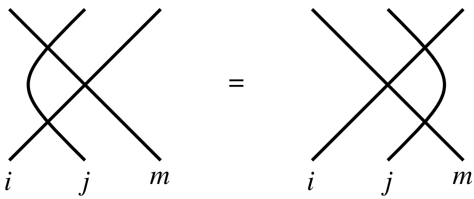
for $i + k \neq j$ (4.1e)



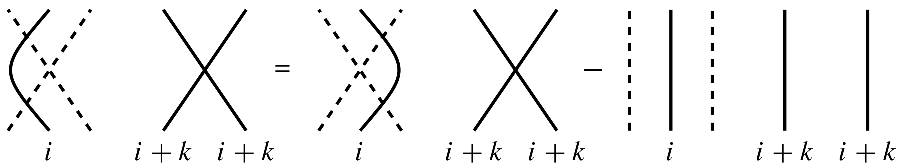
(4.1f)



(4.1g)



(4.1h)



(4.1i)

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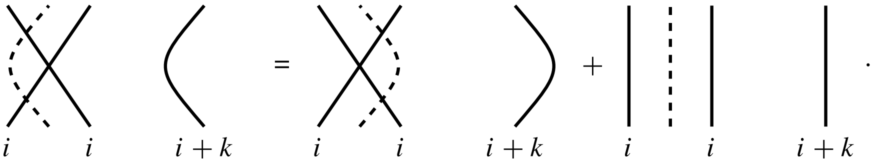
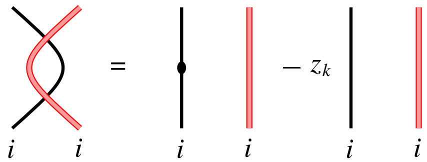
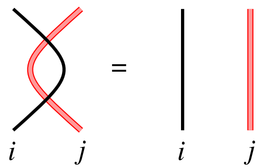
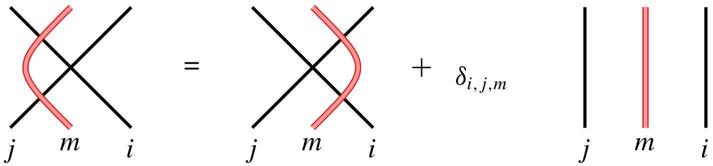


(4.1j)

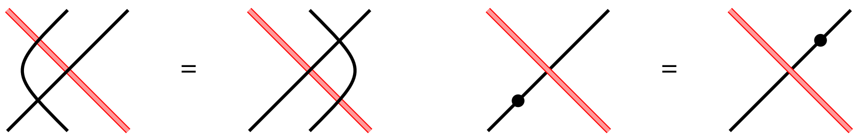


(4.1k)





(4.1l)



(4.1m)

For the relations (4.1m), we also include their mirror images.

Some care must be used when understanding what it means to apply these relations locally. In each case, the LHS and RHS have a dominant term which are related to each other via an isotopy through a disallowed diagram with a tangency, triple point or a dot on a crossing. You can only apply the relations if this isotopy avoids tangencies, triple points and dots on crossings everywhere else in the diagram; one can always choose isotopy representatives sufficiently generic for this to hold.

This algebra is graded if S is graded with h, z_i having degree 2. This is satisfied if $S = \Bbbk$ or $S = \Bbbk[h, z_1, \dots, z_\ell]$.

- As usual, the dot has degree 2.

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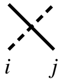
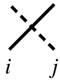
- The crossing of two strands has degree 0, unless they have the same label, in which case it is −2.
- The crossing of a strand with label i from right of a ghost to left of it has degree 1 if the ghost has label $i+k$ and degree 0 otherwise.
- Such a crossing from left to right has degree 1 if the ghost has label $i+k$ and degree 0 otherwise.
- A crossing of red and black strands with the same label has degree 1, and one with different labels has degree 0.

That is,

deg  $= -2\delta_{i,j}$ deg  $= \delta_{j,i-k}$ deg  $= \delta_{j,i+k}$

deg  $= 2$ deg  $= \delta_{j,i}$ deg  $= \delta_{j,i}$

This algebra has a *reduced steadied quotient*, which we denote T^{ϑ}. This is obtained by:

- Killing all idempotents where the strands can be broken into two groups separated by a blank space of size $>|\kappa|$ (so no ghost from the right group can be left of a strand in the left group and *vice versa*) and all red strands in the right group; we call such idempotents *unsteady*.
- Killing all dots on the strand with label ∞.

We just remind the reader that we allow the case where $k \in \mathbb{Z}$ (so $e = 1$). In this case, the graph U is just elements of $\mathbb{C}/\mathbb{Z}$ equal to one of the r_i, connected to itself by a loop and the equations $i = j$, $i = j + k$, $i = j - k$ are all equivalent.

REMARK 4.3. We should note that unlike in the tensor product algebras for $\widehat{\mathfrak{sl}}_e$ in [**Weba**, Section 3], a black line being left of a red is not enough to conclude the diagram is 0; it must be far enough left to avoid all entanglements with ghosts. See Example 4.4 below.

We can associate the elements of U to roots of $\mathfrak{g}_U$. As in [**Weba**], we let T^{ϑ}_{ν} for ν a weight of $\widehat{\mathfrak{gl}}_e$ be the subalgebra where the sum of the weights λ_i minus the sum of the roots labeling the black strands is ν. For $e \neq 1$, it is sufficient to consider the $\mathfrak{g}_U$-weight, but for $e = 1$, it is not quite clear what this means. The

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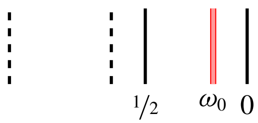
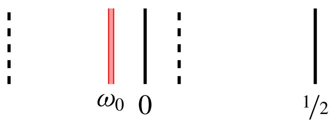
algebra $\widehat{\mathfrak{gl}}_1$ has a 'Cartan algebra' which is 2-dimensional with basis c, ∂; we let ω, α be the dual basis. The weights of the highest weight Fock representation are $\omega, \omega - \alpha, \omega - 2\alpha, \dots$.

EXAMPLE 4.4. Let $k = -1/2$, $Q_1 = 0$. Rather than list idempotents up to equivalence, which is still a bit redundant, let us implicitly identify idempotents easily found to be isomorphic using the relations above. If we have one black strand, then we can see that we obtain the trivial algebra if it is labeled $1/2$, and a 1-dimensional algebra if it is labeled 0 (in both cases, this is just the corresponding cyclotomic quotient). Similarly, if we have two black strands with the same label we get the trivial algebra again.

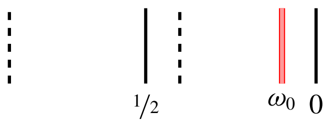
On the other hand, for one strand labeled 0 and one labeled $1/2$, the picture is more interesting. We get 2 interesting idempotents, which can be represented visually by

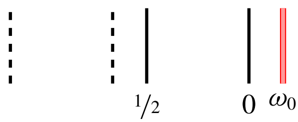
$e_1 =$ 

$e_2 =$ 

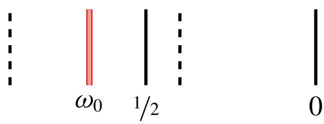
One can easily calculate that $e_1 T^{\vartheta} e_1 \cong \Bbbk$ and that $e_2 T^{\vartheta} e_2 \cong \Bbbk[y_2]/(y_2^2)$ where y_2 represents the dot on the rightward strand.

Note that e_1 is not unsteady (and in fact is nonzero in the steadied quotient), even though it contains a black strand left of a red one, since that strand is 'protected' by a ghost. The idempotents



and


are unsteady, and thus sent to 0. Note that the idempotent



is not unsteady, but it is isomorphic to the left-hand unsteady idempotent above, by relation (4.1k).

Savvy representation theorists will have already guessed that we have arrived at the familiar highest weight category with these endomorphism rings for its

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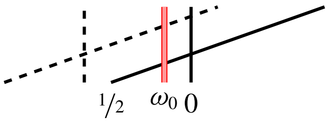
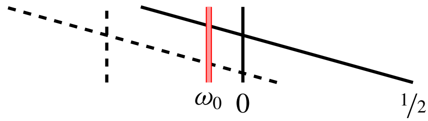
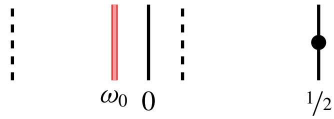
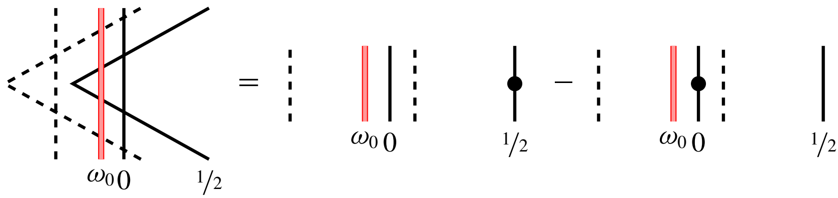
projectives; for example, this is given by a regular integral block of category $\mathcal{O}$ for $\mathfrak{sl}_2$. A basis of this ring is given by e_1, e_2 as above and

$a =$ 

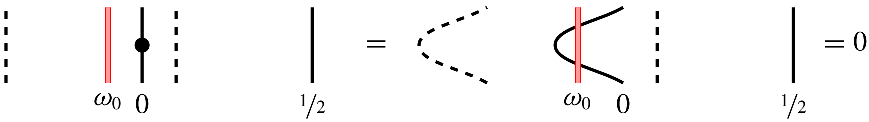
$b =$ 

$x =$ 

The product $x = ab$ follows from the relation (4.1f) since

$ab =$ 

The latter term is 0, by the calculation



since the second diagram factors through an unsteady idempotent. One can similarly calculate that $ba = ax = xa = bx = xb = 0$.

If $\kappa = 0$, then we recover the tensor product algebra $T^{\underline{\lambda}}$ described in [Weba, Section 3] for the Lie algebra $\mathfrak{g}_U$. We view moving to $\kappa \neq 0$ as passing from $\widehat{\mathfrak{sl}}_e$ to $\widehat{\mathfrak{gl}}_e$ in a way that we shall make more precise. This idea has appeared in several places, for example, the work of Frenkel and Savage on quiver varieties [FS03].

We are generally interested in the category T^{ϑ} -mod of *graded* T^{ϑ}-modules. When we consider the category of modules without a grading (for a graded or ungraded algebra), we use the symbol T^{ϑ} -$\underline{\text{mod}}$.

Assuming that $e \neq 1$, the category T^{ϑ} -mod carries a categorical action of $\mathfrak{g}_U$, via functors $\mathcal{F}_i$ and $\mathcal{E}_i$, which basically correspond to the addition and removal of

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a black line with label $i \in U$, defined in [**Webc**, Section 3.1]. We can view these are the induction and restriction functors for the map of rings $T^{\vartheta}_{\nu} \to T^{\vartheta}_{\nu-\alpha_i}$ which adds a black strand with label i at least κ units right of any other strand.

If $e = 1$, we still have functors $\mathcal{F}_i$ and $\mathcal{E}_i$ corresponding to the different points in $i \in U$ given by induction and restriction functors for the same inclusion $T^{\vartheta}_{\nu} \to T^{\vartheta}_{\nu-\alpha_i}$. The functors $\mathcal{E}_i$ and $\mathcal{F}_i$ have a structure reminiscent of, but not identical with a categorical Heisenberg action in the sense of Cautis and Licata [**CL12**]. In particular, they do categorify a level ℓ-Fock space representation of $U_q(\mathfrak{g}_U)$, as we prove later.

There is a symmetry of this picture:

PROPOSITION 4.5. *The map on a weighted KLR diagrams which keeps all red and black strands in the same place, reindexes their labels sending* $\omega_i \mapsto \omega_{-i}$, $\alpha_i \mapsto \alpha_{-i}$ *and sends* $\kappa \mapsto -\kappa$ *is an isomorphism.*

In terms of Uglov weightings, this sends $\vartheta^{\pm}_{\underline{\mathbf{s}}} \mapsto \vartheta^{\mp}_{\underline{\mathbf{s}}^{\star}}$, where $\underline{\mathbf{s}}^{\star} = (-s_{\ell}, \dots, -s_1)$.

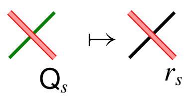
4.2. An algebra isomorphism. We use the same parameters as in Section 3.2. Let $\mathscr{D}$ be some collection of sets, and let B be the collection of all loadings on the graph U where the underlying set is in $\mathscr{D}$.

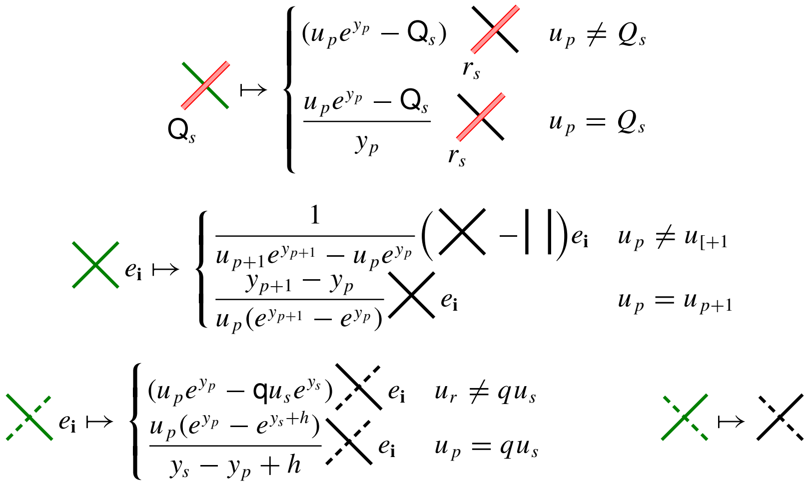
If $\mathfrak{C}^{\vartheta}_{\mathscr{D}}$ is the WF Hecke algebra as defined earlier over $\mathscr{R}$, then the spectrum of the action of a square lies in this set by Lemma 2.29. Now, consider a U-valued loading on a set $D \in \mathscr{D}$, that is, a function $\mathbf{i}\colon D \to U$; we use $u_1, \dots, u_m$ be the list of values of this function in increasing order. By abstract Jordan decomposition, there is an idempotent $\epsilon_{\mathbf{i}}$ which projects to the $\mathbf{i}(d)$ generalized eigenspace of X_d for $d \in D$. We let $\mathbb{T}^{\vartheta}_B(\mathscr{R})$ denote the deformed steadied weighted KLR algebra attached to the elements $r_i = ks_i \in U$ and the set of loadings B, base changed by the natural map $\Bbbk[h, z_1, \dots, z_{\ell}] \to \mathscr{R}$.

We now define an algebra isomorphism between the WF Hecke algebras and steadied weighted KLR algebras. This isomorphism will be local in nature: on each diagram, it operates by replacing every crossing of strands or ghosts and every square with a linear combination of diagrams in the weighted KLR algebra.

THEOREM 4.6. *We have an isomorphism of* $\mathscr{R}$*-algebras* $\mathfrak{C}^{\vartheta}_{\mathscr{D}} \cong \mathbb{T}^{\vartheta}_B$ *sending*

$$\epsilon_{\mathbf{i}} \mapsto e_{\mathbf{i}} \qquad X_p \mapsto \sum_{\mathbf{i}} u_p e^{y_p} e_{\mathbf{i}}$$





where the solid strand shown is the pth (and $p+1$st in the first line), and the ghost is associated to the sth from the left, or the red line is sth from the left.

COROLLARY 4.7. *The algebra $\mathbb{T}^{\vartheta}$ is flat over $\mathscr{R}$.*

We can combine this theorem with Theorem 3.9 to compare category $\mathcal{O}$ over a Cherednik algebra to weighted KLR algebras. Let B_d° be the set of all loadings on sets in $\mathscr{D}_d^{\circ}$.

THEOREM 4.8. *There is an equivalence of highest weight categories $T_{B_d^{\circ}}^{\vartheta_{\mathbf{s}}^{\sigma}}$ -$\underline{\text{mod}} \cong \mathcal{O}_d^{\mathbf{s}}$, where σ is the sign of κ.*

Thus, considering the category $T_{B_d^{\circ}}^{\vartheta_{\mathbf{s}}^{\sigma}}$ -mod, we obtain a graded lift of the category $\mathcal{O}_d^{\mathbf{s}}$ as a highest weight category.

4.3. Basis vectors. Now, we use the combinatorics described above to give a cellular basis of $\mathbb{T}_{\nu}^{\vartheta}$ and T_{ν}^{ϑ}, generalizing those of [**HM10**, **SW**]. This basis, and the variants of it we construct are the key to understanding the structure of these quotients and their representation theory.

For each $\mathbf{i}$-tableau S of fixed shape ν, we draw a diagram $B_{\mathsf{S}} \in e_{\nu} T^{\vartheta} e_{\mathbf{i}}$ which has no dots, and connects the point connected to a box in $\mathbf{i}_{\eta}$ at $y = 1$ to the point on the real line which labels it in S at $y = 0$; put another way, the strands are in bijection with boxes, with each strand ending just right of the x-coordinate of the

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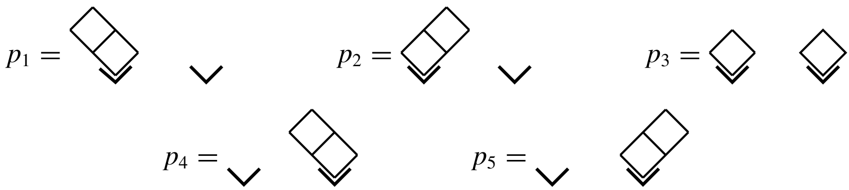
box, and starting at the real number labeling the box. Note the similarity to the definition of $\mathcal{C}_{\mathsf{S},\mathsf{T}}$ given in Section 2.6.

The permutation w_{S} traced out by the strands when read from the top is the unique one which puts the Russian reading word of the tableau into order. As usual, letting $(-)^*$ be the anti-automorphism which flips diagrams, let $C_{\mathsf{S},\mathsf{T}} = B^*_{\mathsf{S}} B_{\mathsf{T}}$. These vectors will be shown to be a cellular basis. This will perhaps be clarified a little by an example:

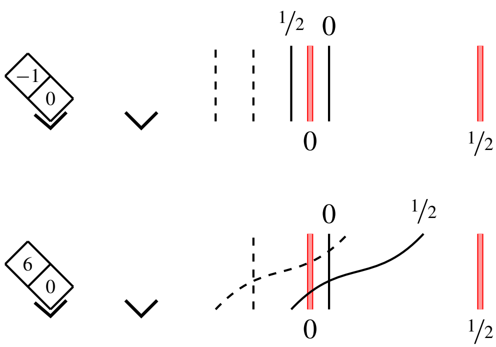
EXAMPLE 4.9. Now, we consider the example where $k = {-9}/2$, with $Q_0 = 0$ and $Q_1 = 1/2$, so $U = \{0, 1/2\}$. Consider the algebra attached to $\mu = \omega_0 + \omega_{1/2} - \delta$. We label the new edges so that e_i connects to the node i. The only resulting category, weighted order, and basis only depend on the difference of the weights $\vartheta_1 - \vartheta_2$. In fact, there are only 3 different possibilities; the category changes when this value passes $\pm 9/2$.

There are 5 multipartitions of the right residue:



The basis vectors we draw will look exactly like those of Example 2.19, except that now we draw black lines instead of green and must label with the black strands with simple roots. Since the pictures are so similar, let us specialize to the case with $\vartheta_1 = 0$, $\vartheta_2 = 9$. In this case, our order is $p_1 > p_2 > p_3 > p_4 > p_5$.

A loading in this case is given by specifying the position the point a labeled 0 and the point b labeled $1/2$. We denote this loading $\mathbf{i}_{a,b}$. As we see later, every projective is a summand of that for one of $\mathbf{i}_{(1,-1)}, \mathbf{i}_{(1,6)}, \mathbf{i}_{(1,10)}, \mathbf{i}_{(8,10)}, \mathbf{i}_{(15,10)}$. For these loadings, the tableaux with their corresponding B_{S}'s are:



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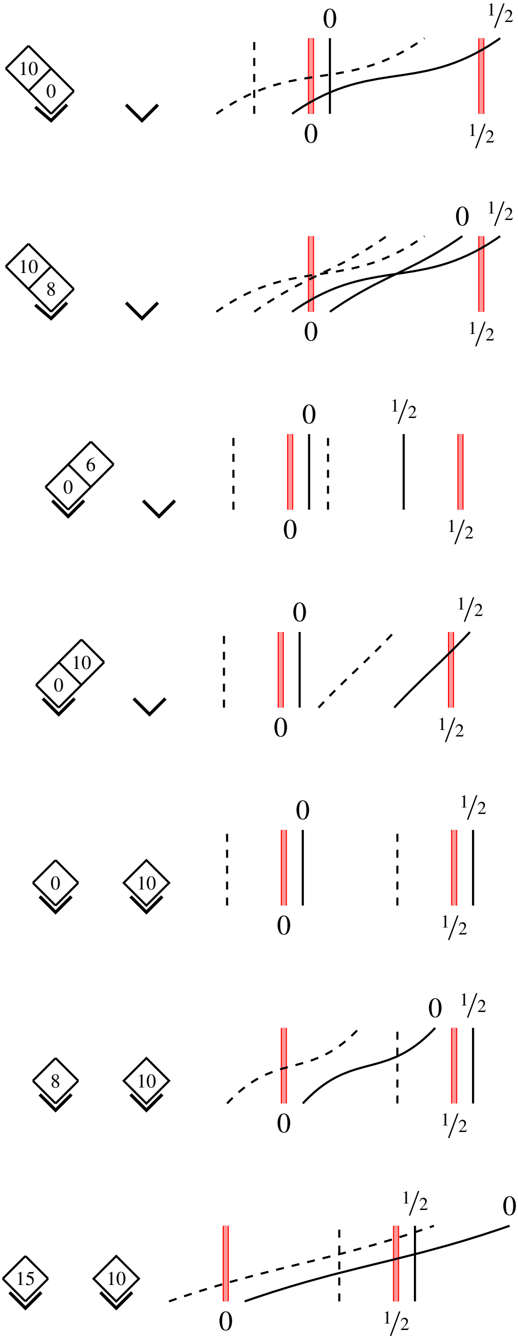
10
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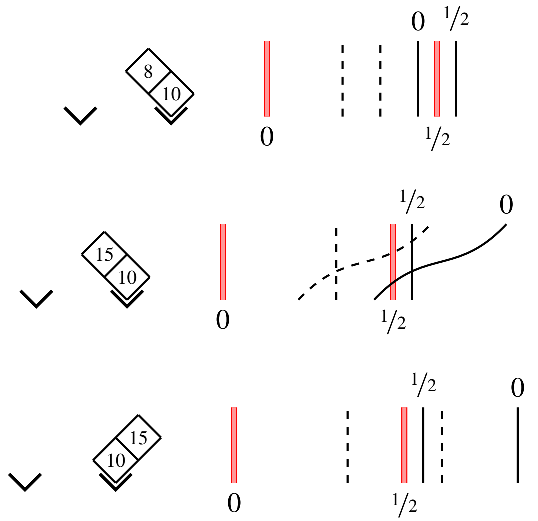

PROPOSITION 4.10. *The elements* $C_{\mathsf{S},\mathsf{T}}$ *are homogeneous of degree* $\deg(\mathsf{S}) + \deg(\mathsf{T})$.

Proof. These elements are defined as a product of homogeneous elements, and thus obviously homogeneous. In order to determine the degree, we must count:

- Crossings of like-labeled black strands with degree -2: these correspond to pairs of boxes with the same residue which are not in the same column, such that the rightward one is filled with a smaller number than the leftward.
- Crossings of like-labeled red and black strands with degree 1: these correspond to pairs of boxes and nadirs of tableaux where the box is to the left of the nadir, but is filled with a higher number than the nadir's x-coordinate.
- Between strands and ghosts of adjacent strands with degree 1: these correspond to pairs of boxes with adjacent residue more than κ units apart, such that the rightward one is filled with a smaller number than the leftward.

We organize counting these by the leftward box, whose residue we call i; if the entry there is h, we look at all boxes to the right of this one with the same or adjacent residue. These naturally form into strips around each vertical line of residue i. This is not quite true when $e = 1, 2$, but our argument goes through there as well, simply noting that we double count every strip of residue $i \pm k$.

In each such strip, there are 3 possibilities: relative to h either there is an addable box of residue i, a removable box of residue i or neither. Assume for

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now that this strip does not lie above a nadir of residue i. Then, if there is no removable or addable box, the number of boxes with label $<h$ of residue i is one less than those of residue $i-k$ and one more than those residue $i+k$, or *vice versa*. Thus, the degree contributions of the boxes of residue i and those of residue $i \pm k$ exactly cancel, and there is no total contribution to the degree.

If there is an addable box of residue i, then there is one more box of adjacent residue than in the first case, and there is a total contribution of 1 to the degree; if there is a removable box of residue i, then there is one fewer box of adjacent residue than in the first case, and there is a total contribution of -1 to the degree.

Finally, if the strip we consider lies above a nadir of residue i, then we have one fewer adjacent box than expect, and so the contribution to the degree is increased by 1, as we expected from the red and black crossing. This completes the proof. □

4.4. Graded cellular structure. Fix any set B of loadings for the weighting ϑ. For a multipartition ξ, let $M_B(\xi)$ be the set of all $\mathbf{i}$-tableaux on ξ for $\mathbf{i} \in B$. The elements $C_{\mathsf{S},\mathsf{T}}$ define a map $C\colon M_B(\xi) \times M_B(\xi) \to T^\vartheta_B$, where T^ϑ_B is the reduced steadied quotient of the weighted KLR algebra on the loadings B, and similarly for $\mathbb{T}^\vartheta_B$.

THEOREM 4.11. *The algebra $\mathbb{T}^\vartheta_B$ has a cellular structure with data given by $(\mathcal{P}_\ell, M_B, C, *)$.*

Proof. Consider the axioms of a cellular algebra, as given in Definition 2.23. Condition (1) is manifest.

Condition (2) is that a basis is formed by the vectors $C_{\mathsf{S},\mathsf{T}}$ where S and T range over tableaux for loadings from B of the same shape. First, note that it suffices to prove this for any set of loadings containing the original B, so we can always add new loadings. By the graded Nakayama's lemma, it suffices to check this after base change to $\Bbbk$. In this case, we can essentially just transfer structure from the algebra $\mathfrak{C}^\vartheta$ using Theorem 4.6. We have an isomorphism $\gamma\colon \mathfrak{C}^\vartheta_{\mathscr{D}} \otimes_{\mathscr{R}} \Bbbk \cong T^\vartheta_B$ where after possibly adding more loadings to B, we may assume that it is the set of all loadings on some collection of sets $\mathscr{D}$.

Thus, any D-tableau for $D \in \mathscr{D}$ can be turned into a tableau for a loading in B by simply labeling points with the content of the box they fill in the Young diagram. This shows that the number of $C_{\mathsf{S},\mathsf{T}}$ is the same as the number of basis vectors $\mathcal{C}_{\mathsf{S},\mathsf{T}}$ from $\mathfrak{C}^\vartheta_{\mathscr{D}} \otimes_{\mathscr{R}} \Bbbk$. Thus, it suffices to show that the $C_{\mathsf{S},\mathsf{T}}$ span T^ϑ_B.

First, note that when we consider $\mathfrak{C}^\vartheta_{\mathscr{D}}$ just as a module over the squares, as we calculated in the proof of 2.29, action of a square is upper triangular in the basis vectors $\mathcal{C}_{\mathsf{S},\mathsf{T}}$: if as before X_d denotes a square at $d \in \mathbb{R}$, then

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$X_d\mathcal{C}_{\mathsf{S},\mathsf{T}} = Q_p q^{\sigma(i-j)}\mathcal{C}_{\mathsf{S},\mathsf{T}} + \cdots$ where (i, j, p) is the box of diagram containing d and as before, σ denotes the sign of κ; the higher order terms are either in higher cells, or have fewer crossings. In particular, replacing each $\mathcal{C}_{\mathsf{S},\mathsf{T}}$ with its projection to this generalized eigenspace $e_{\mathbf{i}_\mathsf{S}}\mathcal{C}_{\mathsf{S},\mathsf{T}}$ still yields a basis of $\mathfrak{C}^{\vartheta}_{\mathscr{D}} \otimes_{\mathscr{R}} \Bbbk$. Under the isomorphism γ, this diagram is sent to a linear combination of $C_{\mathsf{S},\mathsf{T}} + \cdots$ where the other terms either have fewer crossings, or lie in a higher cell. This upper triangularity shows that the $C_{\mathsf{S},\mathsf{T}}$ form a basis.

Condition (3) is clear from the calculation

$$C^*_{\mathsf{S},\mathsf{T}} = (B^*_\mathsf{S} B_\mathsf{T})^* = B^*_\mathsf{T} B_\mathsf{S} = C_{\mathsf{T},\mathsf{S}}.$$

Thus, we need only check the final axiom, that for all x, we have an equality

$$xC_{\mathsf{S},\mathsf{T}} \equiv \sum_{\mathsf{S}' \in M_B(\xi)} r_x(\mathsf{S}', \mathsf{S}) C_{\mathsf{S}',\mathsf{T}} \qquad (\star)$$

modulo the vectors associated to partitions higher in dominance order. The numbers $r_x(\mathsf{S}', \mathsf{S})$ are just the structure coefficients of x^* acting on the basis of S_ξ given by B_S. Since we have that $xB^*_\mathsf{S} \equiv \sum_{\mathsf{S}' \in M_B(\xi)} r_x(\mathsf{S}', \mathsf{S}) B_{\mathsf{S}'}$ modulo diagrams factoring through loadings that are higher in weighted dominance order, the equation ($\star$) holds. This completes the proof. □

It is a standard fact about cellular algebras that any projective module over them has a cell filtration; a graded version of this is proven by Hu and Mathas [**HM10**, 2.14], showing that each projective P has a cell filtration where the graded multiplicity space of S_ξ is $\dot{S}_\xi \otimes_{T^\vartheta} P$.

PROPOSITION 4.12. *The projective $P_\mathbf{i}$ has a standard filtration, where the graded multiplicity of S_ξ is exactly the number of* $\mathbf{i}$*-tableaux on ξ, weighted by their degree.*

Proof. Since $\dot{S}_\xi \otimes_{T^\vartheta} P_\mathbf{i} \cong e_\mathbf{i} S_\xi$, this follows instantly from the result of Hu and Mathas mentioned above. □

EXAMPLE 4.13. Let us return to the case of Example 4.9. In this case, if we let B be the collections of loadings given there, every simple module is 1-dimensional, and so $T^\vartheta_B e_\mathbf{i}$ is already indecomposable. Thus, the multiplicities of standard modules in the indecomposable projectives are easily calculated from the bases of standard modules given in Example 2.19.

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The decomposition matrix in the 3 cases are given by

$$\begin{bmatrix} 1 & q^{-1} & q^{-2} & q^{-1} & 0 \\ 0 & 1 & q^{-1} & 0 & 0 \\ 0 & 0 & 1 & q^{-1} & q^{-2} \\ 0 & 0 & 0 & 1 & q^{-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & q^{-2} & q^{-1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 1 & 0 & q^{-1} \\ 0 & 0 & q^{-1} & 1 & q^{-2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ q^{-1} & q^{-2} & 1 & 0 & 0 \\ q^{-1} & 0 & q^{-1} & 1 & q^{-1} \\ 0 & 0 & q^{-2} & 0 & 1 \end{bmatrix}.$$

4.5. Generalization to bimodules. As defined in [Webc], there are natural bimodules $B^{\vartheta,\vartheta'}$ attached to each pair of weightings; these bimodules have steadied quotients $\mathscr{B}^{\vartheta,\vartheta'}$, which are T^{ϑ} - $T^{\vartheta'}$-bimodules. We call the functors $\mathscr{B}^{\vartheta,\vartheta'} \overset{L}{\otimes} -: T^{\vartheta'}\text{-mod} \to T^{\vartheta}\text{-mod}$ *change-of-charge functors*; these are quite interesting functors. In particular, we eventually show that they induce equivalences of derived categories.

In this bimodule, we can construct analogues of the elements $C_{\mathsf{S},\mathsf{T}}$, which we also denote $C_{\mathsf{S},\mathsf{T}}$ by abuse of notation (the original elements $C_{\mathsf{S},\mathsf{T}}$ will be a special case of these where $\vartheta = \vartheta'$). These are similar in form and structure to the basis $\mathcal{C}_{\mathsf{S},\mathsf{T}}$ defined in Section 3.4.

Let us first describe the basis which is cellular for the right module structure. Let D_{S} be the element of the bimodule $\mathscr{B}^{\vartheta,\vartheta'}$ defined analogously with B_{S}. Its bottom is given by $\mathbf{i}_\eta$ (for the weighting ϑ'). Its top is given by the entries of S, with each entry determining the position on the real line of a point in the top loading, labeled with the root associated to that box. The diagram proceeds by connecting the points in the loading associated to the same box in the top and bottom, while introducing the smallest number of crossings. As usual, this diagram is not unique; we choose any such diagram and fix it from now on.

DEFINITION 4.14. The right cellular basis for $e_{\mathbf{i}}\mathscr{B}^{\vartheta,\vartheta'}e_{\mathbf{j}}$ is given by $D_{\mathsf{S}}B^*_{\mathsf{T}}$ for S a $\mathbf{i}$-tableau for some loading $\mathbf{i}$ and the weighting ϑ (upon which the definition of $\mathbf{i}$-tableau depends), and T a $\mathbf{j}$-tableau for some loading $\mathbf{j}$ and the weighting ϑ'.

The left cellular basis for $e_{\mathbf{j}}\mathscr{B}^{\vartheta',\vartheta}e_{\mathbf{i}}$ is given by the reflections of these vectors, that is by $B_{\mathsf{T}}D^*_{\mathsf{S}}$.

LEMMA 4.15. *The vectors* $D_{\mathsf{S}}B^*_{\mathsf{T}}$ *are a basis for the bimodule* $\mathscr{B}^{\vartheta,\vartheta'}$. *Furthermore, the sum of vectors attached to partitions* $\leqslant \xi$ *in* ϑ'*-weighted order is a right submodule. In particular, as a right module,* $\mathscr{B}^{\vartheta,\vartheta'}$ *is standard filtered.*

Similarly, the left cellular basis shows that the bimodule $\mathscr{B}^{\vartheta,\vartheta'}$ *is standard filtered as a left module.*

Proof. First, we wish to show that these elements are a basis. This follows from Lemma 3.16 by the same argument as the proof of Theorem 4.11. That they are standard filtered follows from the calculation

$$B^*_{\mathsf{T}}x \equiv \sum_{\mathsf{S}' \in M_B(\xi)} B^*_{\mathsf{T}'} r_x(\mathsf{T}', \mathsf{T}) + \cdots$$

where the additional terms are in higher cells; multiplying on the left by D_{S}, we obtain the desired result. □

5. The structure of the categories

5.1. Highest weight categorifications. We let $\mathcal{S}^{\vartheta}_{\nu}$ denote the category of finite-dimensional representations of the reduced steadied quotient T^{ϑ}_{ν}; we let $\mathcal{S}^{\vartheta}$ denote the sum of these over all ν. As shown in Corollary 3.7, if $e \neq 1$, this category carries a categorical $\mathfrak{g}_U$-action induced from that on projective modules.

It follows immediately from Lemma 2.25 that:

PROPOSITION 5.1. *The category $\mathcal{S}^{\vartheta}_{\nu}$ is highest weight with standards S_ξ and partial order given by weighted dominance order. The category $\mathbb{T}^{\vartheta}_{\nu}$-mod is also highest weight, in the sense given by Rouquier* [**Rou08**, Section 4.1.3].

LEMMA 5.2. *The module $\mathcal{F}_i S_\xi$ carries a filtration $M_1 \subset M_2 \subset \cdots \subset M_j$ indexed by addable boxes of residue i in ξ from left to right. The quotient M_h/M_{h-1} is $S_{\xi(h)}(\deg(\mathsf{T}_h))$, where $\xi(h)$ is ξ with the hth addable box of residue i added, and T_h is obtained by putting the tautological tableau in ξ, and ∞ in the new box.*

Proof. We induct on the partial order; if ξ is maximal, then $S_\xi = P_{\mathbf{i}_\xi}$ and the only $\mathbf{i}_\xi$-tableau on ξ is the tautological one. Thus, the result follows from Proposition 4.12 in this case.

Now, we induct. Since $\mathcal{F}_i$ is exact, $\mathcal{F}_i P_{\mathbf{i}_\xi}$ is filtered by the images under $\mathcal{F}_i$ of standards, with multiplicity given by counting $\mathbf{i}_\xi$-tableaux of a given shape. On the other hand, the usual standard filtration is indexed by counting $\mathbf{i}_\xi \circ i$-tableaux; thus the kernel K of the map $\mathcal{F}_i P_{\mathbf{i}_\xi} \to \mathcal{F}_i S_\xi$ is of the expected dimension.

Furthermore, the basis vectors attached to any tableau $\mathbf{i}_\xi \circ i$-tableau which is not the tautological tableau on ξ with one new box added is killed. Thus the remaining basis vectors, where T is the tautological tableau of ξ with a box added give a basis of $\mathcal{F}_i S_\xi$. Furthermore, we can define a filtration compatible with this basis given by the span M_h of vectors where the new box on T is equal to or left of the hth addable box.

This defines the desired filtration, and we have an isomorphism $S_{\xi(h)}(\deg(\mathsf{T}_h)) \to M_h/M_{h+1}$ sending the basis vector B_{S} to the basis vector $C_{\mathsf{S},\mathsf{T}_h}$. □

For simplicity, we let δ_h denote $\deg(\mathsf{T}_h)$; is precisely the number of i-addable boxes right of the hth, minus the number of such which are removable. On the other hand, let δ^h denote the number of i-removable boxes *left* of the hth, minus the number of such which are removable.

Note that we have $\dot{S}_\eta \overset{L}{\otimes} (\mathcal{F}_i)S_\xi \cong (\mathcal{E}_i \dot{S}_\eta) \overset{L}{\otimes} S_\xi$. Combining this with the usual criterion that M is a standard filtered if and only if $\mathrm{Tor}^i(\dot{M}, S_\xi) = 0$ for all ξ, this shows that $\mathcal{E}_i S_\xi$ is standard filtered.

The functors $\mathcal{E}_i$ and $\mathcal{F}_i$ are biadjoint up to shift. Thus they also commute with duality. The result above also implies that:

COROLLARY 5.3.

(1) *The module $\mathcal{E}_i S_\xi$ carries a filtration $N_m \subset N_{m-1} \subset \cdots \subset N_j$ indexed by removable boxes of residue i in ξ from left to right. The quotient N_h/N_{h+1} is $S_{\xi\{h\}}(\delta^h)$, where $\xi\{h\}$ is ξ with the hth removable box of residue i removed, and T_h is obtained by putting the tautological tableau in ξ, and ∞ in the new box.*

(2) *The module $\mathcal{F}_i S^\star_\xi$ carries a filtration $O_j \subset O_{j-1} \subset \cdots \subset O_1$ indexed by addable boxes of residue i in ξ from left to right. The quotient O_h/O_{h+1} is $S^\star_{\xi(h)}(-\delta_h)$.*

(3) *The module $\mathcal{E}_i S^\star_\xi$ carries a filtration $Q_1 \subset Q_2 \subset \cdots \subset Q_m$ indexed by addable boxes of residue i in ξ from left to right. The quotient Q_h/Q_{h-1} is $S^\star_{\xi\{h\}}(-\delta^h)$.*

Losev has defined a notion of a *highest weight categorification* [Los13, 4.1]; this consists of the data of a:

(i) a highest weight category $\mathcal{C}$ with index set Λ for its simples/standards/indecomposable projectives, together with a function $c\colon \Lambda \to \mathbb{C}$;

(ii) a partition of Λ into subsets Λ_a with index set $\mathfrak{A}$;

(iii) integers n_a for each $a \in \mathfrak{A}$ and a function $d_a\colon \{1, \ldots, n_a\} \to \mathbb{C}$;

(iv) an isomorphism $\sigma_a\colon \{+, -\}^{n_a} \to \Lambda_a$, identifying Λ_a with signed sequences of length n_a.

Now, consider the highest weight category $\mathcal{S}^\vartheta$; we aim to show that it is, in fact, a highest weight categorification in the sense above. The combinatorics of this structure are almost exactly the same as those described by Losev for rational Cherednik algebras [Los13, Section 3.5].

(i) The indexing set $\Lambda = \mathcal{P}_\ell$ is the set of ℓ-multipartitions, and the function c is the sum over all boxes of the partition of the x-coordinate of the box.

(ii) The set $\mathfrak{A}$ is the set of partitions with no removable boxes of residue i, and Λ_a is the set of all partitions that contain a with only boxes of residue i added.

(iii) The number of addable boxes of residue i is n_a. The function d_a records, from the left to right, the x-coordinates of the addable boxes.

(iv) The isomorphism $\Lambda_a \to \{+,-\}^{n_a}$ sends a partition ξ to the sign vector where the first sign is $+$ if the leftmost addable box of residue i in a is present in ξ and $-$ if it is not, and similarly for the other addable boxes in order from left to right.

THEOREM 5.4. *When $e \neq 1$, the categorical $\mathfrak{g}_U$-module $\mathcal{S}^\vartheta$ is a highest weight categorification in the sense of Losev.*

Proof. Conditions 1, 3 and 4 from Losev's definition [Los13, 4.1] are clear. Thus we need only check conditions 0 and 2.

Condition 0 is that $\mathcal{F}_i$ and $\mathcal{E}_i$ preserve the categories of standard filtered objects; by exactness, we need only check that the image of standards has a standard filtration. This follows from Lemma 5.2 and Corollary 5.3.

Condition 2 is that these images $\mathcal{F}_i S_\xi$ and $\mathcal{E}_i S_\xi$ have certain images in the Grothendieck group, which are exactly those determined by Lemma 5.2 and Corollary 5.3. Thus, the result follows. □

Each simple module is the unique quotient of a unique standard module, so we can index these by multipartitions as well; we denote the simple quotient of S_ξ by L_ξ, and its projective cover by P_ξ. These simple modules (and also the projectives) carry a natural crystal structure for $\mathfrak{g}_U$, induced by taking the unique simple quotient of $\mathcal{F}_i L_\xi$ or $\mathcal{E}_i L_\xi$. This gives a crystal structure on multipartitions determined by the weighting ϑ.

DEFINITION 5.5. The *$\boldsymbol{\vartheta}$-weighted crystal structure* on the space of ℓ-multipartitions is defined as follows: drawing the partitions in Russian style, one places a close parenthesis over each addable box of residue i, and an open parenthesis over each removable box of residue i.

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- The Kashiwara operator $\tilde{e}_i$ removes the box under the leftmost uncanceled open parenthesis and sends the partition to 0 if there is no uncanceled open parenthesis.

- The Kashiwara operator $\tilde{f}_i$ adds a box under the rightmost uncanceled closed parenthesis and sends the partition to 0 if there is no uncanceled closed parenthesis.

In the Uglov case, this crystal structure is precisely that described by Tingley [Tin08, 3.2] in terms of abaci; in general, this crystal will coincide with that of the Uglovation.

COROLLARY 5.6. *The map sending a multipartition to L_ξ intertwines the ϑ-weighted crystal structure with that defined by the categorification functors.*

Proof. This is an instant consequence of [Los13, 5.1]. □

5.2. Decategorification. With this cellular basis in hand, we can extend all the results showing how quiver Schur algebras categorify Fock spaces to this more general case.

For our purposes, the *Fock space* $\mathbf{F}_\vartheta$ *of level* ℓ is the $\mathbb{C}[q, q^{-1}]$ module freely spanned by ℓ-multipartitions. For each multipartition ξ, we denote the corresponding vector u_ξ. Now, we choose weighting for our partitions; as before, this corresponds to choosing a weighting on $U_{\mathbf{w}}$, with all edges in the cycle given weight κ, and an ordering on the new edges (which is arbitrary), to put them in bijection with the constituents of the multipartition.

The affine Lie algebra $U_q(\mathfrak{g}_U)$ acts in a natural way on this higher level Fock space. We let

$$F_i u_\xi = \sum_{\mathrm{res}(\eta/\xi)=i} q^{-m(\eta/\xi)} u_\eta \quad E_i u_\xi = \sum_{\mathrm{res}(\xi/\eta)=i} q^{n(\xi/\eta)} u_\eta.$$

As usual:

- the sums are over all ways of adding (respectively removing a box) of residue i;

- $m(\eta/\xi)$ is the number of addable boxes of residue i right of the single box η/ξ minus the number of such boxes which are removable; and

- $n(\xi/\eta)$ is the number of addable boxes of residue i *left* of the single box ξ/η minus the number of such boxes which are removable.

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Note that as long as the weights of the partitions are generic, no two addable or removable boxes will be at the same horizontal position, so for each pair, the first is left of the second, or *vice versa*.

PROPOSITION 5.7. *If the weighting is Uglov for the charge s_*, then the resulting Fock space will agree with Uglov's Fock space for that charge s_*.*

THEOREM 5.8. *The Grothendieck group of the category of representations of T^ϑ_ν is isomorphic as a $U_q(\mathfrak{g}_U)$ representation to the corresponding level ℓ Fock space under the isomorphism $[S_\xi] \mapsto u_\xi$. In particular, we have that $[P_{\mathbf{i}}]$ maps to the sum over ℓ-multipartitions of the graded multiplicity of $\mathbf{i}$-tableaux.*

Proof. That the classes $[S_\xi]$ are a basis of the Grothendieck group holds because $\mathcal{S}^\vartheta$ is highest weight. Thus, we need only check how categorification functors act on these classes, which follows immediately from Lemma 5.2 and Corollary 5.3. □

The q-Fock space $\mathbf{F}_\vartheta$ has a natural symmetric bilinear form $(-,-)$ where the u_ξ are an orthonormal basis. Furthermore, it can be endowed with a sesquilinear form by

$$\langle u, v\rangle := (\bar{u}, v).$$

On the other hand, the Grothendieck group $K^0_q(T^\vartheta)$ also carries canonical bilinear and sesquilinear forms: we let

$$([M],[N]) = \dim_q(\dot{M} \overset{L}{\otimes} N) \quad \langle M, N\rangle = \dim_q \mathbb{R}\mathrm{Hom}(M, N).$$

PROPOSITION 5.9. *Under the isomorphism $\mathbf{F}_\vartheta \cong K^0_q(T^\vartheta)$, the forms $(-,-)$ match.*

Proof. We need only check that they are correct on standard modules; this follows from the orthonormality of the classes S_ξ. □

While the notation suggests that the forms $\langle -,-\rangle$ will coincide as well, this is not an easy statement to prove. It is one of the consequences of Proposition 5.22.

5.3. Change-of-charge functors: KLR case. The bimodules $\mathscr{B}^{\vartheta,\vartheta'}$ induce functors between the categories $\mathcal{S}^\vartheta$ and $\mathcal{S}^{\vartheta'}$. We call the groupoid of functors generated by these *change-of-charge functors*. One should see these as analogous with the twisting functors on category $\mathcal{O}$; this connection can be made precise by realizing $\mathcal{S}^\vartheta$ as a version of 'category $\mathcal{O}$' for an affine quiver variety.

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These functors are particularly useful since they show that up to derived equivalence, all the categories $\mathcal{S}^\vartheta$ only depend on λ, up to derived equivalence. They thus allow us to transport structure from one category to another.

LEMMA 5.10. *The functor $\mathscr{B}^{\vartheta,-\vartheta} \overset{L}{\otimes} -$ sends projective modules to tilting modules and tilting modules to injective modules.*

Proof. We already know that $\mathscr{B}^{\vartheta,-\vartheta} \overset{L}{\otimes} -$ is standard filtered, so if we prove it is self-dual, that will show it is tilting.

For a fixed loading $\mathbf{i}$, choose a basepoint b which is less that b_i for all i, and a real number $\gamma \gg 0$, sufficiently large so that the loading $\mathbf{i}'$ where we move the points of the loading by the automorphism of $\mathbb{R}$ given by $x \mapsto \gamma(x-b)+b$ is Hecke (it always will be for γ sufficiently large). There is a natural (generating) element $g_{\mathbf{i}} \in e_{\mathbf{i}} T^{-\vartheta} e_{\mathbf{i}'}$ and similarly for $g_{\mathbf{j}} \in e_{\mathbf{j}} T^{\vartheta} e_{\mathbf{j}'}$.

Each vector in the basis $C_{\mathsf{S},\mathsf{T}}$ for the bimodule $e_{\mathbf{j}}\mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{i}}$ factors as $g_{\mathbf{j}}C_{\mathsf{S}',\mathsf{T}'}g_{\mathbf{i}}^*$ for S', T' the obvious associated tableaux of type $\mathbf{j}'$ and $\mathbf{i}'$. Thus we have a surjective maps

$$\pi: e_{\mathbf{j}'}\mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{i}'} \to e_{\mathbf{j}}\mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{i}} \quad \pi(a) = g_{\mathbf{j}}ag_{\mathbf{i}}^*.$$

Similarly, as we range over all S, T, the elements $g_{\mathbf{j}}^*C_{\mathsf{S},\mathsf{T}}g_{\mathbf{i}}$ are linearly independent, giving an injective map

$$\iota: e_{\mathbf{j}}\mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{i}} \to e_{\mathbf{j}'}\mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{i}'} \quad \iota(b) = g_{\mathbf{j}}^*bg_{\mathbf{i}}.$$

For two elements $g_{\mathbf{j}}ag_{\mathbf{i}}^*$ and $g_{\mathbf{j}}bg_{\mathbf{i}}^*$, for $a, b \in e_{\mathbf{j}'}\mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{i}'}$, we define a pairing

$$\langle g_{\mathbf{j}}ag_{\mathbf{i}}^*, g_{\mathbf{j}}bg_{\mathbf{i}}^*\rangle = \tau(a^*g_{\mathbf{j}}^*g_{\mathbf{j}}bg_{\mathbf{i}}^*g_{\mathbf{i}}) = \tau(g_{\mathbf{i}}^*g_{\mathbf{i}}a^*g_{\mathbf{j}}^*g_{\mathbf{j}}b)$$

where $\tau: e_{\mathbf{i}'}T^{-\vartheta}e_{\mathbf{i}'} \cong e_{\mathbf{i}'}T^{\lambda}e_{\mathbf{i}'} \to \Bbbk$ is the Frobenius trace of [Weba, 2.26] if $e \neq 1$ (we abuse notation and also use $\mathbf{i}$ to denote the unloading of this loading). If $e = 1$, then we can use an explicit trace on $\Bbbk[S_m] \wr \Bbbk[x]/(x^\ell)$. As noted in [Weba, 2.27], we can modify this trace to make it symmetric; it is a bit more convenient to use this less-canonical trace, but symmetric, trace.

This pairing is well defined, since if b is in the kernel of π, then

$$a^*g_{\mathbf{j}}^*g_{\mathbf{j}}bg_{\mathbf{i}}^*g_{\mathbf{i}} = a^*g_{\mathbf{j}}^* \cdot 0 \cdot g_{\mathbf{i}} = 0;$$

the same statement for a follows by a symmetrical argument. Now, assume that $\pi(b) \neq 0$; by injectivity, $\iota\pi(b) = g_{\mathbf{j}}^*g_{\mathbf{j}}bg_{\mathbf{i}}^*g_{\mathbf{i}} \neq 0$ as well. Thus, by the nondegeneracy of τ [Weba, 2.26] , there exists an a, such that $\langle g_{\mathbf{j}}ag_{\mathbf{i}}^*, g_{\mathbf{j}}bg_{\mathbf{i}}^*\rangle \neq 0$; so this new pairing is nondegenerate as well.

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Note that furthermore, the adjoint under this action of right multiplication by c is left multiplication by c^* since

$$\langle cg_{\mathbf{j}}ag_{\mathbf{i}}^*, g_{\mathbf{j}}bg_{\mathbf{i}}^*\rangle = \tau(a^*g_{\mathbf{j}}^*c^*g_{\mathbf{j}}bg_{\mathbf{i}}^*g_{\mathbf{i}}) = \langle g_{\mathbf{j}}ag_{\mathbf{i}}^*, c^*g_{\mathbf{j}}bg_{\mathbf{i}}^*\rangle$$

and similarly for right multiplication. Since this is a nondegenerate invariant pairing, we have proven the self-duality of this module.

The statement on tiltings and injectives is equivalent to the adjoint $\mathbb{R}\operatorname{Hom}(\mathscr{B}^{\vartheta,-\vartheta}, -)$ sending injectives to tiltings. This functor sends the duals of projectives to the duals of tiltings, so we are done. □

COROLLARY 5.11. *The Ringel dual of $\mathcal{S}^{\vartheta}_{\nu}$ is $\mathcal{S}^{-\vartheta}_{\nu}$.*

Note that in our notation for Uglov weightings, this implies that $\mathcal{S}^{\vartheta^+_{\underline{s}}}_{\nu}$ is Ringel dual to $\mathcal{S}^{\vartheta^-_{\underline{s}}}_{\nu}$. By Proposition 4.5, this is in turn isomorphic to $\mathcal{S}^{\vartheta^+_{\underline{s}^\star}}_{\nu^\star}$ where $\nu^\star$ is the image of ν under the diagram automorphism induced by $i \mapsto -i$ on $\mathbb{C}/\mathbb{Z}$.

LEMMA 5.12. *The functor $\mathscr{B}^{\vartheta,\vartheta'} \overset{L}{\otimes} -$ induces an equivalence of categories.*

Proof. We can reduce to the case where $\vartheta = -\vartheta'$; since any weighting is between this pair, functors of this form factor through $\mathscr{B}^{\vartheta,\vartheta'} \overset{L}{\otimes} -$ on the right and the left. Thus, if all the functors when $\vartheta' = -\vartheta$ are equivalences, the desired result will follow.

Since $\mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{i}}$ is a tilting module by Lemma 5.10, its Ext algebra is concentrated in homological degree 0 (that is there are no higher Ext's). The functor $\mathscr{B}^{\vartheta,-\vartheta} \overset{L}{\otimes} -$ induces a map

$$e_{\mathbf{i}}T^{\vartheta}e_{\mathbf{j}} \to \operatorname{Hom}(\mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{j}}, \mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{i}}),$$

and by the vanishing of higher Exts, it suffices to prove that this map is an isomorphism.

We already know that the dimension of the left-hand side is

$$\dim(e_{\mathbf{i}}T^{\vartheta}e_{\mathbf{j}}) = \sum_{\xi}[T^{\vartheta}e_{\mathbf{i}} : S_{\xi}][T^{\vartheta}e_{\mathbf{j}} : S_{\xi}]$$

by BGG reciprocity. On the other hand, the dimension of the right hand side is

$$\dim\operatorname{Hom}(\mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{j}}, \mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{i}}) = \sum_{\xi}[S'_{\xi} : \mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{i}}][S'_{\xi} : \mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{j}}]$$

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since the multiplicities of the standard and costandard filtrations on a tilting coincide. Thus, the equality of dimensions follows immediately from the fact that the (co)standard multiplicities of $\mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{j}}$ coincide with those of $T^{\vartheta}e_{\mathbf{j}}$ by Corollary 3.17.

Thus, we need only show that this map is injective. Since the projective cover of the socle of $T^{\vartheta}e_{\mathbf{i}}$ is a sum of Hecke loadings, if this map is not injective, then there is an element of the kernel where $\mathbf{i}$ and $\mathbf{j}$ are Hecke loadings. But, in this case $\mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{j}} \cong T^{-\vartheta}e_{\mathbf{j}}$, so we just obtain the induced isomorphism $e_{\mathbf{i}}T^{\vartheta}e_{\mathbf{j}} \cong \mathrm{Hom}(\mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{j}}, \mathscr{B}^{\vartheta,-\vartheta}e_{\mathbf{i}})$. This completes the proof. □

This shows that the derived category of T^{ϑ} -mod only depends on the highest weight λ and not on ϑ itself (though these different categories are not *canonically equivalent*). Combining Lemma 5.12 and Theorem 4.8 implies that:

COROLLARY 5.13. *If the charges $\underline{\mathbf{s}}$ and $\underline{\mathbf{s}}'$ are in the same orbit of $\widehat{B}_{\ell}$, that is their KZ functors land in the same block of the Hecke algebra, then the categories $D^b(\mathcal{O}^{\underline{\mathbf{s}}})$ and $D^b(\mathcal{O}^{\underline{\mathbf{s}}'})$ are equivalent.*

Recall that if we have an exceptional collection $(\Delta, \leqslant)$, and we choose a new order $\leqslant'$ on the collection, there is a unique new exceptional collection $(\Delta', \leqslant')$ with the same indexing set, such that Δ'_i lies in the triangulated category generated by $\{\Delta_j\}_{j\geqslant' i}$ and $\Delta'_i \equiv \Delta_i$ modulo the triangulated category generated by $\{\Delta_j\}_{j>' i}$. We call this the *mutation* of the exceptional collection by this change of partial order. Let $d_{\xi}^{\vartheta,\vartheta'}$ be the degree of the basis vector D_{T} for the tautological tableau.

LEMMA 5.14. *The image of the standard exceptional collection in $\mathcal{S}^{\vartheta'}$ under $\mathscr{B}^{\vartheta',\vartheta} \overset{L}{\otimes} -$ is the mutation of the shifted standard collection $S_{\xi}(-d_{\xi}^{\vartheta,\vartheta'})$ in $\mathcal{S}^{\vartheta}$ for the induced change of partial order.*

Proof. We prove this by induction on the partial order for ϑ', which we denote $\leqslant'$ (matching the role it plays in the definition of mutation above). If ξ is maximal, then S_{ξ} is projective, and $\mathscr{B}^{\vartheta',\vartheta} \overset{L}{\otimes} S'_{\xi} = S_{\xi}(-d_{\xi}^{\vartheta,\vartheta'})$.

For ξ arbitrary, we have that by induction, the image of the category generated by S'_{η} with $\eta >' \xi$ is the same that generated by S_{η} with $\eta >' \xi$. Since $P'_{\xi} \equiv S'_{\xi}$ modulo the subcategory generated by S'_{η} with $\eta \geqslant' \xi$, we have that $\mathscr{B}^{\vartheta',\vartheta} \overset{L}{\otimes} P'_{\xi} \equiv \mathscr{B}^{\vartheta',\vartheta} \overset{L}{\otimes} S'_{\xi}$ modulo S_{η} with $\eta \geqslant' \xi$.

On the other hand, the standard filtration on $\mathscr{B}^{\vartheta',\vartheta} \overset{L}{\otimes} P'_{\xi}$ makes it clear that it lies in the subcategory generated by S_{η} with $\eta \geqslant' \xi$ and is equivalent to $S_{\xi}(-d_{\xi}^{\vartheta,\vartheta'})$

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modulo S_η with $\eta >' \xi$. Thus, the same statements hold for $\mathscr{B}^{\vartheta',\vartheta} \overset{L}{\otimes} S'_\xi$, and we are done. □

Following Bezrukavnikov [Bez03, Proposition 1], we can reconstruct the entire t-structure of $D^b(T^\vartheta\text{-mod})$ just from the exceptional collections S_* and $S^\star_*$; there is a unique t-structure containing both of these sets of modules in its heart. This gives us a description of the image of the standard t-structure on $D^b(T^{\vartheta'}\text{-mod})$ under $\mathscr{B}^{\vartheta',\vartheta} \overset{L}{\otimes} -$.

PROPOSITION 5.15. *The equivalence* $\mathscr{B}^{\vartheta',\vartheta} \overset{L}{\otimes} -$ *sends the standard* t*-structure on* $D^b(T^{\vartheta'}\text{-mod})$ *to the unique* t*-structure whose heart contains the mutation of* S_ξ *and inverse mutation of* S^*_ξ *for the new ordering* $>'$.

Note that if we replace a weighting by its Uglovation, no boxes with the same residue switch order, so weighted partial order does not change. Thus, we have that:

COROLLARY 5.16. *If* $\vartheta_{\underline{s}}$ *is the Uglovation of* ϑ*, the bimodule* $\mathscr{B}^{\vartheta_{\underline{s}},\vartheta}$ *induces a Morita equivalence.*

Let V be a free $\mathbb{Z}[q, q^{-1}]$-module of finite rank, equipped with a sesquilinear form $\langle -, -\rangle$ and an antilinear bar involution such that $\langle \bar{u}, \bar{v}\rangle = \langle v, u\rangle$. A *semiorthonormal basis* of V is a partially ordered basis $\{v_i\}_{i\in(I,\leqslant)}$ as a $\mathbb{Z}[q, q^{-1}]$-module such that $\langle v_i, v_j\rangle = 0$ if $j \nleqslant i$, and $(v_i, v_i) = 1$.

If $\leqslant'$ is another partial order on I, then v_* possesses a unique *mutation* to another semiorthogonal basis $\{v'_i\}$ indexed by I, this time endowed with $\leqslant'$, such that

$$v'_i \in \operatorname{span}\{v_j\}_{j\geqslant' i} \quad v'_i \equiv v_i \pmod{\operatorname{span}\{v_j\}_{j>' i}}.$$

It follows from Uglov's formula for the bar involution that the standard basis u_ξ is semiorthogonal for the weighted dominance order for ϑ.

PROPOSITION 5.17. *The classes* $[\mathscr{B}^{\vartheta',\vartheta} \overset{L}{\otimes} S'_\xi]$ *are the mutation of the semiorthogonal basis* $q^{-d_\xi^{\vartheta,\vartheta'}}[S_\xi]$ *from* ϑ*- to* ϑ'*-weighted dominance order.*

5.4. The affine braid action. The affine braid group $\widehat{B}_\ell$ of rank ℓ acts on the set of Uglov weightings; it leaves κ unchanged, and acts on the weights

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$(\vartheta_1, \ldots, \vartheta_\ell)$ of the new edges by

$$\sigma_i \cdot (\vartheta_1, \ldots, \vartheta_\ell) = (\vartheta_1, \ldots, \vartheta_{i+1} - \kappa e/\ell, \vartheta_i + \kappa e/\ell, \ldots, \vartheta_\ell)$$
$$\sigma_0 \cdot (\vartheta_\ell - \kappa e(-1 + 1/\ell), \vartheta_2, \ldots, \vartheta_{\ell-1}, \vartheta_1 + \kappa e(-1 + 1/\ell)).$$

For each element of the affine Weyl group, we have an induced isomorphism between the sets of weighted multipartitions, permuting them in the obvious manner.

We can lift this action of the affine Weyl group to the Fock spaces, at the cost of making it an action of the braid group; recall that the sum $\oplus_\vartheta \mathbf{F}_\vartheta$ where the sum is over all Uglov weightings carries an action of the quantum group $U_q(\widehat{\mathfrak{sl}}_\ell)$ which commutes with the $U_q(\mathfrak{g}_U)$ action. There is a natural map of the affine braid group $\hat{B}_\ell \to U_q(\widehat{\mathfrak{sl}}_\ell)$ called the *quantum Weyl group*. This map sends σ_i to the element t_i that acts on an element v of weight μ by

$$t_i \cdot v = \sum_{\substack{a,b,c \geqslant 0 \\ a-b+c=\mu^i}} (-1)^{a+c} q^{ac-b} E_i^{(a)} F_i^{(b)} E_i^{(c)} v.$$

LEMMA 5.18. *Each of the generators t_i for $i = 0, \ldots, \ell - 1$ induces an isometry $\mathbf{F}_\vartheta \to \mathbf{F}_{\sigma_i \cdot \vartheta}$ that sends the standard basis of $\mathbf{F}_\vartheta$ to the mutation of the shifted standard basis $\{q^{-d^{\vartheta, s_i \cdot \vartheta}} u_\xi\}$ of $\mathbf{F}_{\sigma_i \cdot \vartheta}$ for the order change from $\sigma_i \cdot \vartheta$-weighted dominance order to ϑ-weighted dominance order.*

Proof. Note that switching from ϑ to $\sigma_i \cdot \vartheta$ has the effect of crossing the bundle of strands corresponding to each column of the ith and $i + 1$st components of the multipartition (where as usual, one considers the eth and first components with a shift for σ_0). The contribution to the degree is 0 if in the abacus model, both corresponding positions are vacant or both are filled, or -1 if one is vacant, and the other filled. Thus, the scalar $m = -d_\xi^{\vartheta, \sigma_i \cdot \vartheta}$ is exactly the number of such positions where a bead can be pushed from the ith to $i + 1$st runner of the abacus, or *vice versa*.

Since t_i lies inside the completion of the quantum universal enveloping algebra of the root $\mathfrak{sl}_2$ for i, we need only study the action of this subalgebra on $\oplus_\vartheta \mathbf{F}_\vartheta$. This is easiest to see when basis vectors are described in terms of abaci; in this case, F_i pushes beads from the ith runner to the $i + 1$st (or from the ℓth to the 1st with a shift by n in the case of F_0), and E_i pushes them in the opposite direction. As usual, as an $U_q(\mathfrak{sl}_2)$ module, $\mathbf{F}_\vartheta$ is a sum of tensor products of the standard representation. Each summand is indexed by a tableau with no beads that can be pushed from the $i + 1$st runner to the ith, with tensor factors corresponding to positions where a bead can be pushed up (of which there are m). The standard basis of the Fock space matches the standard basis of the tensor product.

Thus, we need only compute how the only quantum Weyl group generator of $\mathfrak{sl}_2$ acts on this basis. The usual formula for the quasi-R-matrix in terms of the quantum Weyl group shows that $\Theta = \Delta(t)(t^{-1} \otimes t^{-1})$, or alternatively $\Theta(t \otimes t) = \Delta(t)$. We can generalize this to the formula

$$\Theta^{(m)}(t \otimes \cdots \otimes t) = \Delta^{(m)}(t).$$

On the standard basis, $t \otimes \cdots \otimes t$ acts by

$$(t \otimes \cdots \otimes t) \cdot v_\xi = q^m v_{\sigma_i \cdot \xi}$$

if the weight of ξ for $\widehat{\mathfrak{sl}}_\ell$ is μ. By the form of the m-fold quasi-R-matrix, we have that

$$t_i \cdot v_\xi = \Theta^{(m)} q^m v_{\sigma_i \cdot \xi} = q^m v_{\sigma_i \cdot \xi} + \sum_{\xi < \xi'} a_{\xi'} v_{\sigma_i \cdot \xi'}$$

for some coefficients $a_{\xi'}$ where $a_{\xi'}$ can only be nonzero if ξ' agrees with ξ outside the i and $i+1$st rows.

Since the element t_i acts as an isometry in the pairing $\langle -, - \rangle$, this is again an semiorthonormal basis, and thus agrees with the mutation. □

THEOREM 5.19. *The functors* $\mathbb{B}_{\sigma_i} = \mathscr{B}^{\vartheta, \sigma_i \cdot \vartheta} \overset{L}{\otimes} -$ *define a strong action of the affine braid group on the categories* $D(\mathcal{S}_\vartheta)$ *where* ϑ *is summed over all Uglov weightings, categorifying the action of the quantum Weyl group of* $\widehat{\mathfrak{sl}}_\ell$.

Proof. We apply Lemma 4.15 in order to check the braid relations. For any positive lift w of an element of the affine symmetric group, and any factorization $w = w'w''$ into positive elements, we have that $w''\vartheta$ is between $w\vartheta$ and ϑ. Thus, by Lemma 4.15, we have that $\mathbb{B}_{w'}\mathbb{B}_{w''} \cong \mathscr{B}^{\vartheta, w \cdot \vartheta} \overset{L}{\otimes} -$. This implies the braid relations and the associativity of these isomorphisms shows that this action is strong.

Thus, we need only check the action on the Grothendieck group is correct. This follows immediately from comparing Lemmas 5.18 and 5.14; the action of $\mathbb{B}_{\sigma_i}$ and of the quantum Weyl group both send the standard basis to its mutant by the same change of order, so they coincide. □

REMARK 5.20. The same tensor product also induces actions on the categories $D^b(T^\vartheta\text{-}\underline{\text{mod}})$ of ungraded modules (by forgetting the grading) and on $D^b(T^\vartheta\text{-dg-mod})$ by considering all graded algebras and modules as complexes with trivial differential. In both these cases, the conclusions of Theorem 5.19 still hold.

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Note that we have a sort of dual braid group action, that arising from Rickard complexes, as in the work of Chuang and Rouquier [CR08, 6.1]. This is an action of the affine braid group $\widehat{B}_e$ categorifying the quantum Weyl group action from $\mathfrak{g}_U$. We denote the functor associated to $\sigma \in \widehat{B}_e$ by Θ_σ.

LEMMA 5.21. *Consider any highest weight categorical $\mathfrak{sl}_2$-action. Then Θ_s for the unique simple reflection s sends the exceptional collection of standard modules S_ξ to the mutation of this exceptional collection $S_\xi[n_\xi]$ where we reverse order on every piece of the filtration, and n_ξ as before is the integer attached to each standard as part of the data of a highest weight categorification.*

Proof. We need only check this for the unique highest weight categorification of $(\mathbb{C}^2)^{\otimes n}$, since every highest weight categorification has a filtration (compatible with standards and categorification functors) with these as subquotients. The standards in this case are naturally indexed by sign sequences. We let $\bar{\xi}$ of a sign sequence denote the same sequence with $+$ and $-$ switched.

This categorification has a deformation; the representation theoretically inclined can think of as passing to deformed category $\mathcal{O}$. At the generic point, we obtain a semisimple category; it is, in fact, just the representations of a naive tensor product of categorifications of $\mathbb{C}^2$. Since the result is obvious in this case, we obtain that $\Theta(S_\xi)$ is a flatly deformable shift of a module which is generically isomorphic to $S_{\bar{\xi}}[n]$. In particular, it has the same composition factors as $S_{\bar{\xi}}[n]$, and thus is in the subcategory generated by $S_{\bar{\eta}}$ for $\eta \geqslant \xi$, and equivalent to $S_{\bar{\xi}}[n]$ modulo that generated by $S_{\bar{\eta}}$ for $\eta > \xi$. □

5.5. Canonical bases. There is a natural duality ψ on projective objects in S^ϑ, given by the anti-automorphism $*$. More categorically, we can think of this as $\mathrm{Hom}(-, T^\vartheta)$, which is naturally a right module, given a left module structure via $*$. We can extend this to derived categories in the obvious way.

PROPOSITION 5.22.

(1) *The functor ψ categorifies the bar involution of Fock space.*

(2) *The sesquilinear inner products denoted $\langle -, - \rangle$ on Fock space and the Grothendieck group coincide.*

(3) *The affine braid group action of Theorem 5.19 categorifies the quantum Weyl group action.*

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Proof. Since

$$\langle [M], [N]\rangle = ([\psi M], [N]) \quad \langle u, v\rangle = (\bar{u}, v)$$

and we already know that the forms $(-,-)$ coincide by Proposition 5.9, the statements (1) and (2) are equivalent.

For each ϑ and ν, there exists some element of the affine braid group π, such that $\pi \cdot \vartheta$ is well-separated (in the sense of [**Webc**, Section 3.3]) for ν; that is, the weights ϑ_i are sufficiently far apart that the category will not change as we separate them further. As proven in [**Webc**, 3.6], this algebra is Morita equivalent to the *quiver Schur algebra* of [**SW**]. We let $\ell(\vartheta, \nu)$ be the minimal length of such an element. We prove the statements above by induction on $\ell(\vartheta, \nu)$. More precisely, our inductive hypothesis will be:

(h_n) the inner products $\langle -, -\rangle$ agree for all ϑ and ν such that $\ell(\vartheta, \nu) \leqslant n$, and for any generator $\mathbb{B}_i$, the action when both $\ell(\vartheta, \nu) \leqslant n$ and $\ell(\sigma_i\vartheta, \nu) \leqslant n$ agrees with the quantum Weyl group action.

When $n = 0$, the category $\mathcal{S}^{\vartheta}_{\nu}$ agrees with the representations of a quiver Schur algebra as in [**SW**]; thus, statement (1) and thus (2) hold by [**SW**, 7.19]. Since we have checked that the sesquilinear forms coincide, Proposition 5.17 and Lemma 5.18 describe the effect of the change-of-charge functor and the quantum Weyl group action in terms of the same mutations, so they coincide. This principle is the key of the proof: once we know that the forms $\langle -, -\rangle$ coincide on the image category, we know that the action of $\mathbb{B}_i$ agrees with the quantum Weyl group.

Thus, we move to the inductive step $(h_{n-1}) \Rightarrow (h_n)$. We consider ϑ, ν with $\ell(\vartheta, \nu) = n$. Then, for some generator σ_i, we have $\ell(\sigma_i\vartheta, \nu) = n - 1$. The action of $\mathbb{B}_i : \mathcal{S}^{\vartheta}_{\nu} \to \mathcal{S}^{\sigma_i\vartheta}_{\nu}$ sends the standard modules S_ξ to the mutation of the shifted standard modules. We also already know that these are also the image under the quantum Weyl group $t_i : \mathbf{F}_\vartheta \to \mathbf{F}_{\sigma_i\cdot\vartheta}$. Thus, we have that

$$\langle u_\xi, u_\eta\rangle = \langle t_i u_\xi, t_i u_\eta\rangle = \langle \mathbb{B}_i S_\xi, \mathbb{B}_i S_\eta\rangle = \langle S_\xi, S_\eta\rangle,$$

where we use in turn that t_i is an isometry, that we have already checked the coincidence of classes and of forms for $\mathcal{S}^{\sigma_i\vartheta}_{\nu}$, and that $\mathbb{B}_i$ induces an isometry on Grothendieck groups.

This establishes claims (1–2), and claim (3) for reflections that decrease $\ell(\vartheta, \nu)$; however, the cases where σ_i increases or keeps $\ell(\vartheta, \nu)$ unchanged follow immediately by the same argument. We already know that the forms $\langle -, -\rangle$ coincide in the target, so we may use the same argument as above. This establishes the theorem. □

The structure of q-Fock spaces together with their bar involution leads to the definition of a canonical basis. This basis $\{b_\xi\}$ is defined to be the unique bar invariant basis such that $b_\xi \in u_\xi + \sum_{\xi' < \xi} q^{-1}\mathbb{Z}[q^{-1}]u_{\xi}$.

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THEOREM 5.23. *The basis in $K^0(\mathcal{S}^\vartheta)$ given by the indecomposable projectives P_ξ is identified under the isomorphism to twisted Fock space with Uglov's canonical basis $\{b_\xi\}$, and thus the basis of simples with the dual canonical basis.*

Proof. Obviously, the projectives P_ξ are invariant under ψ: the modules $T^\vartheta e_{\mathbf{i}}$ are, and when $\mathbf{i} = \mathbf{i}_\xi$, the indecomposable P_ξ appears as a summand exactly once. The highest weight structure shows that $b_\xi \in u_\xi + \sum_{\xi'<\xi} \mathbb{Z}[q, q^{-1}]u_\xi$. Thus, we need only establish these coefficients are positive. That is, that only positive shifts of standard modules appear in the standard filtration. For this, it suffices to check that $\operatorname{Hom}(P_{\xi'}, P_\xi)$ is positively graded for $\xi' \neq \xi$. This is a consequence of [**Webc**, 4.4]; by this corollary, the sum $\oplus_\xi \tilde{P}_\xi$ is a summand of a graded projective generator whose endomorphisms are positively graded. Since $\operatorname{Hom}(\tilde{P}_{\xi'}, \tilde{P}_\xi) \to \operatorname{Hom}(P_{\xi'}, P_\xi)$ is a surjection, the latter is positively graded as well. □

This shows a diagrammatic analogue of Rouquier's conjecture. By BGG reciprocity, we have that the multiplicities $[S_\xi : P_\eta] = [L_\eta : S_\xi]$ agree; thus, it follows that have that:

COROLLARY 5.24. *The graded decomposition numbers for T^ϑ agree with the coefficients of Uglov's canonical basis of Fock space $\mathbf{F}_\vartheta$ in terms of standard modules. That is, for all η, we have that*

$$b_\eta = \sum_\xi [S_\xi : P_\eta]u_\xi = \sum_\xi [L_\eta : S_\xi]u_\xi.$$

Transferring structure via the equivalence of Theorem 4.8, we find that Corollary 5.24 implies that:

COROLLARY 5.25 (Rouquier's conjecture). *The multiplicities of standard modules in projectives in $\mathcal{O}^{\underline{\mathbf{s}}}$; and thus by BGG reciprocity, the multiplicities of simples in standards, are the same as the coefficients of Uglov's canonical basis of a Fock space, specialized at $q = 1$.*

6. Koszul duality

Unlike the earlier sections, the results in this section depends on the 'categorical dimension conjecture' of Vasserot and Varagnolo [**VV10**, 8.8] that $\mathcal{O}^\vartheta$ is equivalent to a truncated parabolic category $\mathcal{O}$ for an affine Lie algebra; as mentioned in the introduction, this is proven in [**RSVV16**, **Los16**].

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This conjecture shows, amongst other things, that $\mathcal{O}^\vartheta$ and thus T^ϑ possess a Koszul grading. *A priori*, it is not clear that this Koszul grading is Morita equivalent to the one that we have already defined; in fact, a general uniqueness property of Koszul gradings shows this. To clarify:

DEFINITION 6.1. We call a finitely dimensional graded algebra A *Koszul* if it is graded Morita equivalent to a positively graded algebra A' which is Koszul in the usual sense; we call a graded abelian category *Koszul* if it is equivalent to the category of graded modules over a Koszul algebra.

THEOREM 6.2. *The usual grading on T^ϑ is Koszul, and the equivalence of Theorem 4.8 induces an equivalent graded lift of $\mathcal{O}^\vartheta$ to the grading on category $\mathcal{O}$. In particular, $\mathcal{S}^\vartheta$ is standard Koszul and balanced.*

Proof. By the numerical criterion of Koszulity [**BGS96**, 2.11.1], if an algebra has one Koszul grading, then any other grading with the same graded Cartan matrix is again Koszul, and in fact graded Morita equivalent to the first Koszul grading. Thus, any grading on T^ϑ whose Cartan matrix is the matrix expressing Uglov's canonical basis in terms of its dual is a Koszul grading, since the grading induced from the truncated category $\mathcal{O}$ has this property holds in the truncated parabolic category $\mathcal{O}$ by [**VV10**, 8.2]. By Corollary 5.24, this is the case for the diagrammatic grading on T^ϑ as well, so this grading is Koszul. Similarly, T^ϑ is balanced and standard Koszul (the latter being part of the definition of the former) by [**SVV14**, 4.3]. □

In the case where T^ϑ is Morita equivalent to a quiver Schur algebra (as shown in [**Webc**, Theorem A]), this Koszulity has been established independently by Maksimau in forthcoming work [**Mak14**]. We give an independent and different proof in [**Web17**].

Now we turn to describing the Koszul dual of T^ϑ; for simplicity, we only do this in the case where U is a e-cycle, so $\mathfrak{g}_U = \widehat{\mathfrak{sl}}_e$. Consider a $\ell \times e$ matrix of integers $U = \{u_{ij}\}$, and let $s_i = \sum_{j=1}^{e} u_{ij}$ and $t_j = \sum_{i=1}^{\ell} u_{ij}$ and an integer w. We wish to consider the former as an Uglov weighting for $\widehat{\mathfrak{sl}}_e$, and the latter for $\widehat{\mathfrak{sl}}_\ell$.

Associated to each row of U, we have a charged e-core partition; we fill an abacus with beads at the positions $(u_{ij} - a)e + j$ for $j = 1, \dots, e$ and all $a \in \mathbb{Z}_{\geqslant 0}$, and take the partition described by this abacus. Let v_i be the unique integer such that $v_i - w$ is the total number of boxes of residue i in all these partitions. We wish to consider the algebra $T^{\vartheta^+_{\underline{\mathbf{s}}}}_{\underline{\mathbf{t}},w} := T^{\vartheta^+_{\underline{\mathbf{s}}}}_{\mu}$ and $\mathcal{S}^{\vartheta}_{\underline{\mathbf{t}},w} := \mathcal{S}^{\vartheta}_{\mu}$ with weight $\mu := \lambda - \sum v_i \alpha_i$. We note that by Proposition 4.5, we have an equivalence $\mathcal{S}^{\vartheta^\pm_{\underline{\mathbf{s}}}}_{\underline{\mathbf{t}},w} \cong \mathcal{S}^{\vartheta^\mp_{\underline{\mathbf{s}}^\star}}_{\underline{\mathbf{t}}^\star,w}$.

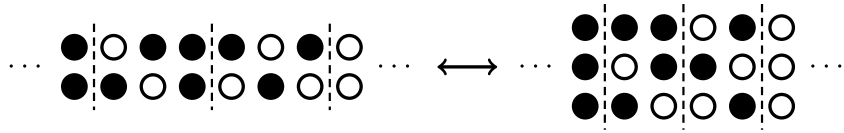

Figure 3. The Koszul duality bijection.

THEOREM 6.3. *The Koszul dual of* $\mathcal{S}^{\vartheta^{\pm}_{\underline{\mathbf{s}}}}_{\underline{\mathbf{t}},w}$ *is* $\mathcal{S}^{\vartheta^{\mp}_{\underline{\mathbf{t}}}}_{\underline{\mathbf{s}},w} \cong \mathcal{S}^{\vartheta^{\pm}_{\underline{\mathbf{t}}^\star}}_{\underline{\mathbf{s}}^\star,w}$.

Proof. Previous mentioned results [**RSVV16**, 7.4] imply that $\mathcal{O}^{\mathbf{s}}_{\underline{\mathbf{t}},w}$ and $\mathcal{O}^{\mathbf{t}^\star}_{\underline{\mathbf{s}}^\star,w}$ are Koszul dual. Translating to diagrammatic algebras, this implies that $\mathcal{S}^{\vartheta^{+}_{\underline{\mathbf{s}}}}_{\underline{\mathbf{t}},w}$ and $\mathcal{S}^{\vartheta^{+}_{\underline{\mathbf{t}}^\star}}_{\underline{\mathbf{s}}^\star,w}$ are Koszul dual. □

We can visualize the combinatorial bijection between simple modules in these two Koszul dual categories. To a simple in $\mathcal{S}^{\vartheta^{+}_{\underline{\mathbf{s}}}}$, we can associate a charged ℓ-multipartition, and thus an ℓ-runner abacus. We place the runner for the new edge e^1 at the bottom, and the list them in ascending order. The duality map works by cutting this abacus into rectangles along the vertical lines between ae and $ae+1$ for $a \in \mathbb{Z}$, and then flipping along the SW/NE diagonal. An example of this operation is shown in Figure 3. That is, the runner corresponding to e^j becomes the beads in the positions $ae + j$. This reverses the roles of ℓ and e.

PROPOSITION 6.4. *This bijection between multipartitions matches that on simples induced by Koszul duality.*

Proof. In order to understand this duality, we must give the correspondence between our combinatorics and that for affine Lie algebras as in the work of Vasserot–Varagnolo [**VV10**].

We associate a weight of an affine Lie algebra to an abacus diagram as follows: we cut off the diagram at some point to the far left of all boxes of the partition (that is left of which the abacus is solid). After simultaneously shifting all s_i, we might as well assume that we cut off all beads at negative positions, so we have exactly s_i dots remaining on the ith runner. We read the x-coordinates of the dots on each runner in turn (all on the first, then all on the second, and so forth), which gives us an N tuple (for N the total number of dots) which we denote $(a_1, \dots, a_N)$ (this matches the notation in [**Los16**, Section 2.3]). The affine Weyl group $\widehat{S}_N$ acts on this set with the level e-action (that is the 'translation' adding e to one coordinate and subtracting e from another is an element of the Weyl group). We let y be the unique minimal length element of this group that sends

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the sequence $(a_1, \ldots, a_N)$ to an element of the fundamental alcove (all entries are increasing and between 1 and e). Visually, we can think of y the element that switches:

- from the order induced on dots by reading leftward on each runner in order;
- to that induced by reading across the runners from the first to the ℓth, first reading all dots in position $\ldots, 2e+1, e+1, 1, -e+1, -2e+1, \ldots$ starting at the greatest x position that appears, then at x-coordinates congruent to 2 (mod e), and so forth.

By [**SVV14**, 2.16], the weight of the Koszul dual simple is obtained by applying the element y in the level ℓ action to the element of the fundamental alcove given by s_1 instances of 1, then s_2 instances of 2, and so forth. This is given by the flip map we have described, since this switches the reading down runners used to obtain $(a_1, \ldots, a_N)$ with the reading across runners that gives y, and preserves how shifted from the fundamental alcove a dot is (this matches with taking the inverse since we have gone from level e to level ℓ). □

Alternatively, we can describe this map by decomposing this abacus further into one with runners corresponding to each entry of an $\ell \times e$ matrix; the runners of our previous description correspond to the rows, and the runner for the jth column is gotten by taking the beads (or lack of beads) at positions $ae + j$. In this case, the duality map is gotten by transposing the matrix of runners. Similarly, to each ℓ-multipartition, we can associate a matrix U, by creating the associated $\ell \times e$ matrix of abaci, and then taking the charge of each; that is, the number of boxes in the diagram of the partition associated to the runner.

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