Can. J. Math., Vol. XXIV, No. 5, 1972, pp. 957–966

AN L^p SATURATION THEOREM FOR SPLINES

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1. Let $\Delta_n : 0 = x_0^{(n)} < x_1^{(n)} < \ldots < x_n^{(n)} = 1$ be a subdivision of $[0, 1]$, and let $\mathscr{S}_k(\Delta_n)$ denote the class of functions whose restriction to each sub-interval $[x_{i-1}^{(n)}, x_i^{(n)})$ is a polynomial of degree at most k. Gaier [**1**] has shown that for uniform subdivisions Δ_n (that is, subdivisions for which $x_i^{(n)} = i/n$)

$$||f - \mathscr{S}_k(\Delta_n)||_p = o(n^{-k-1})$$

if and only if f is a polynomial of degree at most k. Here, and subsequently, $||\cdot||_p$ denotes the usual norm in $L^p[0, 1]$, $1 \leqq p \leqq \infty$, and we should emphasize that functions differing only on a set of Lebesgue measure zero are identified.

One of the authors [**4**] has recently characterized those functions f for which

$$||f - \mathscr{S}_k(\Delta_n)||_\infty = O(n^{-k-1}).$$

In this paper we solve the corresponding problem for the L^p norms, $1 \leqq p < \infty$. Let

$$\mathrm{Lip}(1, L^p) = \left\{f : \left(\int_0^1 |f(x+\delta) - f(x)|^p dx\right)^{1/p} = O(\delta)\right\}$$

(f assumed to be identically zero outside $[0, 1]$) and define

$$\mathscr{L}_p^k = \{f : f \in C^{k-1}[0, 1], f^{(k-1)} \text{ is absolutely continuous}, f^{(k)} \in \mathrm{Lip}(1, L^p)\}.$$

Our main result is the following

THEOREM. *Let f be a real-valued function on* $[0, 1]$ *and let* $\{\Delta_n\}_{n=1}^\infty$ *be uniform subdivisions. Then*

$$||f - \mathscr{S}_k(\Delta_n)||_p = O(n^{-k-1}), \tag{1}$$

if and only if $f \in \mathscr{L}_p^k$.

Acknowledgement. The authors wish to thank Professor Z. Ditzian for some helpful comments.

2. In this section we shall demonstrate the sufficiency part of (1); in fact we shall establish the following more general result, namely,

LEMMA 3. *Let $f \in \mathscr{L}_p^k$. Then, given any sequence of arbitrary subdivisions $\{\Delta_n\}_{n=1}^\infty$, there exists a sequence of spline functions $\{S_n\}_{n=1}^\infty$ of degree k with knots at the points of Δ_n* (i.e., $S_n \in \mathscr{S}_k(\Delta_n) \cap C^{k-1}[0, 1]$) *satisfying*

$$||f - S_n||_p = O(||\Delta_n||^{k+1}), \tag{2}$$

where $||\Delta_n|| = \max_{1 \leqq i \leqq n} (x_i^{(n)} - x_{i-1}^{(n)})$.

Received September 28, 1971 and in revised form, February 29, 1972.

https://doi.org/10.4153/CJM-1972-096-1 Published online by Cambridge University Press

We must first prove two other lemmas.

LEMMA 1. *If* $f \in \mathrm{Lip}(1, L^p)$, *then*

$$\|f - \mathscr{S}_0(\Delta_n)\|_p = O(\|\Delta_n\|) \tag{3}$$

Proof. We first remark that $f \in \mathrm{Lip}(1, L^p)$ implies via the Hölder inequality that $f \in \mathrm{Lip}(1, L^1)$, so that f is of bounded variation [**2**] and hence $f \in L^p[0, 1]$. Let us define

$$D_n = \int_0^{\|\Delta_n\|} \sum_{i=0}^{n-1} \int_0^{x_{i+1}-x_i} |f(t + x_i) - f(x + x_i)|^p \, dx \, dt.$$

With the change of variables

$$u = x + x_i, \qquad v = t - x,$$

and since $f \in \mathrm{Lip}(1, L^p)$, we have

$$\begin{aligned} D_n &\leqq \sum_{i=0}^{n-1} \int_{-\|\Delta_n\|}^{\|\Delta_n\|} \int_{x_i}^{x_{i+1}} |f(u + v) - f(u)|^p \, du \, dv \\ &\leqq K\|\Delta_n\|^{p+1}, \end{aligned}$$

where K is independent of n. Thus there exists τ_n, $0 \leqq \tau_n \leqq \|\Delta_n\|$, such that

$$\sum_{i=0}^{n-1} \int_0^{x_{i+1}-x_i} |f(\tau_n + x_i) - f(x + x_i)|^p \, dx \leqq K\|\Delta_n\|^p.$$

Choosing σ_n to be the step function taking the value $f(\tau_n + x_i)$ on $[x_i, x_{i+1})$, it follows that

$$\|f - \sigma_n\|_p = O(\|\Delta_n\|).$$

LEMMA 2. *Let* $f \in L^p[0, 1]$ *and* ν *be a natural number. If* $s_n \in \mathscr{S}_{\nu-1}(\Delta_n) \cap C^{\nu-2}$, $n = 1, 2, \ldots,$ *and* ϕ *is a positive function on the natural numbers such that*

$$\|f - s_n\|_p = O(\phi(n)), \tag{4}$$

then there exists a sequence $\{S_n\}_{n=1}^{\infty}$, $S_n \in \mathscr{S}_\nu(\Delta_n) \cap C^{\nu-1}$, *satisfying*

$$\|F - S_n\|_p = O(\|\Delta_n\|\phi(n)) \tag{5}$$

where

$$F(x) = \int_0^x f(t)dt.$$

(C^{-1} *is interpreted to be the space of all real-valued functions.*)

Proof. Consider the B-spline $M_i(x)$ of degree $\nu - 1$ for the subdivision Δ_n, $i = 0, 1, \ldots, n - \nu$, defined in [**6**]. It is known [**6**] that $M_i(x)$ is a non-negative function having support in $[x_i, x_{i+\nu}]$, $M_i \in \mathscr{S}_{\nu-1} \cap C^{\nu-2}$ and

$$\int_0^1 M_i(x)dx = 1. \tag{6}$$

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Following [3], define

$$A_i = \int_{x_i}^{x_{i+1}} (f(t) - s_n(t))dt, \qquad i = 0, 1, \ldots, n-1,$$

$$\tilde{A}_i = \int_{x_i}^{x_{i+1}} |f(t) - s_n(t)|dt, \qquad i = 0, 1, \ldots, n-1,$$

and

$$S_n(x) = \int_0^x s_n(t)dt + \sum_{i=0}^{n-\nu} A_i \int_0^x M_i(t)dt.$$

Suppose that $x_i \leqq x \leqq x_{i+1}$. Then

$$F(x) - S_n(x) = \int_0^x (f(t) - s_n(t))dt - \sum_{j=0}^{n-\nu} A_j \int_0^x M_j(t)dt$$
$$= \int_{x_i}^x (f(t) - s_n(t))dt + \sum_{j=0}^{i-1} A_j - \sum_{j=0}^{n-\nu} A_j \int_0^x M_j(t)dt.$$

By (6),

$$\int_0^x M_j(t)dt = 1 \quad \text{for} \quad j \leqq i - \nu \quad \text{and} \quad \int_0^x M_j(t)dt = 0 \quad \text{for} \quad j \geqq i + 1.$$

Thus

$$|F(x) - S_n(x)| \leqq \left| \int_{x_i}^x (f(t) - s_n(t))dt \right| + \left| \sum_{j=i-\nu+1}^{i-1} A_j\left(1 - \int_0^x M_j(t)dt\right) \right| + \left| A_i \int_0^x M_i(t)dt \right|$$
$$\leqq \tilde{A}_i + \sum_{j=i-\nu+1}^{i} |A_j| \leqq K\left[\sum_{j=i-\nu+1}^{i} \tilde{A}_j^p \right]^{1/p}$$

for some constant $K = K(\nu)$. Hence

$$\text{(7)} \qquad \int_{x_i}^{x_{i+1}} |F(x) - S_n(x)|^p dx \leqq K^p \sum_{j=i-\nu+1}^{i} \tilde{A}_j^p (x_{i+1} - x_i).$$

But by Hölder's inequality,

$$\tilde{A}_j^p \leqq (x_{j+1} - x_j)^{p-1} \int_{x_j}^{x_{j+1}} |f(t) - s_n(t)|^p dt$$
$$\leqq ||\Delta_n||^{p-1} \int_{x_j}^{x_{j+1}} |f(t) - s_n(t)|^p dt;$$

thus

$$\int_0^1 |F(x) - S_n(x)|^p dx$$
$$\leqq K^p ||\Delta_n||^{p-1} \sum_{i=0}^{n-1} \sum_{j=i-\nu+1}^{i} (x_{i+1} - x_i) \int_{x_j}^{x_{j+1}} |f(t) - s_n(t)|^p dt$$

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and by reversing the order of summation, we obtain

$$(8)\quad \int_0^1 |F(x) - S_n(x)|^p dx \leqq K^p ||\Delta_n||^{p-1} \sum_{j=0}^{n-1} (x_{j+\nu} - x_j) \int_{x_j}^{x_{j+1}} |f(t) - s_n(t)|^p dt$$

$$\leqq K_1^p ||\Delta_n||^p \int_0^1 |f(t) - s_n(t)|^p dt$$

for some constant $K_1 = K_1(\nu)$. The lemma follows on applying (4) in (8).

Lemma 3 is an easy consequence of Lemmas 1 and 2.

3. We now seek to establish the necessity part of (1). In this section, all subdivisions Δ_n are assumed to be uniform.

It will be convenient to state at this point the L^p version of Markoff's inequality and of two inequalities due to Gaier.

LEMMA 4 (Markoff [**1**]). *Let P be a polynomial of degree k on $[a, b]$ and $0 \leqq j \leqq k$. Then there exists a constant $K = K(p, k, j)$ such that*

$$(9)\qquad \left[\int_a^b |P^{(j)}(x)|^p dx \right]^{1/p} \leqq K(b - a)^{-j} \left[\int_a^b |P(x)|^p dx \right]^{1/p}.$$

LEMMA 5 (Gaier [**1**]). *Suppose P is a function on $[-a, b]$ which reduces to a polynomial of degree k on each of $[-a, 0]$ and $(0, b]$, and define $h = P(0+) - P(0-)$. Then there exists a constant $K = K(p, k)$ such that*

$$(10)\qquad |h| \leqq K(\min (a, b))^{-1/p} \left[\int_{-a}^b |P(x)|^p dx \right]^{1/p}.$$

LEMMA 6 (Gaier [**1**]). *Let $T_j \in \mathscr{S}_k(\Delta_j)$, $j = n, n + 1$, and let $h_i^{(\nu)}$ denote the jumps of $T_n^{(\nu)}$ at $x_i = i/n$, $i = 1, \ldots, n - 1$, i.e., $h_i^{(\nu)} = T_n^{(\nu)}(x_i+) - T_n^{(\nu)}(x_i-)$. Let $0 < \epsilon < 1/2$. Then there exists a constant $C = C(\epsilon)$ such that*

$$(11)\qquad |h_i^{(\nu)}| \leqq C n^{\nu+1/p} \left[\int_{i/(n+1)}^{(i+1)/(n+1)} |T_{n+1} - T_n|^p dx \right]^{1/p}$$

if $\epsilon \leqq i/n \leqq (1 - \epsilon)$.

Next we prove

LEMMA 7. *Let*

$$(12)\qquad ||f - S_n||_p = O(n^{-k-1})$$

for some sequence $S_n \in \mathscr{S}_k(\Delta_n)$, $n = 1, 2, \ldots$

Then

$$(13)\qquad f \in C^{k-1}[0, 1], f^{(k-1)} \text{ is absolutely continuous}$$

and

$$(14)\qquad ||f^{(k)} - S_{2^n}^{(k)}||_p = O(2^{-n}).$$

Proof. Since $S_{2^n} - S_{2^{n+1}} \in \mathscr{S}_k(\Delta_{2^{n+1}})$, we may apply (9) to obtain

$$(15)\qquad \begin{aligned} ||S_{2^n}^{(\nu)} - S_{2^{n+1}}^{(\nu)}||_p &\leqq K(2^{n+1})^{\nu}||S_{2^n} - S_{2^{n+1}}||_p \\ &\leqq K_1 2^{-n(k+1-\nu)}, \qquad \nu = 1, 2, \ldots, k, \end{aligned}$$

where $K = K(p, k, \nu)$, $K_1 = K_1(p, k, \nu)$ are constants.

Hence there are functions $f_\nu \in L^p[0, 1]$ satisfying

$$(16)\qquad ||S_{2^n}^{(\nu)} - f_\nu||_p \leqq K_2 2^{-n(k+1-\nu)}$$

for some constant $K_2 = K_2(p, k, \nu)$.

Therefore $S_{2^n}{}^{(\nu)} \to f_\nu$ a.e. on $[0, 1]$ and so for almost all ϵ in $[0, 1]$, we have

$$(17)\qquad S_{2^n}^{(\nu-1)}(\epsilon) \to f_{\nu-1}(\epsilon).$$

Let $h_i{}^{(\nu)}$ denote the jump of $S_{2^n}{}^{(\nu)}$ at $x_i = i2^{-n}$, and let

$$x_+{}^0 = \begin{cases} 1, & x \geqq 0, \\ 0, & x < 0. \end{cases}$$

For $x \in [\epsilon, 1 - \epsilon]$, we define

$$(18)\qquad \begin{aligned} P_{2^n}(x) &= \int_\epsilon^x S_{2^n}^{(\nu)}(t)dt \\ &= S_{2^n}^{(\nu-1)}(x) - S_{2^n}^{(\nu-1)}(\epsilon) - \sum_i{}' h_i^{(\nu-1)}[x - i/2^n]_+{}^0, \end{aligned}$$

where $\sum_i{}'$ means that we sum over those i for which $\epsilon \leqq i2^{-n} \leqq 1 - \epsilon$.

By (11),

$$(19)\qquad \begin{aligned} \left[\sum_i{}' |h_i^{(\nu-1)}|^p\right]^{1/p} &\leqq C(\epsilon)2^{n(\nu-1+1/p)}||S_{2^n} - S_{2^{n+1}}||_p \\ &\leqq C_1(\epsilon)2^{-n(k+2-\nu-1/p)}. \end{aligned}$$

Let

$$(20)\qquad \bar{f}(x) = \int_\epsilon^x f_\nu(t)dt + f_{\nu-1}(\epsilon).$$

Then,

$$(21)\qquad \begin{aligned} &\left[\int_\epsilon^{1-\epsilon} |f_{\nu-1}(x) - \bar{f}(x)|^p dx\right]^{1/p} \\ &\leqq ||f_{\nu-1} - S_{2^n}^{(\nu-1)}||_p + \left[\int_\epsilon^{1-\epsilon} |S_{2^n}^{(\nu-1)}(x) - P_{2^n}(x) - f_{\nu-1}(\epsilon)|^p dx\right]^{1/p} \\ &\qquad + \left[\int_0^1 |P_{2^n}(x) + f_{\nu-1}(\epsilon) - \bar{f}(x)|^p dx\right]^{1/p} \\ &= \alpha_1{}^n + \alpha_2{}^n + \alpha_3{}^n, \text{ say.} \end{aligned}$$

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From (17)–(19),

$$(22)\quad \alpha_2{}^n \leqq \left[\int_\epsilon^{1-\epsilon} |S_{2^n}^{(\nu-1)}(\epsilon) - f_{\nu-1}(\epsilon)|^p dx\right]^{1/p} + \left[\sum_i{}' |h_i^{(\nu-1)}|^p\right]^{1/p} \to 0 \text{ as } n \to \infty.$$

Using (16), (18) and (20), we have

$$(23)\quad \alpha_3{}^n \leqq \left[\int_0^1 \int_\epsilon^x |S_{2^n}^{(\nu)}(t) - f_\nu(t)|^p dt\, dx\right]^{1/p} \leqq ||S_{2^n}^{(\nu)} - f_\nu||_p \to 0 \text{ as } n \to \infty.$$

Hence, letting $n \to \infty$ in (21) and using (16), (22) and (23), we obtain

$$f_{\nu-1}(x) = \bar{f}(x) = \int_\epsilon^x f_\nu(t)dt + f_{\nu-1}(\epsilon)$$

a.e. on $[\epsilon, 1 - \epsilon]$, $\nu = 1, 2, \ldots, k$, for almost all ϵ in [0, 1]. Since $f_0 = f$, it follows that $f_\nu(x) = f^{(\nu)}(x)$ a.e. on [0, 1], and this together with (16) establishes (14). If $k = 0$, (13) is redundant. If $k = 1$, we have

$$f(x) = \int_\epsilon^x f_1(t)dt + f(\epsilon) \text{ a.e. on } [\epsilon, 1 - \epsilon]$$

for almost all ϵ in $[0, \frac{1}{2}]$, and from (16), $f_1 \in L^p[0, 1]$, and hence $f_1 \in L^1[0, 1]$.

Thus we may find $x_0 \in (0, 1)$ and a sequence $\epsilon_1 > \epsilon_2 > \ldots \to 0$ such that

$$(x_0) = \int_{\epsilon_i}^{x_0} f_1(t)dt + f(\epsilon_i).$$

It follows that $f(\epsilon_i) \to \phi_0$, say, as $i \to \infty$ and defining

$$\phi(x) = \int_0^x f_1(t)dt + \phi_0,$$

we have $f(x) = \phi(x)$ a.e. on [0, 1], and so f is equivalent to an absolutely continuous function on [0, 1], which is (13).

Essentially similar arguments enable us to establish (13) for general values of k. This completes the proof of the lemma.

LEMMA 8. *Let ξ_i, $i = 1, 2, \ldots, m$, be rational numbers with a common denominator q such that* $\min(\xi_i, 1 - \xi_i) \leqq 1/4$, $i = 1, 2, \ldots, m$. *Let h_i, $i = 1, 2, \ldots, m$, be real numbers such that*

$$(24)\quad \sum_{i=1}^m |h_i|^p q^{p-1} \geqq N > 0,$$

and let n be any integral multiple of q. Then there exists an integer l with $n + 1 \leqq l \leqq 2n$, such that

$$\text{(25)} \qquad \sum_{i=1}^{m} |h_i|^p [\rho(\xi_i, \Delta_l)]^{kp+1} \geqq cNl^{-p(k+1)}$$

for some absolute constant c, where $\rho(\xi_i, \Delta_l) = \inf_{0 \leqq j \leqq l} |\xi_i - j/l|$.

Proof. Assume first that $1/4 \geqq \xi_1 > \xi_2 > \ldots > \xi_m \geqq 1/q$. Define t_i to be $n\xi_i$, $i = 1, 2, \ldots, m$. Then t_i is a natural number no greater than $n/4$. Fix i and let r be a natural number with $1 \leqq r \leqq n$. Then either we have (i) $h/\xi_i \leqq r < (h + \frac{1}{2})/\xi_i$ for some natural number h, or we have (ii) $(h - \frac{1}{2})/\xi_i \leqq r < h/\xi_i$ for some natural number h.

If case (i) applies, then, since $\xi_i = t_i/n$, we have $hn/t_i \leqq r < (h + \frac{1}{2})n/t_i$, and so

$$0 \leqq \frac{t_i}{n} - \frac{t_i + h}{n + r} = \frac{t_i r - hn}{n(n + r)} \leqq \frac{1}{2(n + r)}.$$

Thus

$$\text{(26)} \quad \rho(\xi_i, \Delta_{n+r}) = (t_i r - hn)/(n(n + r)) = \xi_i s/(n + r), \text{ where } s = r - hn/t_i$$

If case (ii) applies, we have

$$(h - \tfrac{1}{2})\frac{n}{t_i} \leqq r < \frac{hn}{t_i}, \text{ and so}$$

$$0 < \frac{t_i + h}{n + r} - \frac{t_i}{n} = \frac{hn - t_i r}{n(n + r)} \leqq \frac{1}{2(n + r)}.$$

Hence

$$\text{(27)} \qquad \rho(\xi_i, \Delta_{n+r}) = \frac{hn - t_i r}{n(n + r)} = \frac{\xi_i s}{n + r},$$

where $s = hn/t_i - r$. Therefore we have

$$\begin{aligned} \sum_{r=n+1}^{2n} [\rho(\xi_i, \Delta_r)]^{kp+1} &= \sum_{r=1}^{n} [\rho(\xi_i, \Delta_{n+r})]^{kp+1} \\ &= \sum_{h=0}^{t_i - 1} \sum_{hn/t_i \leqq r < (h+\frac{1}{2})n/t_i} [\rho(\xi_i, \Delta_{n+r})]^{kp+1} \\ &\qquad + \sum_{h=1}^{t_i} \sum_{(h-\frac{1}{2})n/t_i \leqq r < hn/t_i} [\rho(\xi_i, \Delta_{n+r})]^{kp+1} \\ &\geqq 2 \sum_{h=0}^{t_i - 1} \sum_{s=1}^{[\frac{1}{2}n/t_i]} \left(\frac{\xi_i s}{2n}\right)^{kp+1} \end{aligned}$$

where we have used (26) and (27) to estimate the sums over r.

Using the integral test to estimate the inner sum above, we obtain

$$\sum_{r=1}^{n} [\rho(\xi_i, \Delta_{n+r})]^{kp+1} \geqq \frac{2t_i}{kp + 2} \left(\frac{\xi_i}{2n}\right)^{kp+1} \left(\frac{n}{2t_i} - 1\right)^{kp+2}.$$

Since $n/2t_i = 1/2\xi_i \geqq 2$, we have $n/2t_i - 1 \geqq n/4t_i$. Hence

$$(28) \qquad \sum_{r=1}^{n} [\rho(\xi_i, \Delta_{n+r})]^{kp+1} \geqq (kp+2)^{-1}2^{-3kp-4}t_i\left(\frac{\xi_i}{n}\right)^{kp+1}\left(\frac{n}{t_i}\right)^{kp+2} = C_1 n^{-kp}.$$

Thus from (24) and (28),

$$\sum_{r=n+1}^{2n} \sum_{i=1}^{m} |h_i|^p[\rho(\xi_i, \Delta_r)]^{kp+1} = \sum_{i=1}^{m} \sum_{r=n+1}^{2n} |h_i|^p[\rho(\xi_i, \Delta_r)]^{kp+1} \geqq C_1 N n^{-kp} q^{1-p}.$$

Hence there exists l with $n + 1 \leqq l \leqq 2n$ such that

$$\sum_{i=1}^{m} |h_i|^p[\rho(\xi_i, \Delta_l)]^{kp+1} \geqq c_1 N n^{-1-kp} q^{1-p} \geqq cNl^{-p(k+1)}$$

for some constant c, since $q < l \leqq 2n$. It is obvious that the proof may easily be modified for the slightly more general statement of the lemma.

LEMMA 9. *Suppose*

$$(29) \qquad \left[\int_0^1 |f - S_n(t)|^p dt\right]^{1/p} \leqq Kn^{-k-1}$$

for some sequence $S_n \in \mathscr{S}_k(\Delta_n)$, $n = 1, 2, \ldots$. *Then there is a constant* C *such that for* $n = 1, 2, \ldots$,

$$\sum_{i=1}^{n} |h_i^{(k)}|^p n^{p-1} \leqq C,$$

where $h_i^{(k)}$, $i = 1, 2, \ldots, n$ *are the jumps of* $S_n^{(k)}$.

Proof. Applying Lemma 6 with $T_j = S_j, j = n, n+1$, and with $\nu = k$, $\epsilon = 1/4$, we deduce that

$$\sum_i{}' |h_i^{(k)}|^p n^{-1-pk} \leqq K \int_0^1 |S_{n+1} - S_n|^p \leqq K_1 n^{-p(k+1)}$$

for some constants K, K_1. If Lemma 9 is to be false, we may suppose, without loss of generality, that given $N > 0$, there exists a natural number q such that

$$\sum_{i=1}^{m} |h_i^{(k)}| q^{p-1} \geqq N,$$

where $1/4 \geqq \xi_m > \xi_{m-1} > \ldots > \xi_1 = 1/q$ are the points of $[0, 1/4] \cap \Delta_q$ and $h_i^{(k)}$, $i = 1, 2, \ldots, m$, are the corresponding jumps of $S_q^{(k)}$. By Lemma 8 therefore, we can find l with $2q + 1 \leqq l \leqq 4q$ satisfying

$$(30) \qquad \sum_{i=1}^{m} |h_i^{(k)}|^p[\rho(\xi_i, \Delta_l)]^{kp+1} \geqq cNl^{-p(k+1)}.$$

Let $\xi_i \in [r_i/l, (r_i + 1)/l] = I_{r_i}$, where r_i is an integer. Applying Lemmas 4 and 5 to the function $S_q(x) - S_l(x)$ on I_{r_i}, we have

$$|h_i^{(k)}| \leqq K_2[\rho(\xi_i, \Delta_l)]^{-1/p}\left[\int_{I_{r_i}} |S_q^{(k)} - S_l^{(k)}|^p dt\right]^{1/p}$$
$$\leqq K_3[\rho(\xi_i, \Delta_l)]^{-k-1/p}\left[\int_{I_{r_i}} |S_q - S_l|^p dt\right]^{1/p}$$

so that, since the intervals I_{r_i} are pairwise disjoint,

$$\int_0^1 |S_q - S_l|^p dt \geqq K_4 \sum_{i=1}^{m} |h_i^{(k)}|^p [\rho(\xi_i, \Delta_l)]^{kp+1}$$
$$\geqq K_5 N l^{-p(k+1)}$$

by (30). Thus

$$||f - S_l||_p \geqq ||S_q - S_l||_p - ||f - S_q||_p$$
$$\geqq K_5 N l^{-k-1} - K q^{-k-1}$$
$$\geqq (K_5 N - 4^{k+1} K) l^{-k-1}$$

which contradicts (29) if N is sufficiently large. This proves the lemma.

We are now in a position to prove the necessity part of the theorem which we state as

LEMMA 10. *If* $||f - \mathscr{S}_k||_p = O(n^{-k-1})$, *then* $f \in \mathscr{L}_k^{\,p}$.

Proof. Suppose $K > 0$ and $S_n \in \mathscr{S}_k(\Delta_n)$ satisfy

(31) $$\left[\int_0^1 |f - S_n|^p dt\right]^{1/p} \leqq K n^{-k-1}, \qquad n = 1, 2, \ldots .$$

By Lemma 7, $f \in C^{k-1}[0, 1]$, $f^{(k-1)}$ is absolutely continuous on $[0, 1]$, and

(32) $$\left[\int_0^1 |f^{(k)} - \sigma_{2^n}|^p dt\right]^{1/p} \leqq K' 2^{-n}, \qquad n = 1, 2, \ldots$$

where $\sigma_n = S_n^{(k)}$. Let $0 < \delta < 1$ and choose n such that $2^{-n-1} \leqq \delta < 2^{-n}$. Let $h_i^{(\nu)}, i = 1, 2, \ldots, 2^n - 1$ be the jumps of $S_{2^n}^{(\nu)}$, $\nu = 0, 1, \ldots, k$. If $0 \leqq i \leqq 2^n - 2$, we have

(33) $$\int_{i2^{-n}}^{(i+1)2^{-n}} |\sigma_{2^n}(t + \delta) - \sigma_{2^n}(t)|^p dt \leqq 2^{-n} |h_{i+1}^{(k)}|^p,$$

and

(34) $$\int_{1-2^{-n}}^{1-\delta} |\sigma_{2^n}(t + \delta) - \sigma_{2^n}(t)|^p dt = 0.$$

https://doi.org/10.4153/CJM-1972-096-1 Published online by Cambridge University Press

Hence

$$(35)\quad \left[\int_0^{1-\delta} |f^{(k)}(t+\delta) - f^{(k)}(t)|^p dt\right]^{1/p} \leqq 2\|f^{(k)}(t) - \sigma_{2^n}(t)\|_p$$
$$+ \left[\int_0^{1-\delta} |\sigma_{2^n}(t+\delta) - \sigma_{2^n}(t)|^p dt\right]^{1/p} \leqq K'2^{1-n} + 2^{-n/p}\left[\sum_{i=1}^{2^n-1} |h_i^{(k)}|^p\right]^{1/p}$$

by (32)–(34). We now apply lemma 9 to (35) and find a constant C such that

$$\left[\int_0^{1-\delta} |f^{(k)}(t+\delta) - f^{(k)}(t)|^p dt\right]^{1/p} \leqq (2K' + C^{1/p})2^{-n}$$
$$\leqq (4K' + 2C^{1/p})\delta.$$

Hence

$$\left[\int_0^1 |f^{(k)}(t+\delta) - f^{(k)}(t)|dt\right]^{1/p} = O(\delta)$$

and so $f^{(k)} \in \mathscr{L}_k^p$. This completes the proof of the theorem.

Remark. Other characterizations are possible, using the result [**2**] that $f \in \mathscr{L}_p^k$ if and only if $f^{(k)}$ is of *p-bounded variation* on [0, 1], i.e., the supremum over all subdivisions $\delta_n : 0 = x_0 < x_1 < \ldots < x_n = 1$ of the sum

$$\left[\sum_{i=0}^{n-1} |f^{(k)}(x_{i+1}) - f^{(k)}(x_i)|^p (x_{i+1} - x_i)^{1-p}\right]^{1/p}$$

if finite, and, for $p > 1$, the result [**5**] that f is of p-bounded variation on [0, 1] if and only if f is absolutely continuous and $f' \in L^p[0, 1]$.

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