DEFINING-RELATIONS FOR HURWITZ GROUPS

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In memory of John Leech

1. Introduction. With much sadness we note the death of John Leech, on 28 September 1992. Perhaps best known for his discovery of the "Leech Lattice" (which provides the best known sphere-packing in 24 dimensions), John will also be remembered for his contributions to the use of computers in mathematics, and to computational algebra in particular.

Only days before his death he visited St Andrews and discussed with us some questions which remained unanswered from his study of quotients of the $(2,3,7)$ triangle group $\Delta(2,3,7)=\langle x,y \mid x^2=y^3=(xy)^7=1\rangle$; see **[7]** and other references listed there. These questions concerned finite groups obtainable as quotients of $\Delta(2,3,7)$ by the insertion of a fourth relator in its presentation.

Leech himself preferred to use the alternative generators $A=[y,x]=y^{-1}xyx$ and $B=(xy)^3$, in terms of which $x=B^3AB^{-2}$ and $y=B^3AB^{-4}$, the presentation becoming $\Delta(2,3,7)=\langle A,B \mid (AB)^2=(A^{-1}B)^3=B^7=1\rangle$. His main interest was in those groups obtained by inserting a fourth relator w of the form $w=(A^rB^s)^k$ for small non-negative integers r, s and k. For example, he showed in **[7]** that the presentation

$$\langle A,B \mid (AB)^2=(A^{-1}B)^3=B^7=(A^5B^6)^2=1\rangle$$

defines the simple group $L_2(13)$ of order 1092, but could not ascertain whether or not the insertion of $w=(A^4B^4)^3$ as fourth relator defines a finite group. Other cases he asked about included $(A^2B^4)^6$, $(A^3B^4)^4$ and $(A^3B^2)^5$.

We provide the answers in three of these (previously unresolved) cases in this paper, and throw some light on the fourth one, which appears to be quite difficult. Also we look at a few further cases of the same form, with interesting results. Our approach is mostly computational, using coset enumeration by machine to find the order (or at least the index of a subgroup of known finite order) of the group in question.

Many of Leech's computations were done either by hand or on the KDF9 computer at Glasgow (with rather limited storage) in the 1960s. In contrast, on a SUN workstation it is now possible to perform enumerations of several million cosets within a matter of seconds.

2. Further background and some computational results. The triple $(p,q,r)=(2,3,7)$ provides the largest value of the expression $1/p+1/q+1/r$ less than 1 (for positive integers p, q and r), and for this reason the $(2,3,7)$ triangle group and its quotients are significant in the theory of automorphism groups of Riemann surfaces of genus $g>1$ and of regular maps on surfaces; see **[2]** for details and a summary of known examples.

At the time of writing **[2]**, relatively little was known about those quotients obtainable by inserting a fourth relator of the form $w=A^k$ (or equivalently, $w=[x,y]^k$), but now the picture is complete.

The group with presentation $\langle x,y \mid x^2=y^3=(xy)^7=[x,y]^k=1\rangle$ is

(a) trivial when $k=1, 2, 3$, or 5,
(b) isomorphic to $L_2(7)$ of order 168 when $k=4$,
(c) isomorphic to $L_2(13)$ of order 1092 when $k=6$ or 7,
(d) a non-split extension of $(C_2)^6$ by $L_2(7)$ when $k=8$,
(e) infinite when $k \geq 9$.

The cases $k=10$ and $k \geq 12$ were dealt with successfully by Holt and Plesken in [**4**] (using methods which may be applied easily also to large powers of other words in the generators), and independently by Howie and Thomas [**5**]. The last case $k=11$ was completed by Edjvet in [**3**].

If the inserted fourth relator is of the form $w = A^rB^s$, then without loss of generality $1 \leq s \leq 6$ (or otherwise $w = A^r$), and this implies $B \in \langle B^s \rangle \subseteq \langle A \rangle$, making the group cyclic and therefore trivial.

In more general cases, the following lemma (due to Leech) is useful in reducing the number of possibilities for the fourth relator w. We give a purely algebraic proof, but remark that Leech's original proof was almost entirely geometric, based on properties of the hyperbolic tessellation $\{3, 7\}$ (see Figure 1 in [**10**]).

LEMMA. *In the* $(2,3,7)$ *triangle group* $\langle A, B \mid (AB)^2 = (A^{-1}B)^3 = B^7 = 1 \rangle$ *the following elements are mutually conjugate, for every integer* m:
(1) A^mB *and* $A^{m-2}B^6$,
(2) A^mB^2 *and* $(A^{m+1}B^5)^{-1}$,
(3) A^mB^3 *and* $(A^{m-3}B^4)^{-1}$.

Proof. The first part is easy: $A^{-1}(A^mB)A = A^{m-1}BA = A^{m-1}A^{-1}B^{-1} = A^{m-2}B^6$. For the second and third parts, use the original generators x and y to obtain

$$x(A^mB^2)x = x(y^{-1}xyx)^m(xy)^6x = (xy^{-1}xy)^mxy^{-1} = A^{-m}A^{-1}B^2 = A^{-(m+1)}B^2 = (B^5A^{m+1})^{-1},$$

and similarly also

$$\begin{aligned} y^{-1}xyA^{-1}(A^mB^3)Ay^{-1}xy &= y^{-1}xy(y^{-1}xyx)^{m-1}(xy)^2y^{-1}xyxy^{-1}xy = x(y^{-1}xyx)^{m-2}xy^{-1}xy^{-1}xy \\ &= (xy^{-1}xy)^{m-3}(xy)^2 = A^{-(m-3)}B^3 = (B^4A^{m-3})^{-1}. \end{aligned}$$

In particular, A^2B is conjugate to B^{-1}, AB^5 is conjugate to B^{-2}, and A^3B^3 is conjugate to B^{-4}, while A^3B is conjugate to AB^{-1} (of order 3). Similar reduction is possible also for other relators of the form $w = (A^rB^s)^k$ where r and s are small. Specifically:

(4) $A^4B = A^2AB^{-1}A^{-1} = A^2BA^{-1}BA^{-2}$, which is conjugate to $A^{-1}B^2$ and therefore (by the lemma) also to B^{-5},

(5) $AB^3 = B^{-1}A^{-1}B^2$, which is conjugate to $A^{-1}B$,

(6) A^2B^3 is conjugate to B^3A (by the lemma) and so to AB^3 and therefore also to $A^{-1}B$ (by (5)),

(7) A^4B^3 is conjugate to $(AB^4)^{-1}$, but $AB^4 = BA^{-1}BA^{-1}B^{-2} = BA^{-1}B^2AB^{-1}$, so A^4B^3 is conjugate to B^{-2}.

In all these cases (along with the trivial case of AB), the insertion of the relator $w = (A^rB^s)^k$ is either redundant, or sufficient to cause collapse, depending on the value of k. Also two other cases reduce to known ones:

(8) A^5B is conjugate to A^3B^6, but $A^3B^6 = A^3B^{-1} = A^2BA^{-1}BA^{-1} = AB^{-1}A^{-2}BA^{-1}$, therefore A^5B is conjugate to A^{-2},

(9) A^2B^5 is conjugate to $(AB^2)^{-1} = B^{-2}A^{-1} = B^{-1}AB$ and therefore to A.

In particular, the fourth relator w may now be assumed to be of the form $(A^rB^s)^k$, with $s \in \{1,3,5\}$ and $k \geq 2$, and $r \geq 6$ if $s=1$, or $r \geq 5$ if $s=3$, or $r \geq 3$ if $s=5$. Equivalently, and better for the purposes of computation (once B^6 is replaced by B^{-1}), we may assume $s \in \{2,4,6\}$ and $k \geq 2$, with $r \geq 2$ if $s=2$ or 4, or $r \geq 4$ if $s=6$.

In such cases, but again for small r and s and k, coset enumeration on a small computer easily gives results (some of which were known to Leech and Sinkov). For example, if the additional relator is any one of $(A^2B^2)^4$, $(A^2B^4)^3$ or $(A^3B^4)^3$, the presentation defines a group of order 168, which then has to be the simple group $L_2(7)$, the smallest non-trivial quotient of $\Delta(2,3,7)$. These and other results are summarised below.

FOURTH RELATOR w	GROUP
$(A^2B^2)^2$	trivial
$(A^2B^2)^3$	$L_2(13)$, of order 1092
$(A^2B^2)^4$	$L_2(7)$, of order 168
$(A^2B^2)^5$	$L_2(29)$, of order 12 180
$(A^3B^2)^2$	trivial
$(A^3B^2)^3$	$L_2(13)$, of order 1092
$(A^4B^2)^2$	$L_2(8)$, of order 504
$(A^4B^2)^3$	$L_2(29)$, of order 12 180
$(A^5B^2)^2$	trivial
$(A^5B^2)^3$	$L_2(27)$, of order 9828
$(A^6B^2)^2$	$L_2(13)$, of order 1092
$(A^7B^2)^2$	$L_2(29)$, of order 12 180
$(A^8B^2)^2$	trivial
$(A^2B^4)^2$	trivial
$(A^2B^4)^3$	$L_2(7)$, of order 168
$(A^2B^4)^4$	trivial
$(A^2B^4)^5$	$L_2(29)$, of order 12 180
$(A^3B^4)^2$	$L_2(8)$, of order 504
$(A^3B^4)^3$	$L_2(7)$, of order 168
$(A^4B^4)^2$	trivial
$(A^4B^4)^3$	$L_2(13)$, of order 1092
$(A^5B^4)^2$	$L_2(13)$, of order 1092
$(A^6B^4)^2$	$L_2(29)$, of order 12 180
$(A^7B^4)^2$	trivial
$(A^4B^6)^2$	trivial
$(A^4B^6)^3$	$L_2(13)$, of order 1092
$(A^5B^6)^2$	$L_2(13)$, of order 1092
$(A^6B^6)^2$	$L_2(13)$, of order 1092
$(A^7B^6)^2$	non-split extension of $(C_2)^6$ by $L_2(7)$, of order 10 752

Note that already this answers one of the questions raised by Leech: insertion of the additional relator $(A^4B^4)^3$ does indeed define a finite group, namely $L_2(13)$. We will give a formal proof of this fact in the next section.

On a larger machine, further results are possible. In particular, we have found the following (by coset enumeration on a SUN workstation):

(a) when $w=(A^3B^2)^4$ or $(A^4B^6)^4$, the group is $L_2(71)$, of order 178 920,

(a) when $w=(A^5B^4)^3$ the group is $L_2(13)$, of order 1 192 464, following (by coset enumeration on a SUN workstation):

(a) when $w=(A^3B^2)^4$ or $(A^4B^6)^4$, the group is $L_2(71)$, of order 178 920,

(b) when $w=(A^5B^4)^3$, the group is $L_2(13)\times L_2(13)$, of order 1 192 464,

(c) when $w=(A^2B^4)^6$, the group is an extension of $(C_2)^6$ by $L_2(7)\times L_2(13)$, of order 11 741 184.

(d) when $w=(A^3B^4)^4$, the group is an extension of a group of order 2^{15} by $L_2(8)$, of order 16 515 072.

Result (c) provides a surprising answer to another of the questions raised by Leech, who had almost convinced himself the group was infinite. He observed that the relation $(A^2B^4)^6=1$ was a consequence of each of $A^7=1$ and $A^8=1$, or equivalently, that the Hurwitz groups of orders 1092 and 10 752 are both factor groups of

$$\langle A,B \mid (AB)^2=(A^{-1}B)^3=B^7=(A^2B^4)^6=1\rangle.$$

What he did not expect was that the latter group is the direct product of these two Hurwitz groups! We first found this by using the Reidemeister-Schreier process to obtain a presentation for the normal subgroup of index 168 in the group in question, and then performing a coset enumeration on that, but have now confirmed the result by direct enumeration of cosets of the subgroup generated by B. Result (d), also found in answer to a question of Leech, will be verified in the next section.

3. Theoretical results. Three of the computational results mentioned above can be verified using some of the information given by Leech in **[7]**, as follows.

Proposition 3.1. *The group* $\langle A,B \mid (AB)^2=(A^{-1}B)^3=B^7=(A^4B^4)^3=1\rangle$ *is isomorphic to the simple group* $L_2(13)$ *of order* 1092.

Proof. First we note that one of the three normal subgroups of index 1092 in the $(2,3,7)$ triangle group is generated by the conjugates of A^6 $(=(y^{-1}xyx)^6)$. Indeed if $a_0=A^6$ and $a_i=B^{-i}a_0B^i$ for $1\le i\le 6$, and $b_0=Aa_1A^{-1}$ and $c_0=A^2a_5A^{-2}$ and $f_0=A^2a_3A^{-2}a_0$, with $b_i=B^{-i}b_0B^i$ and $c_i=B^{-i}c_0B^i$ and $f_i=B^{-i}f_0B^i$ also for $1\le i\le 6$, then the elements a_i, b_i, c_i and f_i (for $0\le i\le 6$) generate this normal subgroup K. Moreover, letting $e_0=f_0^{-1}b_0^{-1}f_5c_2^{-1}a_2b_3c_3a_6^{-1}$ and $e_i=B^{-i}e_0B^i$ for $1\le i\le 6$, the identical relation $e_0e_2e_4e_6e_1e_3e_5=1$ is a defining relation for K, and the conjugates of the 28 generators by A may be given as follows (see [**7**, §8]):

$$\begin{array}{llll}
a_0^A=a_0, & b_0^A=a_1, & c_0^A=b_2, & f_0^A=e_1a_0,\\
a_1^A=a_6, & b_1^A=a_0^{-1}b_0^{-1}a_6^{-1}, & c_1^A=a_6b_0f_0e_0, & f_1^A=a_6b_0c_0a_3^{-1},\\
a_2^A=b_6, & b_2^A=a_5, & c_2^A=a_5^{-1}b_5^{-1}a_4^{-1}, & f_2^A=a_5^{-1}e_6^{-1}f_6^{-1},\\
a_3^A=a_2^{-1}, & b_3^A=c_6, & c_3^A=c_6^{-1}b_6^{-1}a_5^{-1}c_5f_1^{-1}, & f_3^A=f_2^{-1},\\
a_4^A=e_0, & b_4^A=e_5, & c_4^A=e_5^{-1}e_3^{-1}a_0, & f_4^A=e_5^{-1}f_5^{-1}b_0f_0e_0,\\
a_5^A=a_4^{-1}, & b_5^A=e_2, & c_5^A=e_4, & f_5^A=a_1c_5^{-1}b_5^{-1}a_4^{-1},\\
a_6^A=b_1, & b_6^A=a_4c_1^{-1}b_1^{-1}, & c_6^A=b_1c_1f_4^{-1}b_6, & f_6^A=a_0^{-1}c_0f_3^{-1}b_3^{-1}.
\end{array}$$

Also in terms of these generators, $(A^4B^4)^3=a_0b_6^{-1}a_4$, so the insertion of the extra relation $(A^4B^4)^3=1$ forces $b_6=a_4a_0$ and therefore $b_i=a_{i+5}a_{i+1}$ for all i (modulo 7). But then $a_5=b_2^A=(a_0a_3)^A=a_0a_2^{-1}$, so $a_i=a_{i+5}a_{i+2}$ for all i (modulo 7); in particular this implies $a_1a_0=a_1a_5a_2=a_3a_2$, and it follows that the products $a_{i+1}a_i$ are all equal. Letting their common value be z (say), we find $a_iz=a_ia_{i+6}a_{i+5}=za_{i+5}$ for all i, and further, $e_0=a_4^A=(a_2a_6)^A=b_6b_1=a_4a_0a_6a_2=a_4za_2$, so $e_i=a_{i+4}za_{i+2}$ for all i.

The identícal relation $e_0e_2e_4e_6e_1e_3e_5 = 1$ now gives

$$\begin{aligned}1 &= a_4za_2a_6za_4a_1za_6a_3za_1a_5za_3a_0za_5a_2za_0\\ &= a_4za_4za_6za_1za_3za_5za_0za_0\\ &= a_4a_6a_3a_0a_4a_1a_5a_0z^7 \quad (\text{since } za_{i+5} = a_iz)\\ &= a_4a_1a_2a_3a_0z^7\\ &= a_6a_2a_5z^7.\end{aligned}$$

Conjugating by a_4 we get $1 = a_4^{-1}a_6a_2a_5z^7a_4 = a_1a_2z^7a_5a_4 = a_1a_2z^8$, so that $a_ia_{i+1} = z^{-8}$ for all i. But on the other hand, $a_6a_0 = (a_1a_0)^A = z^A = (a_0a_6)^A = a_0b_1 = a_0a_6a_2$, which gives $z^{-8} = za_2$ and therefore $a_i = z^{-9}$ for all i. Hence the a_i all coincide, and it follows (e.g. from $a_i = a_{i+5}a_{i+2}$) that they are all trivial.

In other words, the group with presentation

$$\langle A, B \mid (AB)^2 = (A^{-1}B)^3 = B^7 = (A^4B^4)^3 = 1\rangle$$

is isomorphic to the quotient of $(2,3,7)$ by the normal subgroup K generated by conjugates of A^6, namely $L_2(13)$.

Proposition 3.2. *The group $\langle A, B \mid (AB)^2 = (A^{-1}B)^3 = B^7 = (A^3B^4)^4 = 1\rangle$ is an extension by $L_2(8)$ of a metabelian 2-group of order 2^{15} and exponent* 4.

Proof. This is similar to the proof of Proposition 3.1. First, the (unique) normal subgroup of index 504 in the $(2,3,7)$ triangle group is generated by the conjugates of $(A^3B^4)^2$; indeed if $a_0 = (A^3B^4)^2$ and $b_0 = A^{-1}a_0A$, then the elements $a_i = B^{-i}a_0B^i$ and $b_i = B^{-i}b_0B^i$(for $0 \le i \le 6$) generate this normal subgroup, subject to a single defining relation

$$a_0b_2^{-1}a_6^{-1}b_6a_5b_0^{-1}a_4^{-1}b_4a_3b_5^{-1}a_2^{-1}b_2a_1b_3^{-1}a_0^{-1}b_0a_6b_1^{-1}a_5^{-1}b_5a_4b_6^{-1}a_3^{-1}b_3a_2b_4^{-1}a_1^{-1}b_1 = 1.$$

This time the conjugates of the generators by A are as follows (see [**7**, §5]):

$$\begin{array}{ll} a_0^A = b_0, & b_0^A = b_6^{-1},\\ a_1^A = a_6b_1^{-1}a_5^{-1}b_5, & b_1^A = a_6,\\ a_2^A = a_1^{-1}, & b_2^A = b_3^{-1}a_0^{-1}b_0,\\ a_3^A = b_0^{-1}a_4^{-1}b_4a_3, & b_3^A = b_0^{-1}a_4^{-1},\\ a_4^A = b_5^{-1}, & b_4^A = b_5^{-1}a_5,\\ a_5^A = b_2^{-1}a_6^{-1}, & b_5^A = b_2^{-1}a_2b_5,\\ a_6^A = a_3^{-1}b_3, & b_6^A = a_3^{-1}. \end{array}$$

The insertion of the extra relation $(A^3B^4)^4 = 1$ is equivalent to forcing any/all of the a_i and b_i to have order 2, and doing so gives the following:

(1) $1 = (a_6^2)^A = (a_3^{-1}b_3)^2 = [a_3, b_3]$, so $[a_i, b_i] = 1$ for all i,
(2) $1 = (a_5^2)^A = (b_2^{-1}a_6^{-1})^2 = [b_2, a_6]$, so $[a_i, b_{i+3}] = 1$ for all i,
(3) $1 = (b_5^2)^A = (b_2^{-1}a_2b_5)^2 = a_2^2(b_2b_5)^2 = [b_2, b_5]$, so $[b_i, b_{i+3}] = 1$ for all i,
(4) $1 = [a_0, b_0]^A = [b_0, b_6^{-1}] = [b_0, b_6]$, so $[b_i, b_{i+1}] = 1$ for all i,
(5) $1 = [b_0, b_3]^A = [b_6^{-1}, b_0^{-1}a_4^{-1}] = [b_6, a_4]$, so $[a_i, b_{i+2}] = 1$ for all i,
(6) $1 = [b_4, b_0]^A = [b_5^{-1}a_5, b_6^{-1}] = [a_5, b_6]$, so $[a_i, b_{i+1}] = 1$ for all i.

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Now define $u_i=[a_i,a_{i+1}]$ and $v_i=[a_i,b_{i+5}]$ for $0\le i\le 6$. We obtain further:

(7) $1=[b_3,b_6]^A=[b_0^{-1}a_4^{-1},a_3^{-1}]=a_4b_0a_3b_0a_4a_3=a_4[b_0,a_3][a_3,a_4]a_4$, so $[a_3,b_0]=[b_0,a_3]^{-1}=[a_3,a_4]$ and therefore $[a_i,b_{i+4}]=u_i$ for all i,

(8) $1=[b_1,b_4]^A=[a_6,b_5^{-1}a_5]=a_6a_5b_5a_6b_5a_5=a_5[a_5,a_6][a_6,b_5]a_5$, so $[a_6,b_5]=[a_5,a_6]^{-1}$ and therefore $[a_i,b_{i+6}]=u_{i+6}^{-1}$ for all i,

(9) $1=[b_5,b_1]^A=[b_2^{-1}a_2b_5,a_6]=[a_2b_5,a_6]=b_5a_2a_6a_2b_5a_6=b_5[a_2,a_6][a_6,b_5]b_5$, so $[a_6,a_2]=[a_2,a_6]^{-1}=[a_6,b_5]=[a_5,a_6]^{-1}$ and therefore $[a_i,a_{i+3}]=u_{i+6}^{-1}$ for all i,

(10) $1=[b_6,b_2]^A=[a_3^{-1},b_3^{-1}a_0^{-1}b_0]=[a_3,a_0b_0]=a_3b_0a_0a_3a_0b_0=b_0[b_0,a_3][a_3,a_0]b_0$, so $u_2=[a_0,a_3]^{-1}=[a_3,a_0]=[b_0,a_3]^{-1}=[a_3,b_0]=u_3$ and therefore the u_i all coincide.

Letting $u=u_i$ (for any i), we find also:

(11) $u^A=[a_4,a_5]^A=[b_5^{-1},b_2^{-1}a_6^{-1}]=[b_5,a_6]=[a_6,b_5]^{-1}=u_5=u$, so u is centralized by both A and B (and therefore by every element of the group),

(12) $u=u^A=[a_0,b_4]^A=[b_0,b_5^{-1}a_5]=[b_0,b_5]=[b_5,b_0]^{-1}$, so $[b_i,b_{i+2}]=u^{-1}$ for all i,

(13) $u=u^A=[a_2,b_6]^A=[a_1^{-1},a_3^{-1}]=[a_1,a_3]$, so $[a_i,a_{i+2}]=u$ for all i,

(14) $u=u^A=[a_6,b_3]^A=[a_3^{-1}b_3,b_0^{-1}a_4^{-1}]=b_3a_3a_4b_0a_3b_3b_0a_4=a_3b_3a_4b_3b_0a_3b_0a_4$
$=a_3a_4[a_4,b_3][b_0,a_3][a_3,a_4]a_4a_3=a_3a_4u_3^{-1}u_3^{-1}u_3a_4a_3=a_3a_4u^{-1}a_4a_3=u^{-1}$, so $u^2=1$,

(15) $u=u^A=[b_2,b_4]^A=[b_3^{-1}a_0^{-1}b_0,b_5^{-1}a_5]=b_0a_0b_3a_5b_5b_3a_0b_0b_5a_5=a_0b_3a_5b_5b_3a_0b_5a_5u$

$$=a_0b_5b_3a_5b_3a_0b_5a_5=a_0b_5a_0b_3a_5b_3b_5a_5u=a_0b_5a_0b_5b_3a_5b_3a_5u=v_0v_5^{-1}u, \text{ so } v_0=v_5$$

and therefore the v_i all coincide.

Finally, letting $v=v_i$ (for any i), we obtain:

(16) $v^A=[a_2,b_0]^A=[a_1,b_6]=v$, so v is central,

(17) $v=v^A=[a_0,b_5]^A=[b_0,b_2^{-1}a_2b_5]=b_0b_5a_2b_2b_0b_2a_2b_5=b_0b_5a_2b_0a_2b_5u=b_0a_2b_0a_2=v^{-1}$, so $v^2=1$, and the identical relation gives

$$\begin{aligned}
(18)\quad 1&=a_0b_2a_6b_6a_5b_0a_4b_4a_3b_5a_2b_2a_1b_3a_0b_0a_6b_1a_5b_5a_4b_6a_3b_3a_2b_4a_1b_1\\
&=a_0b_2a_6b_6a_5b_0a_4b_4a_3b_5a_2b_2a_1b_3a_0b_0a_6a_5b_5a_4b_6a_3b_3a_2b_4a_1v\\
&=a_0b_2a_6b_6a_5b_0a_4b_4a_3b_5a_2b_2a_1a_0b_0a_6a_5b_5a_4b_6a_3a_2b_4a_1u\\
&=a_0b_2a_6b_6a_5b_0a_4b_4a_3b_5b_2a_1a_0b_0a_6a_5b_5a_4b_6a_3b_4a_1v\\
&=a_0b_2a_6b_6a_5b_0a_4b_4a_3b_5b_2a_0b_0a_6a_5b_5a_4b_6a_3b_4u\\
&=a_0b_2a_6b_6a_5b_0a_4b_4a_3b_2a_0b_0a_6a_5a_4b_6a_3b_4uv\\
&=a_0b_2a_6b_6a_5b_0a_4b_4b_2a_0b_0a_6a_5a_4b_6b_4uv\\
&=a_0b_2a_6b_6a_5b_0a_4b_2a_0b_0a_6a_5a_4b_6u\\
&=a_0b_2a_6b_6a_5a_4b_2a_0a_6a_5a_4b_6\\
&=a_0b_2a_6b_6a_5b_2a_0a_6a_5b_6uv\\
&=a_0b_2a_6b_6b_2a_0a_6b_6v\\
&=a_0b_2a_6b_2a_0a_6uv\\
&=a_0a_6a_0a_6uv\\
&=v,
\end{aligned}$$

that is, $v=1$.

As the commutators of all possible pairs of the a_i and b_i are now accounted for, it follows that in the group $\langle A, B \mid (AB)^2 = (A^{-1}B)^3 = B^7 = (A^3B^4)^4 = 1\rangle$, the conjugates of $(A^3B^4)^2$ generate a normal subgroup whose derived group is central and cyclic (generated by u), of order 2. This, together with the coset enumeration described in Section 2 to verify the order, completes the proof.

Proposition 3.3. *The group* $\langle A, B \mid (AB)^2 = (A^{-1}B)^3 = B^7 = (A^7B^6)^2 = 1\rangle$ *is a non-split extension by* $L_2(7)$ *of an elementary Abelian group of order* 2^6.

Proof. First the relations imply $BA^8B^{-1} = BAA^7B^6 = A^{-1}B^{-1}A^7B^6 = A^{-8}(A^7B^6)^2 = A^{-8}$, so that B inverts A^8 by conjugation, but then of course $A^8 = B^7A^8B^{-7} = A^{-8}$, so in fact A^8 is central. This in turn implies that A^8 is trivial (as shown by Leech in [**6**], or by the same sort of argument as in the two proofs above, using the information supplied in [**7**, §4]). On the other hand, if $A^8 = 1$ then clearly $(A^7B^6)^2 = (A^{-1}B^{-1})^2 = 1$, and so the relation $(A^7B^6)^2 = 1$ is equivalent to $A^8 = 1$ (given the three initial relations $(AB)^2 = (A^{-1}B)^3 = B^7 = 1$). In particular, the result now follows from [**10**].

4. Further computational results and final comments. The most difficult of the four cases Leech asked about is that of $(A^3B^2)^5$. As he remarked in [**7**], the group obtained from $\Delta(2,3,7)$ by inserting this word as a fourth relator has $L_2(29)$ as a factor group. But it turns out also to have another simple group as a factor group, namely the group J_1; see presentation 15.7 in [**1**], noting that $A^3B^2 = (y^{-1}xyx)^3y^{-1}x$. In particular, if this group is finite then its order is at least $12\,180 \times 175\,560$, that is, $2\,138\,320\,800$. On the other hand of course it may be infinite, however we have not been able to show this, despite several attempts.

Similarly in the case of $(A^9B^2)^2$, both $L_2(113)$ and J_1 turn out to be quotients. In fact it has the group $G^{3,7,19} = \langle x, y, t \mid x^2 = y^3 = (xy)^7 = t^2 = (xt)^2 = (yt)^2 = (xyt)^{19} = 1\rangle$ as a quotient, for in this group $1 = (y^{-1}xt)^{19} = (y^{-1}xty^{-1}xt)^9y^{-1}xt = (y^{-1}xyx)^9y^{-1}xt$ and thus $A^9B^2 = (y^{-1}xyx)^9y^{-1}x = t$, which has order 2. It is still unknown, however, whether or not the group $G^{3,7,19}$ is finite!

On the other hand, the group $\langle A, B \mid (AB)^2 = (A^{-1}B)^3 = B^7 = (A^5B^6)^3 = 1\rangle$ is easily shown to be infinite: it has the extension of a 6-generator free Abelian group by the simple group $L_2(7)$ as a quotient. To see this, let $a_0 = A^4$ and $a_i = B^{-i}a_0B^i$ for $1 \le i \le 6$, these being generators for the (unique) normal subgroup of index 168 in $\Delta(2,3,7)$, given by Leech in [**7**, §4]. Then $(A^5B^6)^3 = (A^5B^{-1})^3 = A^4AB^{-1}AA^4B^{-1}A^5B^{-1} = A^4BA^{-1}BA^4B^{-1}AB^{-1}BA^4B^{-1} = a_0a_6^{-1}a_0^{-1}a_6$ (noting here that $BA^{-1}BA^4B^{-1}AB^{-1} = BA^{-1}Ba_0B^{-1}AB^{-1} = BA^{-1}a_6AB^{-1} = Ba_0^{-1}a_1^{-1}B^{-1} = a_6^{-1}a_0^{-1}$, by Leech's own calculations in [**7**]). It follows that the relation $(A^5B^6)^3 = 1$ is a consequence of the relations $[a_i, a_j] = 1$, which are enough to define the sort of extension described.

More easily, we can show $\langle A, B \mid (AB)^2 = (A^{-1}B)^3 = B^7 = (A^kB^6)^3 = 1\rangle$ is infinite for all $k \ge 6$, for in each case here the fourth relation $(A^kB^6)^3 = 1$ is a consequence of $A^{k+3} = 1$: if $A^{k+3} = 1$ then $(A^kB^6)^3 = (A^{-3}B^{-1})^3 = (A^{-2}BA)^3 = A^{-1}(A^{-1}B)^3A = 1$. Similarly, for all $k \ge 8$ the group $\langle A, B \mid (AB)^2 = (A^{-1}B)^3 = B^7 = (A^kB^6)^2 = 1\rangle$ is infinite, as $(A^kB^6)^2 = (A^{-1}B^{-1})^2 = 1$ whenever $A^{k+1} = 1$.

Also when the fourth relator is $(A^2B^4)^7$, the group is infinite, but for slightly different reasons. In this case the group has a subgroup of index 42 (generated by AB, $B^2AB^{-2}AB$, $B^{-3}A^2B^2AB^{-2}AB^{-3}$ and $B^{-3}AB^{-2}AB^{-3}AB^{-2}A^2B^{-2}AB^{-3}AB^3AB^{-1}$) with an infinite cyclic quotient. Such a subgroup can be found using the low index subgroups algorithm, and its Abelianization determined by the Reidemeister-Schreier process.

Undoubtedly there are other examples where the group is infinite—indeed one would expect it to be so in all but a few cases where the fourth relator is a proper power—and methods such as those described in [**4**], [**5**] and [**11**] are most likely to be useful in this respect. What is perhaps surprising is the number and type of finite examples that arise, especially those direct products and soluble-by-simple extensions we have found.

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REFERENCES

1. C. M. Campbell and E. F. Robertson, Presentations for the simple groups G, $10^5 < |G| < 10^6$, *Comm. Algebra* **12** (1984), 2643–2663.

2. M. D. E. Conder, Hurwitz groups: a brief survey, *Bull. Amer. Math. Soc.* (*N. S.*) **23** (1990), 359–370.

3. M. Edjvet, An example of an infinite group, in *Discrete groups and geometry* (ed. W. J. Harvey and C. Maclachlan), London Math. Soc. Lecture Note Ser. 173 (Cambridge University Press, 1992), 66–74.

4. D. F. Holt and W. Plesken, A cohomological criterion for a finitely presented group to be infinite, *J. London Math. Soc.* (2) **45** (1992), 469–480.

5. J. Howie and R. M. Thomas, The groups $(2,3,p;q)$; asphericity and a conjecture of Coxeter, *J. Algebra* **154** (1993), 289–309.

6. J. Leech, Some definitions of Klein's simple group of order 168 and other groups, *Proc. Glasgow Math. Assoc.* **5** (1962), 166–175.

7. J. Leech, Generators for certain normal subgroups of $(2,3,7)$, *Proc. Cambridge Philos. Soc.* **61** (1965), 321–332.

8. J. Leech, Note on the abstract group $(2,3,7;9)$, *Proc. Cambridge Philos. Soc.* **62** (1966), 7–10.

9. J. Leech, Computer proof of relations in groups, in *Topics in group theory and computation* (ed. M. P. J. Curran) (Academic Press, 1977), 38–61.

10. J. Leech and J. Mennicke, Note on a conjecture of Coxeter, *Proc. Glasgow Math. Assoc.* **5** (1961), 25–29.

11. R. M. Thomas, Cayley graphs and group presentations, *Math. Proc. Cambridge Philos. Soc.* **103** (1988), 385–387.

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