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Near-Homeomorphisms of Nöbeling Manifolds

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Abstract. We characterize maps between n-dimensional Nöbeling manifolds that can be approximated by homeomorphisms.

1 Introduction

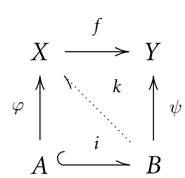
A long standing problem (see, for example, [8, TC 10], [3, Conjecture 5.0.5]) of characterizing topologically universal n-dimensional Nöbeling space, as well as manifolds modeled on it, was solved recently by M. Levin [5] and A. Nagórko [7]. The Theory of Nöbeling manifolds, developed in [5–7] based on completely different approaches, among other things contains various versions of Z-set unknotting theorem, open embedding theorem, n-homotopy classification theorem, etc.

In this note we complete the picture by proving that for n-dimensional Nöbeling manifolds classes of near-homeomorphisms, approximately n-soft maps, fine n-homotopy equivalences and UV^{n-1}-mappings coincide. Recall that an n-dimensional Nöbeling manifold is a Polish space locally homeomorphic to ν^n, the subset of $\mathbb{R}^{2n+1}$ consisting of all points with at most n rational coordinates.

Definition 1.1 For each map f from a space X into a space Y, for each open cover $\mathcal{U}$ of Y and for each integer n, we define the following conditions.

($\mathrm{NH}_{\mathcal{U}}$) There exists a homeomorphism of X and Y that is $\mathcal{U}$-close to f.

($\mathrm{A}n\mathrm{S}_{\mathcal{U}}$) For each at most n-dimensional metric space B, its closed subset A, and maps φ and ψ such that the diagram



commutes, there exists a map $k\colon B \to X$ such that $k|A = \varphi$ and $f \circ k$ is $\mathcal{U}$-close to ψ.

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($\mathrm{F}n\mathrm{HE}_{\mathcal{U}}$) There exists a map g from Y to X such that $f \circ g$ is $\mathcal{U}$-n-homotopic[1] to the identity on Y and $g \circ f$ is $f^{-1}(\mathcal{U})$-n-homotopic to the identity on X (with $f^{-1}(\mathcal{U})$ denoting $\{f^{-1}(U)\}_{U \in \mathcal{U}}$).

($\mathrm{UV}^{n-1}_{\mathcal{U}}$) The star of the image of f in $\mathcal{U}$ is equal to Y, and there is an open cover $\mathcal{W}$ of Y such that for each W in $\mathcal{W}$ there exists U in $\mathcal{U}$ such that the inclusion $f^{-1}(W) \subset f^{-1}(U)$ induces trivial (zero) homomorphisms on homotopy groups of dimensions less than n, regardless of the choice of the base point.

Our main result is the following theorem.

Theorem 1.2 *For each open cover $\mathcal{U}$ of an n-dimensional Nöbeling manifold Y there exists an open cover $\mathcal{V}$ such that for each map f from an n-dimensional Nöbeling manifold into Y, if* ($\mathrm{F}n\mathrm{HE}_{\mathcal{V}}$)*, then* ($\mathrm{NH}_{\mathcal{U}}$).

Theorem 1.2 is an analogue of theorems of Ferry on Hilbert space and Hilbert cube manifolds [4] and of a theorem of Chapman and Ferry on euclidean manifolds [2].

Let ($\mathrm{P}_{\mathcal{U}}$), ($\mathrm{Q}_{\mathcal{U}}$), and ($\mathrm{R}_{\mathcal{U}}$) be any of the predicates stated in Definition 1.1. We are interested in which of the implications

$$\forall_{\mathcal{U}} \exists_{\mathcal{V}} \forall_f (\mathrm{P}_{\mathcal{V}}) \Rightarrow (\mathrm{Q}_{\mathcal{U}})$$

are true. We show that if Y is an $ANE(n)$-space, then, with quantifiers understood to be same as above, ($\mathrm{NH}_{\mathcal{V}}$) $\Rightarrow$ ($\mathrm{A}n\mathrm{S}_{\mathcal{U}}$) (Lemma 2.2), ($\mathrm{A}n\mathrm{S}_{\mathcal{V}}$) $\Rightarrow$ ($\mathrm{F}n\mathrm{HE}_{\mathcal{U}}$) (Lemma 2.4) and ($\mathrm{F}n\mathrm{HE}_{\mathcal{V}}$) $\Rightarrow$ ($\mathrm{UV}^{n-1}_{\mathcal{U}}$) (Lemma 2.5). These implications are standard. To complete the picture, we give an example that shows that ($\mathrm{UV}^{n-1}_{\mathcal{V}}$) does not imply ($\mathrm{F}n\mathrm{HE}_{\mathcal{U}}$), even if X and Y are Nöbeling manifolds (Example 2.6).

Observe that we have the following rule of inference:

$$\big(\forall_{\mathcal{U}} \exists_{\mathcal{V}} \forall_f (\mathrm{P}_{\mathcal{V}}) \Rightarrow (\mathrm{Q}_{\mathcal{U}})\big) \wedge \big(\forall_{\mathcal{U}} \exists_{\mathcal{V}} \forall_f (\mathrm{Q}_{\mathcal{V}}) \Rightarrow (\mathrm{R}_{\mathcal{U}})\big) \Rightarrow \big(\forall_{\mathcal{U}} \exists_{\mathcal{V}} \forall_f (\mathrm{P}_{\mathcal{V}}) \Rightarrow (\mathrm{R}_{\mathcal{U}})\big).$$

Hence the above mentioned implications yield the following theorem.

Theorem 1.3 *For each open cover $\mathcal{U}$ of an n-dimensional Nöbeling manifold Y there exists an open cover $\mathcal{V}$ such that for each map f from an n-dimensional Nöbeling manifold X into Y if*

$$(\mathrm{NH}_{\mathcal{V}}) \text{ or } (\mathrm{A}n\mathrm{S}_{\mathcal{V}}) \text{ or } (\mathrm{F}n\mathrm{HE}_{\mathcal{V}}),$$

then

$$(\mathrm{NH}_{\mathcal{U}}) \text{ and } (\mathrm{A}n\mathrm{S}_{\mathcal{U}}) \text{ and } (\mathrm{F}n\mathrm{HE}_{\mathcal{U}}).$$

Now consider absolute versions of conditions stated in Definition 1.1.

[1]See section 2 for definitions.

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Definition 1.4 For each map f from a space X into a space Y we say that (NH) ((AnS), (FnHE), or (UV^{n-1}), respectively) is satisfied if for each open cover $\mathcal{U}$ of Y (NH$_{\mathcal{U}}$) ((AnS$_{\mathcal{U}}$), (FnHE$_{\mathcal{U}}$), or (UV$^{n-1}_{\mathcal{U}}$), respectively) is satisfied.

If a map satisfies (NH), then we say that it is a *near-homeomorphism*. If it satisfies (AnS), then we say that it is *approximately n-soft*. If it satisfies (FnHE), then we say that it is a *fine n-homotopy equivalence*. If it satisfies (UV^{n-1}), then we say that it is a *UV^{n-1}-map*.

We shall show that if Y is an $ANE(n)$-space, then (UV^{n-1}) ⇒ (FnHE) (Lemma 2.7, which contrasts Example 2.6). Hence we have

$$(\mathrm{NH}) \Rightarrow (\mathrm{A}n\mathrm{S}) \Rightarrow (\mathrm{F}n\mathrm{HE}) \Leftrightarrow (\mathrm{UV}^{n-1}).$$

The above implications combined with Theorem 1.2 yield the following theorem.

Theorem 1.5 *The following conditions are equivalent for each map $f\colon X \to Y$ of n-dimensional Nöbeling manifolds:*

(NH) *f is a near-homeomorphism,*
(AnS) *f is approximately n-soft,*
(FnHE) *f is a fine n-homotopy equivalence,*
(UV^{n-1}) *f is an UV^{n-1}-map.*

2 Preliminaries

Definition 2.1 We say that a metric space X is an *absolute neighborhood extensor in dimension n* if it is a metric space and if every map into X from a closed subset A of an n-dimensional metric space extends over an open neighborhood of A. The class of absolute neighborhood extensors in dimension n is denoted by $ANE(n)$ and its elements are called *$ANE(n)$-spaces*.

Lemma 2.2 *For each open cover $\mathcal{U}$ of an $ANE(n)$-space Y there exists an open cover $\mathcal{V}$ such that if a map into Y satisfies* (NH$_{\mathcal{V}}$)*, then it satisfies* (AnS$_{\mathcal{U}}$).

Proof Choose open covers $\mathcal{V}$ and $\mathcal{W}$ of Y such that the star of $\mathcal{W}$ refines $\mathcal{U}$ and the following condition is satisfied [3, Proposition 4.1.7] for each at most n-dimensional metric space B and its closed subset A.

($*$) *If one of two $\mathcal{V}$-close maps of A into Y has an extension to B, then the other also has an extension to B and we may assume that these extensions are $\mathcal{W}$-close.*

Let A be a closed subset of an at most n-dimensional metric space B, and let maps $\varphi\colon A \to X$ and $\psi\colon B \to Y$ be such that $f\varphi = \psi|A$. By (NH$_{\mathcal{V}}$), there exists a homeomorphism $g\colon X \to Y$, which is $\mathcal{V}$-close to f. By the above stated property of $\mathcal{V}$, there exists a $\mathcal{W}$-close to ψ extension $h\colon B \to Y$ of the composition $g\varphi$. Let $k = g^{-1}h\colon B \to X$. Clearly, $k|A = \varphi$ and fk is $\mathcal{U}$-close to ψ. ■

Definition 2.3 Let $\mathcal{U}$ be an open cover of a space Y. We say that maps $f, g\colon X \to Y$ are *n-homotopic* if for every map Φ from a polyhedron of dimension less than n into

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X, the compositions $f \circ \Phi$ and $g \circ \Phi$ are homotopic by a homotopy whose paths refine $\mathcal{U}$.

Lemma 2.4 *For each open cover $\mathcal{U}$ of an at most n-dimensional ANE(n)-space Y there exists an open cover $\mathcal{V}$ such that if a map into Y satisfies* ($\mathrm{A}n\mathrm{S}_{\mathcal{V}}$)*, then it satisfies* ($\mathrm{F}n\mathrm{HE}_{\mathcal{U}}$).

Proof Choose open covers $\mathcal{V}$ and $\mathcal{W}$ of Y such that $\mathrm{st}_{\mathcal{V}}\,\mathrm{st}\,\mathcal{W}$ refines $\mathcal{U}$ and condition $(*)$ defined in the proof of Lemma 2.2 is satisfied. By ($\mathrm{A}n\mathrm{S}_{\mathcal{V}}$), there exists a map $g\colon Y \to X$ such that $f \circ g$ is $\mathcal{V}$-close to the identity on Y. By $(*)$, any two $\mathcal{V}$-close maps from an at most n-dimensional metric space are $\mathrm{st}\,\mathcal{W}$-n-homotopic. Hence $f \circ g$ is $\mathcal{U}$-n-homotopic to the identity on Y. Let k be a map into X defined on an at most $(n-1)$-dimensional polyhedron K. Let $l = g \circ f \circ k$. Since $f \circ g$ is $\mathcal{V}$-close to the identity on Y, $f \circ k$ is $\mathcal{V}$-close to $f \circ l$. By $(*)$, there exists a $\mathrm{st}\,\mathcal{W}$-homotopy $H\colon K \times [0,1] \to Y$ of $f \circ k$ and $f \circ l$. By ($\mathrm{A}n\mathrm{S}_{\mathcal{V}}$), this homotopy can be lifted to a homotopy of k and l in Y, whose composition with f is $\mathcal{V}$-close to H. Since $\mathrm{st}_{\mathcal{V}}\,\mathrm{st}\,\mathcal{W}$ refines $\mathcal{U}$, this composition is a $\mathcal{U}$-homotopy. Hence H is a $f^{-1}(\mathcal{U})$-homotopy and $g \circ f$ is $f^{-1}(\mathcal{U})$-n-homotopic to the identity on X. ■

Lemma 2.5 *For each open cover $\mathcal{U}$ of an ANE(n)-space Y there exists an open cover $\mathcal{V}$ such that if a map into Y satisfies* ($\mathrm{F}n\mathrm{HE}_{\mathcal{V}}$)*, then it satisfies* ($\mathrm{UV}^{n-1}_{\mathcal{U}}$).

Proof Let $\mathcal{W}$ be an open cover of Y whose star refines $\mathcal{U}$. By Theorem [3, 2.1.12], there exists an open cover $\mathcal{V}$ of Y such that for each V in $\mathcal{V}$ there exists W_V in $\mathcal{W}$, for which the inclusion $V \subset W_V$ induces trivial homomorphisms on homotopy groups of dimensions less than n. Let $k < n$. Let V in $\mathcal{V}$. Let $\varphi\colon S^k \to f^{-1}(V)$. We will show that φ is null-homotopic in $f^{-1}(\mathrm{st}_{\mathcal{V}}\,W_V)$, which will end the proof, as $\mathrm{st}_{\mathcal{V}}\,\mathcal{W}$ refines $\mathcal{U}$. By ($\mathrm{F}n\mathrm{HE}_{\mathcal{V}}$), there exists a map $g\colon Y \to X$ such that $g \circ f$ is $f^{-1}(\mathcal{V})$-n-homotopic with the identity on X and $f \circ g$ is $\mathcal{V}$-close to the identity on Y. In particular, φ is homotopic with $g \circ f \circ \varphi$ in $st_{f^{-1}(\mathcal{V})} f^{-1}(V) \subset f^{-1}(\mathrm{st}_{\mathcal{V}}\,V) \subset f^{-1}(\mathrm{st}_{\mathcal{V}}\,W_V)$. By the assumptions, $f \circ \varphi$ is null-homotopic in W_V. Hence $g \circ f \circ \varphi$ is null-homotopic in $g(W_V) \subset f^{-1}(\mathrm{st}_{\mathcal{V}}\,W_V)$. We are done. ■

Example 2.6 We show that there exists a space Y and an open cover $\mathcal{U}$ of Y such that for each open cover $\mathcal{V}$ of Y there exists a map f from a space X into Y such that ($\mathrm{UV}^{n-1}_{\mathcal{V}}$) is satisfied, but both ($\mathrm{F}n\mathrm{HE}_{\mathcal{U}}$) and ($\mathrm{A}n\mathrm{S}_{\mathcal{U}}$) are not. We give an example for $n > 1$ and the map that we construct is onto Y. For $n = 1$ an example can also be constructed, but the map cannot have a dense image in Y.

Let Y be the unit interval $[0,1]$ and let $\mathcal{U} = \{[0,1]\}$ be the trivial cover of Y. Let $\mathcal{V}$ be an open cover of Y and let V be an element of $\mathcal{V}$ that contains $\frac{1}{2}$. Let $\varepsilon > 0$ such that $[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon] \subset V$. Let $X = [0,1] \times [0,1] \setminus [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon] \times \{\frac{1}{2}\}$. Let $f\colon X \to Y$ be a restriction to X of the projection of $[0,1] \times [0,1]$ onto the first coordinate. We can verify that f satisfies ($\mathrm{UV}^{n-1}_{\mathcal{V}}$) from the definition, taking any $\mathcal{W}$ that refines $\mathcal{V}$ and whose mesh is smaller than 2ε. As X is homotopy equivalent to a circle and Y is contractible, f does not induce a monomorphism on fundamental groups of X and Y. This is easily seen to contradict both ($\mathrm{F}n\mathrm{HE}_{\mathcal{U}}$) and ($\mathrm{A}n\mathrm{S}_{\mathcal{U}}$) for $n > 1$.

It is easy to modify the above example is such a way that X and Y are n-dimensional Nöbeling manifolds.

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Lemma 2.7 *If a map into an at most n-dimensional ANE(n)-space satisfies* (UV^{n-1}), *then it satisfies* (FnHE).

Proof Let f be a UV^{n-1}-map from a space X into an $ANE(n)$-space Y. We will show by induction that for each $0 \le k \le n$, f satisfies (AnS) for polyhedral pairs (A, B) such that $\dim A \setminus B \le k$. Let (P_k) denote the last condition. For $k = 0$ the assertion is obvious, as (UV^{n-1}) implies that f has dense image in Y. Assume that $k > 0$. Let $\mathcal{U}$ be an open cover of Y. By (UV^{n-1}), there exists an open cover $\mathcal{W}$ of Y such that for each $W \in \mathcal{W}$ there exists $U_W \in \mathcal{U}$ such that the inclusion of $f^{-1}(W)$ into $f^{-1}(U_W)$ induces zero homomorphisms on homotopy groups of dimensions less than n. Let $\mathcal{S}$ be an open cover whose star refines $\mathcal{W}$. Let B be a subpolyhedron of an at most n-dimensional polyhedron A such that $\dim A \setminus B \le k$. Let maps $\varphi\colon B \to X$ and $\psi\colon A \to Y$ be such that $f \circ \varphi = \psi_{|B}$. Fix a triangulation of A such that for each simplex δ of this triangulation $\psi(\delta) \subset S$ for some $S \in \mathcal{S}$. By (P_{k-1}), we may extend φ over the $(k-1)$-dimensional skeleton of A to a map k in such a way that $f \circ k$ is $\mathcal{S}$-close to ψ. Consider an k-dimensional simplex δ of A. Observe that k maps boundary of δ into the inverse image $f^{-1}(W)$ of an element W of $\mathcal{W}$. Hence, k extends over δ to a map into the inverse image $f^{-1}(U_W)$. Extend k in this manner over all k-dimensional simplexes of $A \setminus B$ and observe that $f \circ k$ is $\mathcal{U}$-close to φ. This completes the inductive step and a proof that (P_n) holds for f.

Let $\mathcal{U}$ be an open cover of Y. We will show that $(\mathrm{F}n\mathrm{HE}_{\mathcal{U}})$ is satisfied. Choose open covers $\mathcal{V}$ and $\mathcal{W}$ of Y such that $\mathrm{st}_{\mathcal{V}} \mathrm{st}\, \mathcal{W}$ refines $\mathcal{U}$ and condition $(*)$ defined in the proof of Lemma 2.2 is satisfied. By [3, Theorem 2.1.12(vii)], there exist an at most n-dimensional polyhedron A and two maps $q\colon Y \to A$, $p\colon A \to Y$ such that $p \circ q$ is $\mathcal{V}$-close to the identity on Y. By (P_n), there exists a map $r\colon K \to X$ such that $f \circ r$ is $\mathcal{V}$-close to p. Let $g = r \circ q$. By the construction, $f \circ g$ is $\mathrm{st}\, \mathcal{V}$-close to the identity on Y. The rest of the proof follows the proof of Lemma 2.4. ■

3 Proof that $(\mathbf{F}n\mathbf{HE}_{\mathcal{V}})$ implies $(\mathbf{NH}_{\mathcal{U}})$

For definitions of notions used throughout the proof, we refer the reader to [7]. Let $\mathcal{U}$ be an open cover of an n-dimensional Nöbeling manifold Y. Let f be a map from an n-dimensional Nöbeling manifold X into Y. Assume that q is an integer greater that a constant m obtained by [7, Lemma 8.4] applied to n. Additionally assume that q is greater than $36(5N+8)^{n-1}+3$, where N is a constant obtained by [7, Theorem 6.7]. By [7, Lemma 8.1], there exists a closed partition $\mathcal{Q} = \{Q_i\}_{i\in I}$ of Y that is q-barycentric and whose q-th star refines $\mathcal{U}$. Observe that if $\mathcal{P} = \{P_i\}_{i\in I}$ is a closed partition of X that is isomorphic to $\mathcal{Q}$, then by [7, Lemma 8.4], there exists a homeomorphism h of X and Y that maps elements of $\mathcal{P}$ into the corresponding elements of $\mathrm{st}^m \mathcal{Q}$. If $P_i \subset f^{-1}(\mathrm{st}^q_{\mathcal{Q}} Q_i)$ for each i in I, then h is $\mathcal{U}$-close to f. In this case h is a homeomorphism that we are looking for. Let $\mathcal{V}$ be an open cover of Y whose star refines $\mathcal{Q}$. Assume that f satisfies $(\mathrm{F}n\mathrm{HE}_{\mathcal{V}})$. We will show that there exists a closed partition $\mathcal{P}$ of X satisfying the above stated conditions. This will end the proof.

Our first goal is to construct an n-semiregular, closed, interior $\mathcal{N}_n$-cover of X that is isomorphic to $\mathcal{Q}$ and such that f maps its elements into l-th stars of the corresponding elements of $\mathcal{Q}$, for some constant l (we obtain $l = 9$, but the exact value is

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of no importance). By [7, Proposition 6.4], $\mathcal{Q}$ is n-semiregular. Hence there exists an anti-canonical map λ of $\mathcal{Q}$ that is an n-homotopy equivalence. By (FnHE$_{\mathcal{V}}$), there exists a map $Y \to X$ whose composition with f is $\mathcal{V}$-close to the identity. Hence there exists a map $\hat{\lambda}$ into X whose composition with f is $\mathcal{V}$-close to λ. Since X is strongly universal in dimension n, $\hat{\lambda}$ can be approximated by a closed embedding Λ whose composition with f is $\operatorname{st}\mathcal{V}$-close to λ. By the choice of $\mathcal{V}$, λ is $\mathcal{Q}$-close to $f \circ \Lambda$. Hence the composition $\lambda \circ \Lambda^{-1}$ is $\mathcal{Q}$-close to the restriction of f to $\operatorname{im}\Lambda$. Hence $\lambda(\Lambda^{-1}(f^{-1}(Q_i))) \subset \operatorname{st}_{\mathcal{Q}} Q_i$ for each i in I. By Lemma [7, 6.16], there exists an extension of $\lambda \circ \Lambda^{-1}$ to a map g from X to Y such that $g(f^{-1}(Q_i)) \subset \operatorname{st}^7_{\mathcal{Q}} Q_i$ for each i in I. This implies that g is $\operatorname{st}^7 \mathcal{Q}$-close to f and that $g^{-1}(Q_i) \subset f^{-1}(\operatorname{st}^8_{\mathcal{Q}} Q_i)$ for each i in I. Let $R_i = g^{-1}(Q_i)$ for each i in I. By the construction, $\mathcal{R} = \{R_i\}_{i\in I}$ is isomorphic to $\mathcal{Q}$. By [7, Theorem 4.5], there exists a closed interior $\mathcal{N}_n$-cover $\mathcal{P}_0 = \{P^0_i\}_{i\in I}$ of X that is a swelling of $\mathcal{R}$. By taking a small enough swelling, we may require that $P^0_i \subset f^{-1}(\operatorname{st}^9_{\mathcal{Q}} Q_i)$ for each $i \in I$. By the construction, Λ is an anticanonical map of $\mathcal{P}_0$. By [7, Lemma 2.20] and by [7, Corollary 5.1], the composition $f \circ \Lambda$ is an n-homotopy equivalence, since $f \circ \Lambda$ is Q-close to λ and λ is an n-homotopy equivalence. Since f is n-homotopy equivalence, Λ is n-homotopy equivalence and by the definition, $\mathcal{P}_0$ is n-semiregular.

Hence we constructed a cover $\mathcal{P}_0$ that satisfies the following condition.

(0) $\mathcal{P}_0 = \{P^0_i\}_{i\in I}$ is a closed star finite 0-regular n-semiregular interior $\mathcal{N}_n$-cover of X that is isomorphic to $\mathcal{Q}$ and such that $P^0_i \subset f^{-1}(\operatorname{st}^9_{\mathcal{Q}} Q_i)$ for each i in I.

Our second goal is a construction of a sequence $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ of covers of X such that for each $0 < k \leq n$ the following condition is satisfied.

(k) $\mathcal{P}_k = \{P^k_i\}_{i\in I}$ is a closed, star finite, k-regular, n-semiregular, interior $\mathcal{N}_n$-cover of X that is isomorphic to $\mathcal{Q}$ and such that $P^k_i \subset f^{-1}(\operatorname{st}^{9(5N+8)^k}_{\mathcal{Q}} Q_i)$ for each i in I.

Observe that $\mathcal{P} = \mathcal{P}_n$ satisfies the conditions stated at the beginning of the proof. Hence a construction of such a sequence will finish the proof.

Let $\mathcal{F}$ be a cover of Y that refines $\operatorname{st}^l \mathcal{Q}$ for some positive integer $l < q$. Let p be a positive integer such that $2^{p-2} - 1 < l \leq 2^{p-1} - 1$. By the assumption that q is big enough, $\mathcal{Q}$ is p-barycentric. Hence by Lemma [7, Lemmas 6.12 and 6.15], $\operatorname{st}^{2^{p-1}-1} \mathcal{Q}$ is n-contractible in $\operatorname{st}^{2^p-1} \mathcal{Q}$. As $2^p - 1 < 4l + 3$, $\mathcal{F}$ is n-contractible in $\operatorname{st}^{4l+3} \mathcal{Q}$. By (F$nHE_{\mathcal{V}}$), if a cover $\mathcal{F}$ is n-contractible in a cover $\operatorname{st}^{4l+3} \mathcal{Q}$, then $f^{-1}(\mathcal{F})$ is n-contractible in $f^{-1}(\operatorname{st}_{\mathcal{V}} \operatorname{st}^{4l+3} \mathcal{Q}) \prec f^{-1}(\operatorname{st}^{4(l+1)} \mathcal{Q})$. Hence by [7, Theorem 6.7] applied to $\mathcal{P}_k$, there exists a closed k-regular n-semiregular interior $\mathcal{N}_n$-cover $\mathcal{P}_k$ that is isomorphic to $\mathcal{P}_{k-1}$ and that refines $\operatorname{st}^N f^{-1}(\operatorname{st}^{4(9(5N+8)^{k-1}+1)} \mathcal{Q})$. By [7, Remark 3.5], we may require that $\mathcal{P}_k$ is equal to $\mathcal{P}_{k-1}$ on the image of Λ. This implies that $P^k_i \subset \operatorname{st}^{N+1} f^{-1}(\operatorname{st}^{4(9(5N+8)^{k-1}+1)} \mathcal{Q})$, hence $P^k_i \subset f^{-1}(\operatorname{st}^{N+1} \operatorname{st}^{4(9(5N+8)^{k-1}+1)} \mathcal{Q})$. By Lemma- [7, 2.1], $\operatorname{st}^{N+1} \operatorname{st}^{4(9(5n+8)^{k-1}+1)} \mathcal{Q} = \operatorname{st}^{(N+2)(4(9(5N+8)^{k-1})+N+1} \mathcal{Q}$ and clearly the latter exponent is not greater than $9(5N+8)^k$. We are done. ■

4 An Alternative Proof that (UV^{n-1}) Implies (NH)

It is known (see [3, Proposition 5.7.4]) that every n-dimensional Menger manifold M has the pseudo-interior $\nu^n(M)$.

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Lemma 4.1 *The class of n-dimensional Nöbeling manifolds coincides with the class of pseudo-interiors of n-dimensional Menger manifolds.*

Proof Apply [3, Proposition 5.7.5] and the open embedding theorem for Nöbeling manifolds [5, 7]. ■

Next we single out one of the main particular cases in which near-homeomorphisms appear naturally.

Proposition 4.2 *Let A be a σZ-set in a Nöbeling manifold N. Then the inclusion $N \setminus A \hookrightarrow N$ is a near-homeomorphism.*

Proof Apply Lemma 4.1 and [3, Proposition 5.7.7]. ■

If for a map $f\colon X \to Y$ the image $f(X)$ is dense in Y (as is the case for approximately n-soft maps), then the set of nondegenerate values of f, denoted by N_f, consists, by definition, of three types of points of Y: points in $Y \setminus f(X)$, points whose inverse images contain at least two points, and points $y \in Y$ for which although the inverse image $f^{-1}(y)$ is a singleton, the collection $f^{-1}(\mathcal{B})$ does not form a local base at $f^{-1}(y)$ for any local base $\mathcal{B}$ at y in Y. Note that N_f is an F_σ-subset of Y and that the restriction $f|f^{-1}(Y \setminus N_f)\colon f^{-1}(Y \setminus N_f) \to Y \setminus N_f$ is a homeomorphism.

Our next statement extends Proposition 4.2.

Proposition 4.3 *Let $f\colon M \to N$ be an approximately n-soft map of n-dimensional Nöbeling manifolds. If N_f is a σZ-set, then f is a near-homeomorphism.*

Proof We follow the proof of [1, Proposition 3.1]. Let $\{\alpha_k\colon k \in \mathbb{N}\} \subset C(I^n, M)$ be a dense subset of embeddings of the n-dimensional cube into M such that $\alpha_i(I^n) \cap \alpha_j(I^n) = \varnothing$ for each $i, j \in \mathbb{N}$ with $i \neq j$.

The map $f_0\alpha_1\colon I^n \to N$ can be approximated by a Z-embedding $\beta_1\colon I^n \to N \setminus N_{f_0}$ which is $\mathcal{U} - n$-homotopic to $f_0\alpha_1$ for a sufficiently small open cover $\mathcal{U}$ of N. Note that since $f_0\alpha_1$ and $f_0 f_0^{-1}\beta_1$ are n-homotopic via a small "n-homotopy", the approximate n-softness of f_0 implies that α_1 and $f_0^{-1}\beta_1$ are n-homotopic in M (via a small "n-homotopy", where smallness is measured in N). A version of Z-set unknotting Theorem [5, Theorem 2.2] produces a homeomorphism $h_1\colon M \to M$ such that $h_1\alpha_1 = f_0^{-1}\beta_1$ and $f_0 h_1$ is close to f_0. Let $f_1 = f_0 h_1$. Requiring additionally that h_1 is fixed outside of a small neighborhood of $f_0^{-1}(f_0(\alpha_1(I^n)))$ we conclude that $f_1^{-1}(f_1(m)) = m$ for each $m \in \alpha_1(I^n)$. Continuing in this manner we construct the sequence $f_0 = f, f_1, \dots$ of approximately n-soft maps of M into N so that $f_{k+1} = f_k h_{k+1}$, where $h_{k+1}\colon M \to M$ is a homeomorphism fixed outside of a small neighborhood of $f_k^{-1}(f_k(\alpha_{k+1}(I^n)))$ missing $\bigcup\{\alpha_i(I^n)\colon 1 \leq i \leq k\}$. As above, $h_{k+1}\alpha_{k+1} = f_k^{-1}\beta_{k+1}$, where $\beta_{k+1}\colon I^n \to N \setminus N_{f_k}$ is a Z-embedding. Observe also that $\bigcup\{f_{k+1}(\alpha_k(I^n))\colon 1 \leq i \leq k+1\} \subseteq N \setminus N_{f_{k+1}}$, $f_k^{-1}(f_k(m))) = m$ for each $m \in \bigcup\{\alpha_i(I^n)\colon 1 \leq i \leq k\}$ and $f_k|\bigcup\{\alpha_i(I^n)\colon 1 \leq i \leq k-1\} = f_{k-1}|\bigcup\{\alpha_i(I^n)\colon 1 \leq i \leq k-1\}$. If the homeomorphism h_{k+1} is chosen sufficiently close to h_k, then the map $g = \lim\{f_k\}\colon M \to N$ will be approximately n-soft. Note that $g^{-1}(N_g) \cap \bigcup\{\alpha_k(I^n)\colon k \in \mathbb{N}\} = \varnothing$. It then follows from the choice of the set $\{\alpha_k(I^n)\colon k \in \mathbb{N}\}$ that $g^{-1}(N_g)$ is a σZ-set in M.

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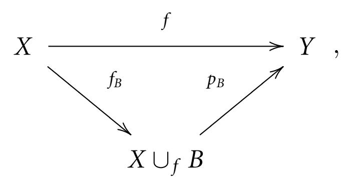
By Proposition 4.2, both inclusions $i\colon M \setminus g^{-1}(N_g) \hookrightarrow M$ and $j\colon N \setminus N_g \hookrightarrow N$ are near-homeomorphisms. Therefore, g (and hence f) can be approximated by a homeomorphism of the form $H_j g_0 H_i^{-1}$, where H_i approximates i, H_j approximates j, and $g_0 = g|(M \setminus g^{-1}(N_g))\colon M \setminus g^{-1}(N_g) \to N \setminus N_g$. ■

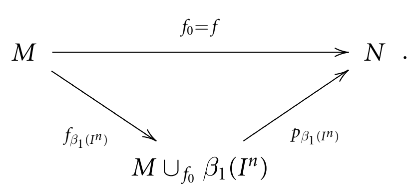
For a map $f\colon X \to Y$ and a closed subset $B \subseteq Y$, the adjunction space $X \cup_f B$ is defined to be the disjoint union of $X \setminus f^{-1}(B)$ and B topologised as follows: $X \setminus f^{-1}(B)$ itself is open and $f^{-1}(U \setminus B) \cup (U \cap B)$ for $U \subseteq Y$ open in Y. Obviously, the map f factors as follows:



where $f_B\colon X \to X \cup_f B$ coincides with the identity on $X \setminus f^{-1}(B)$ and with f on $f^{-1}(B)$ and $p_B\colon X \cup_f B \to Y$ coincides with the identity on B and with f on $X \setminus f^{-1}(B)$. If f is an approximate n-soft map between Polish ANE(n)-spaces and B is a (strong) Z-set in Y, then $X \cup_f B$ is also a Polish ANE(n)-space containing B as a (strong) Z-set and both f_B and p_B are approximately n-soft. If, in addition, X and Y are n-dimensional Nöbeling manifolds, then so is $X \cup_f B$.

Proof of $(\mathrm{UV}^{n-1}) \Rightarrow (\mathrm{NH})$. Proof follows the proof of [1, Characterization Theorem]. Let $\{\beta_k\colon k \in \mathbb{N}\} \subset C(I^n, M)$ be a dense subset of embeddings of the n-dimensional cube into N such that $\beta_i(I^n) \cap \beta_j(I^n) = \varnothing$ for each $i, j \in \mathbb{N}$ with $i \neq j$.

Let $f_0 = f$ and consider the above described factorization of f through the adjunction space, *i.e.*, $f_0 = p_{\beta_1(I^n)} f_{\beta_1(I^n)}$



Note that $N_{f_{\beta_1(I^n)}} \subseteq \beta_1(I^n)$. Since every compact subset of an n-dimensional Nöbeling manifold is a strong Z-set (see [3, Corollary 5.1.6]), it follows from Proposition 4.3 that there exists a homeomorphism $h\colon M \to M \cup_{f_0} \beta_1(I^n)$ approximating $f_{\beta_1(I^n)}$ as close as we wish. Let $f_1 = p_{\beta_1(I^n)} h$. Clearly, f_1 approximates f_0 and is one to one over $\beta_1(I^n)$. Proceeding in this manner we construct a sequence $\{f_k\colon k \in \mathbb{N}\}$ of approximately n-soft maps (of M into N) such that f_{k+1} approximates f_k and is one to one over $\bigcup\{\beta_i(I^n)\colon 1 \leq i \leq k+1\}$. If f_{k+1} is sufficiently close to f_k, then the limit map $g = \lim\{f_k\}\colon M \to N$ will be approximately n-soft. Since g is one to one over $\bigcup\{\beta_k(I^n)\colon k \in \mathbb{N}\}$, it follows from the choice of the collection $\{\beta_k\}$ that N_g is a σZ-set in N. By Proposition 4.3, g, and hence f, is a near-homeomorphism.

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