An Extension of a Formula of Cayley

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1. *Summary.* Let x_{ik} $(i, k = 1, 2, \ldots, n)$ be n^2 independent variables, a_{ik} be n^2 constants, and ξ_{ik} be n^2 operators $a_{ik}\partial/\partial x_{ik}$.

Let $x = (x_{ik})$ be the determinant formed from the n^2 elements x_{ik}, and $\xi = (\xi_{ik})$ the determinantal operator formed from the elements ξ_{ik}.

If α is any constant, then, when $a_{ik} = 1$, so that $\xi_{ik} = \partial/\partial x_{ik}$, Cayley proved that

$$\xi x^{\alpha} = \alpha(\alpha+1)\ldots(\alpha+n-1)\, x^{\alpha-1}. \qquad (1.1)$$

Capelli generalised this result:—

Let x_{IK} be that determinant of order $m \leqslant n$ formed from x by choosing those rows for which $i = i_1, i_2, \ldots, i_m$ and those columns for which $k = k_1, k_2, \ldots, k_m$.

Let x^{IK} be the cofactor of x_{IK}.

Let ξ_{IK} and ξ^{IK} be defined in a similar way.

Capelli generalised (1 . 1) by showing that

$$\xi^{IK} x^{\alpha} = \alpha(\alpha+1)\ldots(\alpha+n-m-1)\, x^{\alpha-1} x_{IK}. \qquad (1.2)$$

In this paper we remove the restriction that $a_{ik} = 1$, and obtain a result corresponding to (1 . 2) assuming only that a_{ik} are constants.

2. We have to define other symbols used in the sequel.

Let I' be a selection of r different numbers from the numbers 1 to n, arranged in ascending order. Let K' be another similar selection. Let I^* be the set I' followed by the set I, and K^* be the set K' followed by the set K. We may form the determinant $x_{I^*K^*}$ whose element x^*_{pq} in the p-th row and q-th column is $x_{\lambda\mu}$ where λ is the p-th number in I^* and μ is the q-th number in K^*. We will say that the determinant x_{IK} has been "bordered" by r rows and r columns, and will write x_{rIK} for $x_{I^*K^*}$. In the special case when $r = 1$ and the "border" consists of elements from the p-th and q-th column of x, we write x_{pIqK} for $x_{I^*K^*}$ or x_{rIK}.

If this "bordering" is repeated s times, the resulting determinant will be called $x_{r_1 r_2 \ldots r_s IK}$ which we shall sometimes abbreviate to $x_{r^s IK}$.

The elements common to the r rows and columns of a "border" form a minor of x which we denote by $x_{(r)}$. The cofactor of this minor in x will be denoted by $x^{(r)}$.

If $x_{(r)}$ is " bordered " by the p-th row and q-th column of x, we denote the resulting determinant by $x_{(pq)(r)}$.

The symbols x^{rIK}, x^{pIqK}, $x^{r^{s}IK}$ and $x^{(pq)(r)}$ denote the cofactors of the corresponding forms when they exist: otherwise they are defined as zero.

We extend this notation to the determinants a and ξ and to the permanents $\|x\|$, $\|a\|$ and $\|\xi\|$, in the obvious way.

We define $P(r)$ by

$$P(r)=\|a\|^{rIK}x_{rIK}\|a\|_{(r)}x^{(r)} \tag{2.1}$$

and

$$P(r_1, r_2, \ldots, r_s)=\|a\|^{r^{(s)}IK}x_{r^{(s)}IK}\prod_{t=1}^{s}\|a\|_{(r_t)}x^{(r_t)}. \tag{2.2}$$

If $s=0$, we define $P=\|a\|^{IK}x_{IK}$.

$Q(r_1, r_2, \ldots, r_s)$ will denote the sum of all such distinct terms as $P(r_1, r_2, \ldots, r_s)$ for fixed numbers $r_1, r_2, \ldots, r_s$, but varying suffixes for the elements in $r_1, r_2, \ldots, r_s$.

In terms of the above notation, our extension of Capelli's result is

$$\xi^{IK}x^{\alpha}=\frac{1}{n-m}\sum_{s=0}^{n-m-1}\alpha(\alpha-1)\ldots(\alpha-s)x^{\alpha-s-1}\sum_{r}\left(n-m-\sum_{t=1}^{s}r_t\right)Q(r_1, r_2 \ldots, r_s) \tag{2.3}$$

where the summation $\sum_r$ is taken over all values of $r_1, r_2, \ldots, r_s$ for which $r_1+r_2+\ldots+r_s<n-m$.

3. We prove (2.3) by induction. An examination shows that it is true when $m=n-1$. Hence, to prove it in general, it will be sufficient to show that if (2.3) is assumed true for $m+1$ numbers in I and K, it must then be true for m numbers in I and K.

Let I and K contain m numbers: then our hypothesis is that

$$\xi^{pIqK}x^{\alpha}=\frac{1}{n-m-1}\sum_{s=0}^{n-m-2}\alpha(\alpha-1)\ldots(\alpha-s)x^{\alpha-s-1}\sum_{r}{}'\left(n-m-1-\sum_{t=1}^{s}r_t\right)Q'$$

(where $r_2+r_2+\ldots+r_s<n-m-1$ in $\sum_r{}'$, and Q' is similar to $Q(r_1, r_2, \ldots, r_s)$, but is formed from $pIqK$ instead of IK), or, what is equivalent,

$$\xi^{pIqK}x^{\alpha}=\frac{1}{n-m-1}\sum_{s=0}^{n-m-1}\alpha(\alpha-1)\ldots(\alpha-s)x^{\alpha-s-1}\sum_{r}\left(n-m-1-\sum_{t=1}^{s}r_t\right)Q' \tag{3.1}$$

with $r_1+r_2+\ldots+r_s<n-m$.

We require two lemmas:—

LEMMA 1. $$\sum_{p=1}^{n} \sum_{q=1}^{n} x_{pq} x^{pIqK} = (n-m) x^{IK}.$$

LEMMA 2. $$\sum_{p=1}^{n} \sum_{q=1}^{n} x_{pq} \|x\|^{pIqK} = (n-m) \|x\|^{IK}.$$

Lemma 1 is proved by Lars Gårding[1]. The proof of Lemma 2 is similar to Gårding's proof of Lemma 1.

We operate on both sides of (3.1) with $\xi_{pq} = a_{pq} \partial/\partial x_{pq}$ and sum for all values of p and q from 1 to n. By Lemma 1,

$$\sum_{p=1}^{n} \sum_{q=1}^{n} \xi_{pq} \xi^{pIqK} x^{\alpha} = (n-m) \xi^{IK} x^{\alpha}. \tag{3.2}$$

This gives the value of the left-hand side after the operation.

The terms on the right-hand side of (3.1) after the operation can be grouped into two classes A and B, where a typical term in the A class is of the form

$$\frac{\alpha(\alpha-1)\ldots(\alpha-s-1)}{n-m-1} x^{\alpha-s-2} \left(n-m-1-\sum_{t=1}^{s} r_t\right) Q'(r_1, \ldots, r_s) a_{pq} x^{pq} \tag{3.3}$$

and a typical term in the B class is of the form

$$\frac{\alpha(\alpha-1)\ldots(\alpha-s-1)}{n-m-1} x^{\alpha-s-1} \left(n-m-1-\sum_{t=1}^{s} r_t\right) \xi_{pq} Q'(r_1, r_2, \ldots, r_s). \tag{3.4}$$

If in (3.3) we break up Q' into its elements P', a typical term is

$$\frac{\alpha(\alpha-1)\ldots(\alpha-s-1)}{n-m-1} x^{\alpha-s-2} \left(n-m-1-\sum_{t=1}^{s} r_t\right) \|a\|^{r^{(s)}pIqK} \times x_{r^{(s)}pIqK} \prod_{t=1}^{s} \|a\|_{(r_t)} x^{(r_t)} a_{pq} x^{pq}. \tag{3.5}$$

By (2.2)

$$\|a\|^{r^{(s)}pIqK} x_{r^{(s)}pIqK} \prod_{t=1}^{s} \|a\|_{(r_t)} x^{(r_t)} a_{pq} x^{pq} = P(r_1, r_2, \ldots, r_s, r_{s+1}),$$

where $r_{s+1} = 1$.

Hence (3.5) becomes

$$\frac{\alpha(\alpha-1)\ldots(\alpha-s-1)}{n-m-1} x^{\alpha-s-2} \left(n-m-\sum_{t=1}^{s+1} r_t\right) P(r_1, r_2, \ldots, r_{s+1})$$

provided $r_{s+1} = 1$.

[1] *Proc. Edinburgh Math. Soc.* (2), 8 (1947), 73-75.

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We replace $s+1$ by s in this, and sum for all values of p and q from 1 to n: then, if n' is the number of times 1 occurs among the numbers $r_1, r_2, \ldots, r_s$, the terms in (3.3) and so the terms in Class A will be the sum of the terms

$$\frac{n' \alpha(\alpha-1)\ldots(\alpha-s)}{n-m-1} x^{\alpha-s-1}\left(n-m-\sum_{t=1}^{s} r_t\right) P(r_1, r_2, \ldots, r_s). \tag{3.6}$$

The terms in (3.4) can be grouped into two classes C and D, where a typical term in C is[1]

$$\frac{\alpha(\alpha-1)\ldots(\alpha-s)}{n-m-1} x^{\alpha-s-1}\left(n-m-1-\sum_{t=1}^{s} r_t\right) \|a\|^{r^{(s)}pIqK} a_{pq} x_{r^{(s)}IK} \times \prod_{t=1}^{s} \|a\|_{(r_t)} x^{(r_t)}, \tag{3.7}$$

and a typical term in D is

$$\frac{\alpha(\alpha-1)\ldots(\alpha-s)}{n-m-1} x^{\alpha-s-1}\left(n-m-\sum_{t=1}^{s} r_t\right) \|a\|^{r^{(s)}pIqK} x_{r^{(s)}pIqK} \times \|a\|_{(r_u)} a_{pq} x^{(pq)(r_u)} \prod_{t=1}^{s}{}' \|a\|_{(r_t)} x^{(r_t)}, \tag{3.8}$$

where r_u is any r and Π' is used instead of Π to denote that the factor $\|a\|_{(r_u)} x^{(r_u)}$ is omitted.

In (3.7), the coefficient of $a_{pq}\|a\|^{r^{(s)}pIqK}$ is independent of p and q. Also, by Lemma 2,

$$\sum_{p=1}^{n} \sum_{q=1}^{n} a_{pq}\|a\|^{r^{(s)}pIqK} = \left(n-m-\sum_{t=1}^{s} r_t\right) \|a\|^{r^{(s)}IK}.$$

Hence the terms in (3.7) and so the terms of class C are all terms of the type

$$\frac{\alpha(\alpha-1)\ldots(\alpha-s)}{n-m-1} x^{\alpha-s-1}\left(n-m-1-\sum_{t=1}^{s} r_t\right)\left(n-m-\sum_{t=1}^{s} r_t\right) P(r_1, \ldots, r_s). \tag{3.9}$$

To evaluate the terms in (3.8) we proceed as follows:—

Let $r_1', r_2', \ldots, r_u', \ldots, r_s'$ be s sets of possible "borders", and suppose that at least one set has $r_u' > 1$.

We choose the sets r_t in (3.8) so that $r_u = r_u'$, $t \neq u$; and r_u so that $(pq)(r_u) = r_u'$. We now sum all the terms in (3.8) for relevant values of p and q. We observe that for these sets the coefficient of $a_{pq}\|a\|_{(r_u)}$ is independent of the separate values of pq and r_u. Again, $\|a\|_{(r_u)}$ is the

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[1] If p occurs in I or q in K, then $\xi_{pq} x_{r^s pIqK} = 0$, but also $\|a\|^{r^s pIqK} = 0$. Hence there is no error resulting from ignoring such terms.

cofactor of a_{pq} in $\|a\|_{(r_u')}$. By Lemma 2, the sum of all such terms as $a_{pq}\|a\|_{(r_u)}$ is $r_u'\|a\|_{(r_u')}$. Again $r_u' = 1+r_u$; and $r_t' = r_t$, $t \neq u$.

Hence the terms in (3.8) which have been thus added give

$$\frac{\alpha(\alpha-1)\dots(\alpha-s)}{n-m-1}\, x^{\alpha-s-1}\left(n-m-\sum_{t=1}^{s} r_t'\right) r_u' \,.\, P(r_1', r_2', \dots, r_s').$$

From this we see that the terms in (3.8) give rise to all possible terms of the type

$$\frac{\alpha(\alpha-1)\dots(\alpha-s)}{n-m-1}\, x^{\alpha-s-1}\left(n-m-\sum_{t=1}^{s} r_t\right)\left(\sum_{t=1}^{s} r_t-n'\right).\, P(r_1, r_2, \dots, r_s), \qquad (3.10)$$

where n' is, as before, the number of times 1 occurs among the numbers $r_1, r_2, \dots, r_s$.

Adding the terms in (3.6), (3.9) and (3.10), we find that, after operation, the right-hand side of (3.1) gives all possible terms of the type

$$\alpha(\alpha-1)\dots(\alpha-s)\, x^{\alpha-s-1}\left(n-m-\sum_{t=1}^{s} r_t\right).\, P(r_1, r_2, \dots, r_s). \qquad (3.11)$$

Hence if we add all possible terms of the type (3.11), the result (2.3) follows from the definition of $Q(r_1, r_2, \dots, r_s)$ and (3.2).

Very similar formulae can be obtained when ξ or x are permanents. The same proofs are valid. The differences in the formulae obtained arise only in that the minors of a and x (whether occurring as minors or their cofactors) in the definition of $P(r_1, r_2, \dots, r_s)$ may occur either as minors of permanents or as minors of determinants. The actual results can be seen from the table below:—

ξ	x	Minors of a	Minors of x
determinant	determinant	permanents	determinants
determinant	permanent	determinants	permanents
permanent	determinant	determinants	determinants
permanent	permanent	permanents	permanents

4. The derivation of Capelli's result from (2.3) is not immediate. The following lemmas, which are extensions of Lemma 1, will be required.

LEMMA 3. *Let $u(r)$ be the number of ways of selecting r different integers from $(n-m)$.*

Let x_{IK} be that determinant of order $m < n$ formed from x as in Section 1, and let $x_{(r)}$, x^{rIK}, x^{IK}, etc. be defined as in Section 1. We note that $x_{(r)}$ may be any minor of x containing r rows and columns.

Then, with the summation convention of the tensor calculus, applied to the sets of numbers in r,

$$x_{(r)}\, x^{rIK} = u(r)\, x^{IK}.$$

We prove this lemma by the method used by Lars Gårding in the proof of his Lemma 1.

Let $i_1', i_2', \ldots, i'_{m'}, i''_1, i''_2, \ldots, i''_{m''}, i_1, i_2, \ldots, i_m$ denote an even permutation of the numbers 1, 2, ..., n.

Let $k_1', k_2', \ldots, k'_{m'}, k''_1, k''_2, \ldots, k''_{m''}, k_1, k_2, \ldots, k_m$ also denote such a permutation, but not necessarily the same. We denote $i_1, i_2, \ldots, i_m$ by I; $k_1, k_2, \ldots, k_m$ by K, with similar meanings for I', K', I'' and K''.

Let $I^*_{m'}$ denote any sequence of m' integers from the set

$$i_1', i_2', \ldots, i'_{m'}, i''_1, i''_2, \ldots, i''_{m''}.$$

Let $I_{m'}$ denote a sequence of m' integers from the full set of n integers $i_1', i_2', \ldots, i'_{m'}, i''_1, i''_2, \ldots, i''_{m''}, i_1, i_2, \ldots, i_m$.

Let $K^*_{m'}$ and $K_{m'}$ have similar meanings with regard to the k sets.

In this notation, using Lagrange's expansion for determinants, we have

$$x_{I_{m'}K'}\, x^{I^*_{m'}IK'K} = x_{I'I''K'K''}. \tag{4.1}$$

If we now define $x^{iIkK} = 0$ if $i \varepsilon I$ or $k \varepsilon K$, we may replace $I^*_{m'}$ by $I_{m'}$ in (4.1). In (4.1) also, we may replace K' by any fixed m' integers from the first $m'+m''$ integers. Adding for all such choices, we get

$$x_{I_{m'}K^*_{m'}}\, x^{I_{m'}IK^*_{m'}K} = u(m')\, x_{I'I''K'K''}.$$

We may now replace $K^*_{m'}$ by $K_{m'}$ giving

$$x_{I_{m'}K_{m'}}\, x^{I_{m'}IK_{m'}K} = u(m')\, x_{I'I''K'K''}$$

which, with a change of notation, is the result of the lemma.

LEMMA 4. *If $u(r_1, r_2, \ldots, r_s)$ is the number of sets of s sequences that can be formed with numbers $r_1, r_2, \ldots, r_s$ in the sequences from $(n-m)$ integers, then*

$$x^{r_1 r_2 \ldots r_s IK} \prod_{t=1}^{s} x_{(r_t)} = u(r_1, r_2, \ldots, r_s)\, x^{IK}.$$

The proof of this lemma is similar to that of Lemma 3, and is omitted.

LEMMA 5. *Defining $u(r_1, r_2, \ldots, r_s)$ as for Lemma 4, we have*

$$x_{r_1 r_2 \ldots r_s IK} \prod_{t=1}^{s} x^{(r_t)} = u(r_1, r_2, \ldots, r_s)\, x^s\, x_{IK}.$$

To prove this, let $y_{ik}=x^{ik}$; then by Jacobi's formula

$$y_{IK}=x^{m-1}x^{IK},$$

$$y^{IK}=x^{n-m-1}x_{IK},$$

and

$$y=x^{n-1}.$$

If we substitute for x in terms of y in Lemma 4, this gives the result stated, with y in place of x.

If now

$$v(r_1, r_2, \ldots, r_s)=\|a\|^{r's)IK}\prod_{t=1}^{s}\|a\|_{(r_t)}$$

$$=\left(n-m-\sum_{t=1}^{s}r_t\right)!\prod_{t=1}^{s}r_t!, \tag{4.2}$$

we have, by Lemma 5 and (2.3),

$$x^{IK}x^{\alpha}=\frac{x^{\alpha-1}x_{IK}}{n-m}\sum_{s=0}^{n-m-1}\alpha(\alpha-1)\ldots(\alpha-s)\Sigma'\left(n-m-\sum_{t=1}^{s}r_t\right)$$

$$\times u(r_1, r_2, \ldots, r_s).v(r_1, r_2, \ldots, r_s) \tag{4.3}$$

(where Σ' denotes the sum for all values of $r_1, r_2, \ldots, r_s$ such that $r_1+r_2+\ldots+r_s<n-m$).

To obtain u we proceed as follows: Select a permutation of R integers from $(n-m)$. This can be done in $(n-m)!/(n-m-R)!$ ways. We arrange the permutation in a line and insert $(s-1)$ divisions between the R integers. This can be done in $\binom{R-1}{s-1}$ ways. This divides the R integers into s sets. Since the order of the sets in u is irrelevant, and the order of the integers in each set is their natural order, we have

$$u(r_1, r_2, \ldots, r_s)=\frac{(n-m)!}{(n-m-R)!}\binom{R-1}{s-1}\frac{1}{s!}\cdot\frac{1}{\prod_{t=1}^{s}r_t!}.$$

From this, (4.2) and (4.3) we obtain

$$x^{IK}x^{\alpha}=x^{\alpha-1}x_{IK}\sum_{s=0}^{n-m-1}\alpha(\alpha-1)\ldots(\alpha-s)\frac{(n-m-1)!}{s!}$$

$$\times\sum_{R=s}^{n-m-1}(n-m-R)\binom{R-1}{s-1}. \tag{4.4}$$

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By equating the coefficients of x^{s-1} in the formal expressions in ascending powers of x of the two sides of the identity

$$(n-m-s)(1+x)^{s-1}+(n-m-s-1)(1+x)^{s}+\ldots+(1+x)^{n-m-2}$$
$$\equiv \frac{(1+x)^{n-m}}{x^2}-\frac{(1+x)^s}{x^2}-\frac{(n-m-s)(1+x)^{s-1}}{x}$$

we have $$\sum_{R=s}^{n-m-1}(n-m-R)\binom{R-1}{s-1}=\binom{n-m}{s+1}.$$

Again, by equating the coefficients of $1/x$ in the formal expressions in ascending powers of x of the two sides of the identity

$$(1+x)^{\alpha-1}\left(1+\frac{1}{x}\right)^{n-m}\equiv\frac{(1+x)^{\alpha+n-m-1}}{x^{n-m}},$$

we obtain

$$\sum_{s=0}^{n-m-1}\alpha(\alpha-1)\ldots(\alpha-s)\frac{(n-m-1)!}{s!}\binom{n-m}{s+1}=\alpha(\alpha+1)\ldots(\alpha+n-m-1).$$

Substituting this is (4.4) we obtain Capelli's result.

5. There are two other cases which lead to simple formulae when $a_{ik}=1$ for all i and k. The one case is when ξ is a determinant and x is a permanent. The other case is when ξ is a permanent and x is a determinant. In both cases the minors of a which occur in the definition of P in (2.2) are determinants and so are zero except when of order 1. Hence, for the non-zero values of P, $a_{r_t}=0$, except when $r_t=1$ in which case $a_{1_t}=1$. Further $a^{r'IK}=0$, except when

$$\sum_{t=1}^{s}r_t+m=n-1, \text{ so that } s=n-m-1.$$

In the case when ξ is a determinant and x a permanent, the non-zero values of P are given by

$$P=a^{I'IK'K}\|x\|_{I'IK'K}\Pi\|x\|^{pq}, \tag{5.1}$$

where I' contains all the numbers, with one exception, from the set 1 to n which are not in I, and K' similarly contains all the numbers, with one exception, from the set 1 to n which are not in K; p, q are numbers in I', K' respectively, no two numbers p being the same and no two numbers q being the same; Π contains $s=n-m-1$ factors. If p_1, q_1, are the missing numbers in I' and K' respectively, we may write (5.1) as

$$P=(-)^{p_1+q_1}\|x\|^{p_1q_1}\Pi\|x\|^{pq}. \tag{5.2}$$

If we write $y_{ik} = \|x\|^{ik}$ and form the determinant y, using the previous notation we write $\Pi\, y_{pq} = \prod_{t=1}^{s} y_{(r_t)}$. (5.2) may be written as

$$P = y_{r^s IK} \prod_{t=1}^{s} y_{(r_t)}. \tag{5.3}$$

We may now apply Lemma 4 and obtain for the non-vanishing Q,

$$Q = \Sigma\, P = u(r_1, r_2, \ldots, r_s)\, y^{IK} = (n-m)\, y^{IK}. \tag{5.4}$$

Since $n-m-\sum_1^s r_t = 1$ we have immediately, from (2.3),

$$\xi^{IK}\|x\|^{\alpha} = \alpha(\alpha-1)\ldots(\alpha-n+m+1)\, x^{\alpha-n+m}\, y^{IK}, \tag{5.5}$$

where $y_{ik} = \|x\|^{ik}$.

The second case when ξ is a permanent and x a determinant may be treated in a similar way. Corresponding to (5.1) we have

$$P = a^{I'IK'K}\, x_{I'IK'K}\, \Pi\, x^{pq} \tag{5.6}$$

and hence, corresponding to (5.2), we have

$$P = x^{p_1 q_1}\, \Pi\, x^{pq}. \tag{5.7}$$

In this case we define $y_{ik} = x^{ik}$ and form the permanent $\|y\|$, which gives $x^{p_1 q_1} = y_{p_1 q_1} = \|y\|^{r^s IK}$ so that corresponding to (5.3) we have

$$P = \|y\|^{r^s IK} \prod_{t=1}^{s} x_{(r_t)}. \tag{5.8}$$

It may be readily proved that, *mutatis mutandis*, Lemmas 3 and 4 are true for permanents as well as determinants, and hence, applying Lemma 4, we have

$$Q = \Sigma\, P = (n-m)\|y\|^{IK} \tag{5.9}$$

and hence (2.3) gives

$$\|\xi\|^{IK} = \alpha(\alpha-1)\ldots(\alpha-n+m+1)\, x^{\alpha-n+m}\|y\|^{IK}. \tag{5.10}$$

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