Ergod. Th. & Dynam. Sys., (2023), **43**, 3448–3459 © The Author(s), 2022. Published by Cambridge University Press.
doi:10.1017/etds.2022.75

The structure of pointwise recurrent expansive homeomorphisms

ENHUI SHI†, HUI XU‡ and ZIQI YU†

† *School of Mathematical Sciences, Soochow University, Suzhou, Jiangsu 215006, China*
(e-mail: ehshi@suda.edu.cn; 20204207013@stu.suda.edu.cn)
‡ *CAS Wu Wen-Tsun Key Laboratory of Mathematics,*
University of Science and Technology of China, Hefei, Anhui 230026, China
(e-mail: huixu2734@ustc.edu.cn)

(*Received* 29 *January* 2022 *and accepted in revised form* 19 *September* 2022)

Abstract. Let X be a compact metric space and let $f : X \to X$ be a homeomorphism on X. We show that if f is both pointwise recurrent and expansive, then the dynamical system (X, f) is topologically conjugate to a subshift of some symbolic system. Moreover, if f is pointwise positively recurrent, then the subshift is semisimple; a counterexample is given to show the necessity of positive recurrence to ensure the semisimplicity.

Key words: recurrence, expansivity, almost periodic point, symbolic system, topological dimension
2020 Mathematics Subject Classification: 37B05 (Primary); 37B20 (Secondary)

1. *Introduction*

By a *dynamical system* (or a *system* for short), we mean a pair (X, f) where X is a compact metric space and f is a homeomorphism on X. Recurrence is one of the most important subjects in the study of dynamical systems. We know that periodic points, distal points, and almost periodic points are all recurrent points. The structure of pointwise recurrent homeomorphisms has been intensively studied by many authors under some specified assumptions on the dynamics of f or on the topology of the phase space X. A classical result due to Montgomery says that every pointwise periodic homeomorphism on a connected manifold is periodic (see [16]). Similar results were established for pointwise recurrent homeomorphisms on some surfaces (see [11, 19]). However, Glasner and Maon showed that even if f possesses very strong recurrence, the dynamics of f can still be complicated (see [7]). Mai and Ye determined the structure of pointwise recurrent maps having the pseudo orbit tracing property (see [14]). It is well known that minimal homeomorphisms are pointwise almost periodic. The structure of minimal distal systems was completely described by Furstenberg in [6]. One may consult [3] for a detailed introduction to the structure theory of general minimal systems.

Expansivity comes from the study of structural stability in differential dynamical systems, which is also a kind of chaotic property. Many important systems are known to be expansive, such as Anosov systems and subshifts of symbolic systems. It is known that the circle and the sphere $\mathbb{S}^2$ admit no expansive homeomorphisms (see [1, 9, 12]) and every compact orientable surface of positive genus admits an expansive homeomorphism (see [18]). Mañé showed that if X admits an expansive homeomorphism, then the topological dimension $\dim(X) < \infty$ (see [15]). One may refer to [1] for a systematic introduction to this property.

The following celebrated result is due to Mañé, which clarifies the structure of minimal expansive homeomorphisms.

THEOREM 1.1. [15] *Let X be a compact metric space. If X admits a minimal expansive homeomorphism f, then (X, f) is topologically conjugate to a minimal subshift of some symbolic system.*

The purpose of the paper is to extend Theorem 1.1 to the case in which f is both pointwise recurrent and expansive. Recently, Artigue in [2] also obtained a kind of generalization for continuum-wise expansive homeomorphisms. Recall that a system (X, f) is *semisimple* if X is the disjoint union of minimal sets (that is, every point of X is almost periodic).

The following is the main theorem of the paper.

THEOREM 1.2. *Let X be a compact metric space and f be an expansive homeomorphism on X. If f is pointwise recurrent, then the system (X, f) is topologically conjugate to a subshift of some symbolic system. If f is pointwise positively recurrent, then the system (X, f) is topologically conjugate to a semisimple subsystem of some symbolic system.*

Here, we give some remarks on the conditions in Theorem 1.2. For $x \in X$, by the recurrence of x, we know that the orbit closure $\overline{\{f^n(x) : n \in \mathbb{Z}\}}$ is topologically transitive. However, we cannot conclude that $\overline{\{f^n(x) : n \in \mathbb{Z}\}}$ is minimal in general, though every point of which is recurrent. In fact, there do exist non-minimal topologically transitive systems which are pointwise recurrent (see [5, 10]). Even if the f in Theorem 1.2 is semisimple, we can only get that the orbit closure of every point is totally disconnected from Theorem 1.1, which does not mean $\dim(X) = 0$. In the last section, a counter example is constructed to show that the positive recurrence is necessary for obtaining the semisimplicity.

The following corollary is immediate.

COROLLARY 1.3. *Let X be a connected compact metric space and let f be an expansive homeomorphism on X. If X is not a single point, then f has a non-recurrent point.*

The following corollary can be deduced from Theorem 1.2 and the main results in [14].

COROLLARY 1.4. *Let X be a compact metric space. If X admits a pointwise positively recurrent homeomorphism f which is expansive and has the pseudo orbit tracing property, then X is finite.*

2. *Preliminaries*

In this section, we will recall some notions and facts around recurrence and expansivity, which will be used in the proof of the main theorem.

2.1. *Recurrence.* Let (X, f) be a system. For $x \in X$, the *orbit* of x is the set $\mathrm{orb}(x, f) := \{f^n(x) : n \in \mathbb{Z}\}$; the α*-limit set* of x is defined to be the set

$$\alpha(x, f) := \{y \in X : \text{there exists } 0 < n_1 < n_2 < \ldots \text{ such that } f^{-n_i}(x) \to y\}$$

and the ω*-limit set* of x is defined to be the set

$$\omega(x, f) := \{y \in X : \text{there exists } 0 < n_1 < n_2 < \ldots \text{ such that } f^{n_i}(x) \to y\}.$$

The point x is *positively recurrent* if x belongs to its ω-limit set; is *negatively recurrent* if x belongs to its α-limit set; is *recurrent* if it is either positively recurrent or negatively recurrent. If $X = \overline{\mathrm{orb}(x, f)}$ for some $x \in X$, then x is called a *transitive point* and f is called *topologically transitive*. Clearly, a transitive point is a recurrent point provided that the space contains no isolated points.

A subset E of X is f*-invariant* (or *invariant* for short) if $f(E) = E$; we use $f|_E$ to denote the restriction of f to E. If E is closed and f-invariant, then we call $(E, f|_E)$ a *subsystem* of (X, f). When $X = \{0, 1, \ldots, k\}^{\mathbb{Z}}$ for some positive integer k and f is the shift on X, we also say that a subsystem of the symbolic system (X, f) is a *subshift*. From the definitions, we see that $\mathrm{orb}(x, f)$ is f-invariant; both $\alpha(x, f)$ and $\omega(x, f)$ are closed and f-invariant. If E is closed, invariant, and contains no proper closed invariant subset, then E is called a *minimal set*. It is clear that E is minimal if and only if for every $x \in E$, the orbit $\mathrm{orb}(x, f)$ is dense in E. A point $x \in X$ is *almost periodic* if $\overline{\mathrm{orb}(x, f)}$ is minimal. By an argument of Zorn's lemma, we know that every subsystem contains a minimal set, and hence contains an almost periodic point. If $\mathrm{orb}(x, f)$ is finite, then x is called a *periodic point*. Periodic points are always almost periodic.

The following proposition can be deduced immediately from the invariance of $\alpha(x, f)$ and $\omega(x, f)$.

PROPOSITION 2.1. *If x is an almost periodic point, then it is both positively recurrent and negatively recurrent.*

The following proposition is due to Gottschalk (see [8, Theorem 1]). Although the recurrence in [8, Theorem 1] is positive recurrence, the following proposition is a direct corollary.

PROPOSITION 2.2. *Let f be a homeomorphism of a compact metric space (X, d) and $n \geq 1$. Then a point $x \in X$ is recurrent with respect to f if and only if it is recurrent with respect to f^n.*

https://doi.org/10.1017/etds.2022.75 Published online by Cambridge University Press

The following proposition is due to Katznelson and Weiss (see [10, Lemma 2.1]). Recall that a compact metrizable space is of dimension 0 if and only if it is totally disconnected.

PROPOSITION 2.3. *Let f be a homeomorphism of a compact metric space X of dimension* 0*. If f is both pointwise positively recurrent and topologically transitive, then f is minimal.*

2.2. *Expansivity and hyperbolic metrics.* Suppose f is a homeomorphism on a compact metric space X with metric d. We say f is *expansive* if there is $c > 0$ such that $\sup_{n\in\mathbb{Z}} d(f^n(x), f^n(y)) > c$, for any distinct points $x, y \in X$; we call c an *expansivity constant* for f. The following proposition can be seen in [1] and is easy to be checked.

PROPOSITION 2.4. *Let n be a positive integer. Then f is expansive if and only if f^n is expansive.*

For $x \in X$ and $r > 0$, let $B_r(x, d)$ denote the open ball of radius r centering at x with respect to metric d, that is, $B_r(x, d) = \{y \in X : d(x, y) < r\}$. For $x \in X$ and $\varepsilon > 0$, let

$$W^s_\varepsilon(x, d) = \{y \in X : d(f^n(x), f^n(y)) \le \varepsilon, \text{ for all } n \ge 0\},$$
$$W^u_\varepsilon(x, d) = \{y \in X : d(f^{-n}(x), f^{-n}(y)) \le \varepsilon, \text{ for all } n \ge 0\}.$$

The sets $W^s_\varepsilon(x, d)$ and $W^u_\varepsilon(x, d)$ are called respectively the *local stable set* and the *local unstable set* of scale ε at x. For $x \in X$, the *stable set* $W^s(x, d)$ and the *unstable set* $W^u(x, d)$ are defined by

$$W^s(x, d) = \{y \in X : \lim_{n\to\infty} d(f^n(x), f^n(y)) = 0\},$$
$$W^u(x, d) = \{y \in X : \lim_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) = 0\}.$$

The following proposition is shown in [15, Lemma I]. (Also see [1, Proposition 2.39].)

PROPOSITION 2.5. *If f is expansive with an expansivity constant c and $0 < \varepsilon < c$, then*

$$W^s(x, d) = \bigcup_{n\ge 0} f^{-n}(W^s_\varepsilon(f^n(x), d), \quad W^u(x, d) = \bigcup_{n\ge 0} f^n(W^u_\varepsilon(f^{-n}(x), d).$$

In [20], Reddy showed that canonical coordinates are hyperbolic for expansive homeomorphisms on compact metric space. Actually, it can be seen that every expansive homeomorphism admits a hyperbolic metric. The following form that we need is also stated and proved in [1, Theorem 2.40].

THEOREM 2.6. [1, Theorem 2.40] *Let f be an expansive homeomorphism on a compact metric space X. Then there is a compatible metric D on X, $\gamma > 0$, $0 < \lambda < 1$, and $a \ge 1$ such that for any $x \in X$:*

(i) *if $y \in W^s_\gamma(x, D)$, then*

$$D(f^n(x), f^n(y)) \le a\lambda^n D(x, y) \quad \text{for any } n \ge 0;$$

https://doi.org/10.1017/etds.2022.75 Published online by Cambridge University Press

(ii) *if* $y \in W^u_\gamma(x, D)$, *then*

$$D(f^{-n}(x), f^{-n}(y)) \leq a\lambda^n D(x, y) \quad \textit{for any } n \geq 0.$$

2.3. *Expansivity and topological dimension.* For $x \in X$ and $\varepsilon > 0$, let $\Sigma^s_\varepsilon(x, d)$ (respectively $\Sigma^u_\varepsilon(x, d)$) denote the connected component of $W^s_\varepsilon(x, d) \cap \overline{B_\varepsilon(x, d)}$ (respectively $W^u_\varepsilon(x, d) \cap \overline{B_\varepsilon(x, d)}$) containing x.

The following two results were established by Mañé in [15].

THEOREM 2.7. *Let X be a compact metric space. If X admits a minimal expansive homeomorphism f, then* $\dim(X) = 0$.

The following lemma is crucial in the proof of Theorem 2.7.

LEMMA 2.8. *Let f be an expansive homeomorphism of a compact metric space* (X, d). *If* $\dim(X) > 0$, *then there is* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$, *there is some point* $a \in X$ *such that*

$$\Sigma^s_\varepsilon(a, d) \cap \partial B_\varepsilon(a, d) \neq \emptyset \quad \textit{or} \quad \Sigma^u_\varepsilon(a, d) \cap \partial B_\varepsilon(a, d) \neq \emptyset.$$

2.4. *Distality.* Let f be a homeomorphism of a compact metric space X with metric d. If for any distinct points $x, y \in X$, we have $\inf_{n\in\mathbb{Z}} d(f^n(x), f^n(y)) > 0$, then f is said to be *distal*.

From the definition, we immediately have the following proposition.

PROPOSITION 2.9. *If f is pointwise periodic, then f is distal.*

The following lemma is well known. It also holds for any finitely generated group actions (see [13]).

PROPOSITION 2.10. [4, Proposition 2.7.1] *If f is both distal and expansive, then X is finite.*

2.5. *Pseudo orbit tracing property.* A sequence of points $\{x_i : a < i < b\}(-\infty \leq a < b \leq +\infty)$ is called a δ *pseudo orbit* for f if $d(f(x_i), x_{i+1}) < \delta$ for each $i \in (a, b-1)$. A sequence $\{x_i : a < i < b\}$ is called to be ε *traced* by $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ for each $i \in (a, b)$. We say that f has the *pseudo orbit tracing property* if for every $\varepsilon > 0$, there is $\delta > 0$ such that every δ pseudo orbit for f can be ε traced by some point of X.

The following theorem is only a part of the main theorem in [14] by Mai and Ye.

THEOREM 2.11. *Let X be a compact metric space and f be a minimal homeomorphism on X. If f has the pseudo orbit tracing property, then it is conjugate to some adding machine; in particular, it is equicontinuous.*

3. *Some auxiliary lemmas*

In this section, we prepare some technical lemmas which will be used later.

Throughout this section, we let f be an expansive homeomorphism on a compact metric space X.

https://doi.org/10.1017/etds.2022.75 Published online by Cambridge University Press

LEMMA 3.1. *There is a compatible metric D on X such that for any $A \geq 1$, there exist $\delta > 0$ and positive integer N such that for any $x \in X$ and any $y \in W^s_{\delta, f^N}(x, D)$, we have*

$$D(f^{-N}(x), f^{-N}(y)) \geq A D(x, y), \quad \textit{and}$$
$$D(f^{iN}(x), f^{iN}(y)) \leq \frac{1}{A^i} D(x, y) \quad \textit{for all } i \geq 0,$$

where $W^s_{\delta, f^N}(x, D) = \{y \in X : D(f^{iN}(x), f^{iN}(y)) \leq \delta, \textit{ for all } i \geq 0\}$.

Proof. Let the metric D, γ, λ, and a be as in Theorem 2.6. Take a positive integer N with $a\lambda^N < 1/A$. Take $\delta > 0$ to be such that for any $u, v \in X$ with $D(u, v) \leq \delta$,

$$\max_{0 \leq n \leq N} D(f^{-n}(u), f^{-n}(v)) \leq \gamma. \tag{1}$$

For each $n \geq 0$, write $n = kN + r$, where $k \geq 1$ and $-N \leq r < 0$. Then for any $y \in W^s_{\delta, f^N}(x, D)$, it follows from equation (1) that

$$D(f^n(f^{-N}x), f^n(f^{-N}y)) = D(f^r(f^{(k-1)N}x), f^r(f^{(k-1)N}y)) \leq \gamma,$$

which means

$$f^{-N}(y) \in W^s_\gamma(f^{-N}(x), D) \quad \text{and} \quad y \in W^s_\gamma(x, D).$$

Thus, for any $y \in W^s_{\delta, f^N}(x, D)$, we have

$$\begin{aligned} D(x, y) &= D(f^N(f^{-N}x), f^N(f^{-N}y)) \\ &\leq a\lambda^N D(f^{-N}(x), f^{-N}(y)) \leq \frac{1}{A} D(f^{-N}(x), f^{-N}(y)), \end{aligned}$$

and for each $i \geq 0$,

$$D(f^{iN}(x), f^{iN}(y)) \leq a\lambda^{iN} D(x, y) \leq (a\lambda^N)^i D(x, y) \leq \frac{1}{A^i} D(x, y).$$

This completes the proof. □

LEMMA 3.2. *If $y, z \in W^s_{\delta/2}(x, D)$, then $z \in W^s_\delta(y, D)$.*

Proof. For each $n \geq 0$, we have

$$D(f^n(y), f^n(z)) \leq D(f^n(y), f^n(x)) + D(f^n(x), f^n(z)) \leq \delta/2 + \delta/2 = \delta.$$

So, the conclusion holds. □

Now we propose the following assumption under which some lemmas are obtained.

Assumption 1. There exist $A > 1$, a compatible metric D on X, and $\delta > 0$, such that for any $x \in X$ and any $y \in W^s_\delta(x, D)$, it holds that

$$D(f^{-1}(x), f^{-1}(y)) \geq A D(x, y), \quad \text{and}$$
$$D(f^n(x), f^n(y)) \leq \frac{1}{A^n} D(x, y) \quad \text{for all } n \geq 0.$$

https://doi.org/10.1017/etds.2022.75 Published online by Cambridge University Press

We should note that the number A in Assumption 1 can be taken arbitrarily large, since we can always replace f by some f^N with N being sufficiently large.

The following corollary is direct from Assumption 1 and Lemma 3.2.

COROLLARY 3.3. *If $y, z \in W^s_{\delta/2}(x, D)$, then $D(f^{-1}(y), f^{-1}(z)) \geq AD(y, z)$.*

COROLLARY 3.4. *If $y, z \in f^{-1}(W^s_{\delta/2}(x, D))$ and $D(y, z) \leq \delta/2$, then $z \in W^s_\delta(y, D)$.*

Proof. For each $n \geq 0$,

$$D(f^{n+1}(y), f^{n+1}(z)) \leq D(f^n(fy), f^n(x)) + D(f^n(x), f^n(fz)) \leq \delta/2 + \delta/2 = \delta.$$

Thus, $z \in W^s_\delta(y, D)$, since $D(y, z) \leq \delta/2 \leq \delta$. □

Borrowing the idea of Mañé from the proof of Lemma III in [15], we may further assume the point in Lemma 2.8 to be almost periodic. Now we strengthen Lemma 2.8 as follows under Assumption 1 with $A \geq 2$.

LEMMA 3.5. *If $\Sigma^s_{\varepsilon_1}(a, D) \cap \partial B_{\varepsilon_1}(a, D) \neq \emptyset$ for some point $a \in X$ and $\varepsilon_1 > 0$, then there is an almost periodic point $x^* \in X$ and $\varepsilon_2 > 0$ such that*

$$\Sigma^s_{\varepsilon_2}(x^*, D) \cap \partial B_{\varepsilon_2}(x^*, D) \neq \emptyset.$$

Proof. Let $\delta > 0$ be as in Assumption 1. We may as well assume that $\delta < \varepsilon_1$. Since $\Sigma^s_{\varepsilon_1}(a, D) \cap \partial B_{\varepsilon_1}(a, D) \neq \emptyset$, we can take a point $y_1 \in \Sigma^s_{\delta/2}(a, D)$ with $D(a, y_1) = \delta/2$ by the boundary bumping lemma [17, Ch. V].

By Corollary 3.3, we have $D(f^{-1}(a), f^{-1}(y_1)) \geq AD(a, y_1) \geq \delta$. Thus, there is a point $y_2 \in \Sigma^s_{\delta/2}(f^{-1}(a), D)$ with $D(f^{-1}(a), y_2) = \delta/2$. Repeating this process, we obtain a sequence (y_n) of points in X satisfying

$$y_{n+1} \in \Sigma^s_{\delta/2}(f^{-n}(a), D) \quad \text{and} \quad D(f^{-n}(a), y_{n+1}) = \frac{\delta}{2}$$

for any $n \geq 0$.

Since the α-limit set $\alpha(a, f)$ of a is a nonempty closed invariant subset of X, there is an increasing sequence $n_1 < n_2 < n_3 < \ldots$ such that $f^{-n_i}(a) \to x^*$ with x^* being an almost periodic point in $\alpha(a, f)$. By passing to some subsequence, we may further assume that $\Sigma^s_{\delta/2}(f^{-n_i}(a), D)$ converges in the hyperspace [17, Ch. IV] to a compact connected subset K of X.

Notice that for any point $p \in K$, there is a sequence $x_{n_i} \in \Sigma^s_{\delta/2}(f^{-n_i}(a), D)$ with $x_{n_i} \to p$. Then for each $i \geq 1$ and $n \geq 0$, $D(f^n(f^{-n_i}a), f^n(x_{n_i})) \leq \delta/2$. Thus, $D(f^n(x^*), f^n(p)) \leq \delta/2$ and hence $p \in W^s_{\delta/2}(x^*, D)$. It follows from the connectedness of K that $K \subset \Sigma^s_{\delta/2}(x^*, D)$. Let q be a limit point of (y_{n_i+1}). Then $q \in K$ and $D(x^*, q) = \delta/2$. By taking a positive $\varepsilon_2 \leq \delta/2$, we have

$$\Sigma^s_{\varepsilon_2}(x^*, D) \cap \partial B_{\varepsilon_2}(x^*, D) \neq \emptyset.$$

Thus, we complete the proof. □

https://doi.org/10.1017/etds.2022.75 Published online by Cambridge University Press

4. *Proof of Theorem 1.2 and Corollary 1.4*

To prove Theorem 1.2, we need only to prove $\dim(X) = 0$. If this is true, by a canonical coding technique, we get that f is conjugate to a subshift of some symbolic system. Precisely, we can take a partition $\{U_1, \ldots, U_k\}$ of X consisting of clopen sets whose diameters are less than the expansivity constant. Set Ω to be the set of $\xi \in \{1, \ldots, k\}^{\mathbb{Z}}$ with $\bigcap_{n\in\mathbb{Z}} f^{-n}U_{\xi(n)} \neq \emptyset$. Now the expansiveness implies that $\bigcap_{n\in\mathbb{Z}} f^{-n}U_{\xi(n)}$ is a singleton for each $\xi \in \Omega$. Then take $\pi : \Omega \to X$ by mapping each $\xi \in \Omega$ to the singleton in $\bigcap_{n\in\mathbb{Z}} f^{-n}U_{\xi(n)}$ and it is straightforward to verify that π is a conjugation between (Ω, σ) and (X, f), where σ is the left shift. For the second part of Theorem 1.2, applying Proposition 2.3, we see that this subshift is semisimple if the recurrence is strengthened to positive recurrence. Thus, we complete the proof.

From Propositions 2.2 and 2.4, we see that f is both pointwise recurrent and expansive if and only if f^n is both pointwise recurrent and expansive for any $n > 1$. Thus, to prove $\dim(X) = 0$, by Lemma 3.1 and by replacing f by some $f^N (N > 1)$ if necessary, we may assume that f satisfies Assumption 1.

Recall that the number A in Assumption 1 can be taken arbitrarily large.

We prove $\dim(X) = 0$ following Mañé's idea. First, assume the dimension is positive and find an almost periodic point x^* whose local stable set $W^s_\varepsilon(x^*, D)$ contains a nondegenerate continuum $\Sigma^s_\varepsilon(x^*, D)$ by Lemmas 2.8 and 3.5. Then, stretch the continuum $\Sigma^s_\varepsilon(x^*, D)$ in backward iterations to produce a small open set and a point whose backward orbit never meets this open set (this is enough to discard minimality in Mañé's proof). Here, the point is chosen in W^s, but far from the positive orbit of x^* so that it cannot be positively recurrent.

Proof of Theorem 1.2. By the discussions at the beginning of this section, we need only to show that $\dim(X) = 0$; and we can suppose that f satisfies Assumption 1 with $A = 7$. Let $\delta > 0$ be as in Assumption 1.

To the contrary, assume that $\dim(X) > 0$. Then by Lemma 2.8, there is a point $a \in X$ and $\varepsilon_1 > 0$ such that

$$\Sigma^s_{\varepsilon_1}(a, D) \cap \partial B_{\varepsilon_1}(a, D) \neq \emptyset \quad \text{or} \quad \Sigma^u_{\varepsilon_1}(a, D) \cap \partial B_{\varepsilon_1}(a, D) \neq \emptyset.$$

By replacing f with f^{-1} if necessary, we may as well assume that $\Sigma^s_{\varepsilon_1}(a, D) \cap \partial B_{\varepsilon_1}(a, D) \neq \emptyset$. Then it follows from Lemma 3.5 that there is an almost periodic point $x^* \in X$ and $0 < \varepsilon_2 \leq \delta/2$ such that

$$\Sigma^s_{\varepsilon_2}(x^*, D) \cap \partial B_{\varepsilon_2}(x^*, D) \neq \emptyset.$$

Let $\Lambda = \overline{\mathrm{orb}(x^*, f)}$, which is totally disconnected by Theorem 2.7. Since $\Sigma^s_{\varepsilon_2}(x^*, D)$ is connected and $\Sigma^s_{\varepsilon_2}(x^*, D) \cap \partial B(x^*, \varepsilon_2) \neq \emptyset$, $\Lambda \cap \Sigma^s_{\varepsilon_2}(x^*, D)$ is a proper closed subset of $\Sigma^s_{\varepsilon_2}(x^*, D)$ in the relative topology. Pick $y \in \Sigma^s_{\varepsilon_2}(x^*, D)$ and $\gamma > 0$ with

$$\Sigma^s_{\varepsilon_2}(x^*, D) \cap B_{2\gamma}(y, D) \subset \Sigma^s_{\varepsilon_2}(x^*, D) \setminus \Lambda. \tag{2}$$

Let Σ_0 be the connected component of $\Sigma^s_{\varepsilon_2}(x^*, D) \cap \overline{B_\gamma(y, D)}$ containing y. Take $z \in \Sigma^s_{\varepsilon_2}(x^*, D) \cap \partial B_\gamma(y, D)$. Since $\varepsilon_2 \leq \delta/2$, $z \in W^s_\delta(y, D)$ by Lemma 3.2.

Set $y_0 = y$ and $z_0 = z$. Then by Corollary 3.3,

$$D(f^{-1}(y_0), f^{-1}(z_0)) \geq AD(y_0, z_0) = 7\gamma.$$

Thus, either $D(f^{-1}(y_0), y) > 3\gamma$ or $D(f^{-1}(z_0), y) > 3\gamma$. Set $y_1 = f^{-1}(y_0)$ if $D(f^{-1}(y_0), y) > 3\gamma$, otherwise set $y_1 = f^{-1}(z_0)$. Let Σ_1 be the connected component of $f^{-1}(\Sigma_0) \cap \overline{B_\gamma(y_1, D)}$ that contains y_1. Take $z_1 \in \Sigma_1$ with $D(y_1, z_1) = \gamma$. Noting that $\gamma \leq \delta/2$, by Corollary 3.4, we have $z_1 \in W^s_\delta(y_1, D)$. Then, $D(f^{-1}(y_1), f^{-1}(z_1)) \geq AD(y_1, z_1) = 7\gamma$. Thus, either $D(f^{-1}(y_1), y) > 3\gamma$ or $D(f^{-1}(z_1), y) > 3\gamma$. Choose y_2 from $\{f^{-1}(y_1), f^{-1}(z_1)\}$ such that $D(y_2, y) > 3\gamma$. Let Σ_2 be the connected component of $f^{-1}(\Sigma_1) \cap \overline{B_\gamma(y_2, D)}$ that contains y_2. Take $z_2 \in \Sigma_2$ with $D(y_2, z_2) = \gamma$. Repeating this process, we obtain a sequence (Σ_i) of compact connected subsets such that $\Sigma_i \cap B_{2\gamma}(y, D) = \emptyset$ for each $i \geq 1$ and

$$\overline{B_\gamma(y, D)} \supset \Sigma_0 \supset f(\Sigma_1) \supset f^2(\Sigma_2) \supset \ldots.$$

Take a point $w \in \bigcap_{i\geq 1} f^i(\Sigma_i)$. Then $w \in \overline{B_\gamma(y, D)}$ and $f^{-i}(w) \notin B_{2\gamma}(y, D)$ for each $i \geq 1$. Thus, w is not negatively recurrent. Since $w \in W^s(x^*, D)$, we have

$$D(f^n(w), \Lambda) \to 0 \quad \text{as } n \to +\infty.$$

By the choice of y, we see that w is not positively recurrent. To sum up, w is not recurrent. This is a contradiction. □

Proof of Corollary 1.4. From Theorems 1.2 and 2.11, we see that for each $x \in X$, the orbit closure $\overline{\mathrm{orb}(x, f)}$ is both expansive and equicontinuous, which implies that x is periodic. Thus, f is pointwise periodic, and so X is finite by Propositions 2.9 and 2.10. □

5. *A non-minimal, pointwise recurrent, and transitive subshift*

In this section, we will construct a non-minimal, pointwise recurrent, and transitive subshift to show that the pointwise positive recurrence in Proposition 2.3 and the second part of Theorem 1.2 cannot be weakened to pointwise recurrence.

We define a sequence ξ as

$$\cdots 0000 \overset{\overset{\xi(0)}{\uparrow}}{1}\ 0100101000101001010000010100101000101001 01 \cdots.$$

Precisely, we let $\xi(n) = 0$ for all $n < 0$ and define $\xi(n)$ for $n \geq 0$ inductively. Set

$$\omega_1 = 1, \omega_2 = \omega_1 0 \omega_1 = 101, \omega_3 = \omega_2 00 \omega_2 = 10100101,$$
$$\ldots, \omega_{n+1} = \omega_n \underbrace{000 \cdots 000}_{n \text{ zeros}} \omega_n, \ldots.$$

Denote by ℓ_n the length of ω_n and let

$$\xi(0)\xi(1)\ldots\xi(\ell_n - 1) = \omega_n.$$

In such a way, we eventually get the sequence ξ. Let $T : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ be the left shift. Let $X = \overline{\mathrm{orb}(\xi, T)}$. It is clear that X is non-minimal, since $\mathbf{0} = \ldots 0000 \ldots$ is in X which is a fixed point. In the remaining part, we will show that (X, T) is pointwise recurrent.

For $x \in \{0, 1\}^{\mathbb{Z}}$, let $\overline{x}$ be the reflection of x with respect to the origin, that is, $\overline{x}(n) = x(-n)$.

CLAIM 1. *For any $x \in X$, we have $\overline{x} \in X$.*

Proof. From the definition of ξ, we see that $\overline{\xi} = \lim_{n\to\infty} T^{\ell_n - 1}\xi$. Then $\xi = \lim_{n\to\infty} T^{-(\ell_n-1)}\overline{\xi}$ and hence $\overline{\xi}$ is also a transitive point in X. Thus for any $x \in X$, we have $\overline{x} \in X$. □

From the construction of ξ, we have the following claim.

CLAIM 2. *For any $n > m > 0$:*
(a) *if $\xi(m) \ldots \xi(n) = 00 \ldots 001$, then $\xi(n) \ldots \xi(n + \ell_{n-m} - 1) = \omega_{n-m}$;*
(b) *if $n - m > \ell_k + k$, then $\xi(m) \ldots \xi(n)$ contains $k - 1$ consecutive 0 terms.*

CLAIM 3. *For any $x \in X \setminus \{\mathbf{0}\}$, 1 occurs in x infinitely often.*

Proof. To the contrary, suppose that there are finitely many 1 terms in x. Then there is $N > 2$ such that $x(n) = 0$ for any $|n| \geq N$. Since $x \neq \mathbf{0}$, we may assume that $T^{n_i}\xi \to x$ for some increasing sequence $0 < n_1 < n_2 < \ldots$. Set $x_i = T^{n_i}\xi$ and take i such that $n_i > 2N$ and x_i coincides with x on $[-2N - \ell_{3N}, 2N + \ell_{3N}]$. Let k be the least integer such that $x(k) = 1$. Then,

$$x_i(-2N) \ldots x_i(k) = \xi(n_i - 2N) \ldots \xi(n_i + k) = 00 \ldots 001.$$

By Claim 2(a), we have

$$x(k) \ldots x(k + \ell_{k+2N} - 1) = x_i(k) \ldots x_i(k + \ell_{k+2N} - 1) = \omega_{k+2N}.$$

Particularly, we have $x(k + \ell_{k+2N} - 1) = 1$. Note that $\ell_n \geq 2^n$ for any $n > 2$. Thus, $k + \ell_{k+2N} - 1 \geq 2^N - N - 1 > N$, since $N > 2$. This is absurd since x takes 0 outside $[-N, N]$. Thus, we have proved the claim. □

Now we are ready to show that (X, T) is pointwise recurrent. It suffices to show that every $x \in X \setminus \{\mathbf{0}\}$ is recurrent. By Claim 1, $\overline{\xi} \in X$. Noting that the recurrence of x is equivalent to the recurrence of $\overline{x}$, by Claims 1 and 3, we may as well assume that there are infinitely many 1 terms in the positive part of x. Let $0 < n_1 < n_2 < \ldots$ be such that $T^{n_i}\xi \to x$.

CLAIM 4. *For any $k > 0$, there exist $m(k) > 0$ such that*

$$x(m(k)) \ldots x(m(k) + k) = 00 \ldots 001.$$

Proof. Let $r > 0$ be such that $r \geq \ell_{k+1} + k + 1$ and $x(r) = 1$. Let $i > 0$ be such that $T^{n_i}\xi$ coincides with x on $[0, r]$. Thus, $x(0) \ldots x(r) = \xi(n_i) \ldots \xi(n_i + r)$. By Claim 2(b), there are k consecutive 0 terms in $x(0) \ldots x(r)$. Since $x(r) = 1$, there is $m(k) \in (0, r)$ such that

$$x(m(k)) \ldots x(m(k) + k) = 00 \ldots 001. \quad \square$$

To show the recurrence of x, it suffices to show that for any $N > 0$, there is $n > 0$ such that

$$x(n) \dots x(n+2N) = x(-N) \dots x(N).$$

Since there is some i such that $T^{n_i}\xi$ coincides with x on $[-N, N]$ and $n_i - N \geq 0$, we have

$$x(-N) \dots x(N) = \xi(n_i - N) \dots \xi(n_i + N).$$

Thus, for any $s > 0$ being such that $\ell_s > n_i + N$, $x(-N) \dots x(N)$ is a subword of ω_s. Set $t = m(n_i + 2N) + n_i + 2N + \ell_{n_i+2N}$. Let j be such that $T^{n_j}\xi$ coincides with x on $[-t, t]$. By Claim 4, $x(m(n_i + 2N)) \dots x(m(n_i + 2N) + n_i + 2N) = 00 \dots 001$. Thus,

$$(T^{n_j}\xi)(m(n_i + 2N)) \dots (T^{n_j}\xi)(m(n_i + 2N) + n_i + 2N) = 00 \dots 001.$$

By Claim 2(a), we have

$$\begin{aligned} &x(m(n_i + 2N) + n_i + 2N) \dots x(m(n_i + 2N) + n_i + 2N + \ell_{n_i+2N} - 1) \\ &\quad = (T^{n_j}\xi)(m(n_i + 2N) + n_i + 2N) \dots (T^{n_j}\xi)(m(n_i + 2N) + n_i + 2N + \ell_{n_i+2N} - 1) \\ &\quad = \xi(n_j + m(n_i + 2N) + n_i + 2N) \dots \xi(n_j + m(n_i + 2N) + n_i + 2N + \ell_{n_i+2N} - 1) \\ &\quad = \omega_{n_i+2N}. \end{aligned}$$

Note that $x(-N) \dots x(N)$ is a subword of ω_s for any $s > 0$ being such that $\ell_s > n_i + N$ and $\ell_{n_i+2N} > n_i + N$. Thus, $x(-N) \dots x(N)$ is a subword of

$$x(m(n_i + 2N) + n_i + 2N) \dots x(m(n_i + 2N) + n_i + 2N + \ell_{n_i+2N} - 1),$$

which implies the recurrence of x.

Acknowledgements. We are grateful to the referee for their helpful and thoughtful comments and suggestions. We would also like to thank Professors Hanfeng Li and Bingbing Liang for helpful comments. The work is supported by NSFC (Nos. 12271388, 12201599, 11771318, 11790274).

References

[1] N. Aoki. Topological dynamics. *Topics in General Topology*. Eds. K. Morita and J. Nagata. North-Holland, Amsterdam, 1989.

[2] A. Artigue. On continuum-wise minimality. *Fund. Math.* **258**(3) (2022), 265–289.

[3] J. Auslander. *Minimal Flows and Their Extensions (North-Holland Mathematics Studies, 153)*. North-Holland, Amsterdam, 1988.

[4] M. Brin and G. Stuck. *Introduction to Dynamical Systems*. Cambridge University Press, Cambridge, 2002.

[5] T. Downarowicz and X. Ye. When every point is either transitive or periodic. *Colloq. Math.* **93**(1) (2002), 137–150.

[6] H. Furstenberg. The structure of distal flows. *Amer. J. Math.* **85** (1963), 477–515.

[7] S. Glasner and D. Maon. Rigidity in topological dynamics. *Ergod. Th. & Dynam. Sys.* **9** (1989), 309–320.

[8] W. Gottschalk, Powers of homeomorphisms with almost periodic properties. *Bull. Amer. Math. Soc. (N.S.)* **50** (1944), 222–227.

[9] K. Hiraide. Expansive homeomorphisms of compact surfaces are pseudo-Anosov. *Osaka J. Math.* **27**(1) (1990), 117–162.

[10] Y. Katznelson and B. Weiss. *When all points are recurrent/generic*. *Ergodic Theory and Dynamical Systems, I (College Park, MD, 1979–1980) (Progress in Mathematics, 10)*. Ed. A. Katok. Birkhäuser, Boston, MA, 1981, pp. 195–210.
[11] B. Kolev and M. Pérouème. Recurrent surface homeomorphisms. *Math. Proc. Cambridge Philos. Soc.* **124** (1998), 161–168.
[12] J. Lewowicz. Expansive homeomorphisms on surfaces. *Bol. Soc. Bras. Mat.* **20**(1) (1989), 113–133.
[13] B. Liang, E. Shi, Z. Xie and H. Xu. The non-coexistence of distality and expansivity for group actions on infinite compacta. *Preprint*, 2021, arXiv:2110.00341.
[14] J. Mai and X. Ye. The structure of pointwise recurrent maps having the pseudo orbit tracing property. *Nagoya Math. J.* **166** (2002), 83–92.
[15] R. Mañé. Expansive homeomorphisms and topological dimension. *Trans. Amer. Math. Soc.* **252** (1979), 313–319.
[16] D. Montgomery. Pointwise periodic homeomorphisms. *Amer. J. Math.* **59**(1) (1937), 118–120.
[17] S. Nadler Jr. *Continuum Theory: An Introduction (Monographs and Textbooks in Pure and Applied Mathematics, 158)*. Marcel Dekker, Inc., New York, 1992.
[18] T. O'Brien and W. Reddy. Each compact orientable surface of positive genus admits an expansive homeomorphism. *Pacific J. Math.* **35** (1970), 737–741.
[19] L. Oversteegen and E. Tymchatyn. Recurrent homeomorphisms on $\mathbb{R}^2$ are periodic. *Proc. Amer. Math. Soc.* **110**(4) (1990), 1083–1088.
[20] W. Reddy. Expansive canonical coordinates are hyperbolic. *Topology Appl.* **15** (1983), 205–210.