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THE EXISTENCE OF HADLEY CONVECTIVE REGIMES OF ATMOSPHERIC MOTION

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Abstract

The solutions of the equations describing deep global convection on a rotating planet are discussed. The existence of generalized steady axisymmetric solutions is established. It is then shown that these are classical solutions when the heat source is sufficiently smooth. The solutions are shown to be unique when the heating is sufficiently weak and asymptotically stable when the shear is sufficiently small. Finally, the application of these results to Earth's atmosphere is discussed, with eddy viscosity replacing molecular viscosity.

1. Introduction

The ideal foundation for a theory of atmospheric climate would be a complete understanding of the possible flow regimes that could appear in response to differential heating in the atmosphere and an analysis of their stability properties. Viewed in a modern mathematical context [4], the problem is to catalog the hierarchy of solutions to the equations of motion that appear as heating and rotation rates are varied, to determine how the solutions exchange stability, and to ascertain the character of the set of limit solutions (as $t \to \infty$) associated with each set of values of the external parameters.

In Earth's atmosphere two distinct, basic regimes of global circulation are observed. The simpler involves an approximately steady axisymmetric flow in

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equatorial regions, which transports heat polewards by convective processes utilizing vertical overturning and which is known as a *Hadley regime* after George Hadley who described it in 1735 in an attempted explanation of the origin of the trade winds. The other involves a more complicated flow occurring in midlatitudes, which is dominated by an asymmetric, meandering eastward current transporting heat polewards by quasi-horizontal processes. It is known as a *Rossby regime* after the pioneering theoretical investigations of C.-G. Rossby in the 1930's [3, 4, 9].

Numerical and rotating annulus simulations of atmospheric flows indicate the occurrence of stable steady axisymmetric flows for small rates of axisymmetric heating; they are replaced by asymmetric oscillatory flows as the heating and rotation are increased (see references in [4, 5, 9]). From the viewpoint of bifurcation theory this suggests that Hadley-like regimes undergo a Hopf bifurcation to Rossby-like regimes. It is our long term aim to verify this mathematically. As a preliminary step, in this paper we investigate the fundamental properties of Hadley-like regimes in a system of equations for deep global convection derived from the compressible Navier-Stokes equations and the first law of thermodynamics using a Boussinesq approximation [4].

In his 1969 review of general circulation, Lorenz [9] noted that the mathematical existence of stable Hadley regimes had not then been established for either the ideal axisymmetric case, or in view of topographic variations, for the more realistic asymmetric case. Here we use the methods of Ladyzhenskaya [8] to establish the existence of generalized steady axisymmetric solutions and we show that these are in fact classical solutions when the heat source is sufficiently smooth. In addition we show that the solutions are unique when the heating is sufficiently weak and are asymptotically stable to both axisymmetric and asymmetric perturbations when the shear in the solution is sufficiently small. Our proofs can be modified readily for asymmetric steady solutions. We note that numerical computations of the above axisymmetric solutions for various physically appropriate heat sources and rotation rates have been carried out by Henderson [5].

In Section 2 we describe the system of equations for deep global convection, and then in Section 3 we state our main results on the existence, uniqueness, smoothness, and stability of steady solutions, after indicating the required functional analytic terminology. The remainder of the paper contains the proofs of our theorems for the axisymmetric case. In Section 4 we prove the existence of generalized steady axisymmetric solutions and in Section 5 we establish their uniqueness and smoothness under appropriate restrictions on the heat source. Then in Section 6 we prove the asymptotic stability of steady solutions with sufficiently small shear. Finally in Section 7 we give dimensional estimates of the above bounds on the heat source and shear for parameters appropriate to the atmosphere of Earth, showing that eddy viscosities rather than molecular viscosities must be used to obtain physically realistic quantities.

2. Equations for global deep convection

The Navier-Stokes equations for compressible flow and the first law of thermodynamics provide a basic system of equations for an atmosphere confined to a rotating planet by gravity and set in motion by differential heating (e.g. [3]). We can simplify this system by using the deep convection version of the Boussinesq approximation relative to the equilibrium state of maximum entropy associated with an atmosphere of given total energy and mass [2]. This equilibrium state has density and potential temperature profiles of the form $\rho_0(z) = \rho_0(0)\exp(-z/H)$ and $\theta_0(z) = \theta_0(0)\exp(Rz/c_p H)$, where z is the height above the planet's surface, R is the gas constant, c_p the specific heat at constant pressure and H the scale height of the atmosphere. The present state of Earth's atmosphere is, despite the differential heating and motion, quite close to its associated equilibrium state.

This approximation was used in [4] to derive a system of equations suitable for modelling global convective processes in the atmosphere. In terms of nondimensional variables this system is

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \Gamma \tau \mathbf{k} - 2\boldsymbol{\omega} \times \mathbf{v} + \alpha_0 \nabla \cdot \nu \rho_0 \nabla \mathbf{v}, \tag{2.1a}$$

$$P\left(\frac{\partial \tau}{\partial t} + \mathbf{v} \cdot \nabla \tau\right) = -\Gamma w + P\Lambda q + \alpha_0 \nabla \cdot \nu \rho_0 \nabla \tau, \tag{2.1b}$$

$$\nabla \cdot \rho_0 \mathbf{v} = 0, \tag{2.1c}$$

where spherical coordinates are used, $\mathbf{k}$ is the unit radial vector and $\boldsymbol{\omega}$ the unit rotational vector. Here $\mathbf{v} = (u, v, w)$ is the velocity, τ the fractional derivation in potential temperature, p a pressure derivation, q the heat source (which contains suitably modelled forms of net radiation, latent heat release and sensible heat transfer), α_0 the reciprocal of the equilibrium density ρ_0, ν the kinematic molecular or eddy viscosity, and P the Prandtl number. The numbers Γ and Λ are positive and depend on the equilibrium state and physical parameters. They are defined in Table 1 where in addition we indicate the dimensional scalings used. In this table θ' is the perturbation about the equilibrium potential temperature θ_0, $\boldsymbol{\Omega}$ the rotational vector with magnitude Ω, and l an appropriate length scale such as the radius of Earth. Note that for the technical convenience of having the diffusive terms formally equivalent and the coupling terms $\Gamma\tau\mathbf{k}$ and $-\Gamma w$ appear with the same coefficient Γ, we have defined τ, Γ and Λ here as, respectively, $P^{-1/2}$, $P^{1/2}$ and $P^{-1/2}$ times those in [4].

TABLE 1. Relations between dimensional and nondimensional variables

Dimensional variable	Scale × nondimensional variable
(x, y)	$l(x, y)$
z	lz
t	$\Omega^{-1}t$
(u, v)	$\Omega l(u, v)$
w	Ωlw
θ'/θ_0	$\Gamma\dfrac{\Omega^2 l}{g}\tau$
ν	$\Omega l^2\nu$

Definitions

$\Gamma = \dfrac{NP^{1/2}}{\Omega}$	$\omega = \dfrac{\Omega}{\omega}$
$\Lambda = \dfrac{g}{N\Omega lP^{1/2}}$	$N^2 = \dfrac{g}{\theta_0}\dfrac{\partial\theta_0}{\partial z} = \dfrac{gR}{c_pH}$

In order to specify boundary conditions for system (2.1) we shall suppose that the part of the atmosphere in which we are interested is confined to a spherical shell

$$\mathbb{S} = \{(\lambda, \phi, z); 0 \leqslant \lambda \leqslant 2\pi, -\pi/2 \leqslant \phi\pi/2, 0 < z < Z\},$$

where z is the dimensionless height above the planet's surface and Z is some finite number corresponding to the height of the top of the atmosphere. We shall use the boundary conditions

$$\mathbf{v} = \mathbf{0} \quad \text{on } z = 0 \text{ and } z = Z, \tag{2.2a}$$

$$\frac{\partial\tau}{\partial z} = 0 \quad \text{on } z = 0, \tag{2.2b}$$

$$\tau = 0 \quad \text{on } z = Z. \tag{2.2c}$$

Here (2.2a) implies that the top and bottom of the atmosphere act as rigid surfaces; (2.2b) eliminates conduction of heat across the planet's surface, so that essentially all thermal forcing is contained in q; and (2.2c) reflects the fact that deviations in potential temperature are small high in the atmosphere, even though this form permits some diffusive heating. Other suitable choices of boundary conditions are indicated in [4] and lead to results similar to those obtained with (2.2).

As Hadley convective regimes essentially involve steady flows we shall make considerable use in the sequel of the steady version of system (2.1). For convenience we rearrange it as

$$\alpha_0\nabla \cdot \nu\rho_0\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{v} - \nabla p + \Gamma\tau\mathbf{k} - 2\boldsymbol{\omega}\times\mathbf{v} = 0, \tag{2.3a}$$

$$\alpha_0\nabla \cdot \nu\rho_0\nabla\tau - P\mathbf{v}\cdot\nabla\tau - \Gamma w + P\Lambda q = 0, \tag{2.3b}$$

$$\nabla \cdot \rho_0\mathbf{v} = 0. \tag{2.3c}$$

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We shall consider in particular the axisymmetric solutions of system (2.3), reflecting the axisymmetric character of Hadley regimes. In this case the longitudinal (λ) dependence can be suppressed in system (2.3) and the appropriate domain for $\mathbf{v}$ and τ is the semi-annular region

$$\mathcal{A} = \{(\phi, z);\ -\pi/2 \leqslant \phi \leqslant \pi/2, 0 < z < Z\}.$$

In view of the spherical coordinates here, the boundary $\partial\mathcal{A}$ of region $\mathcal{A}$ consists of the two rims $z = 0$ and $z = Z$.

3. Hadley convective regimes

We define a *Hadley convective regime* as a solution, in particular an axisymmetric solution, of the steady system of equations (2.3) satisfying the boundary conditions (2.2). The physical relevance of such a regime, of course, depends on the actual form of the heat source q, which should in general have non-vanishing latitudinal gradient and be small in some appropriate sense. By a theorem of Jeffreys [4, 6] the gradient in q implies a nontrivial velocity field, that is, actual motion. When, however, there is no heating the trivial state $(\mathbf{v}, \tau) = (\mathbf{0}, 0)$ is a solution of system (2.1) or (2.3) with boundary conditions (2.2). Moreover it is asymptotically stable in the sense of Lyapunov.

THEOREM 3.1. *When the heat source* $q = 0$, *the trivial solution* $(\mathbf{v}, \tau) = (\mathbf{0}, 0)$ *of system* (2.1) *with boundary conditions* (2.2) *is asymptotically stable.*

PROOF. We define the energy of any solution $(\mathbf{v}, \tau)$ of system (2.1) with boundary conditions (2.2) as

$$E = \tfrac{1}{2}\int \rho_0\{\mathbf{v}\cdot\mathbf{v} + P\tau^2\}\, dV, \tag{3.1}$$

where the integral is over the spherical shell $\mathbb{S}$ and dV is an element of volume in spherical coordinates. We can then show that the time rate of change of E satisfies

$$\begin{aligned}\frac{dE}{dt} &= \int \rho_0\left\{\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} + P\tau\frac{d\tau}{dt}\right\} dV \\ &= -\nu\int\rho_0\left\{|\nabla\mathbf{v}|^2 + |\nabla\tau|^2\right\} dV - B,\end{aligned} \tag{3.2}$$

by substituting for the total derivatives of $\mathbf{v}$ and τ from equations (2.1a) and (2.1b), integrating by parts and using the continuity equation (2.1c) and the boundary conditions (2.2). Here B represents nonnegative dissipative effects on the boundary.

The energy given by (3.1) is positive definite and its time rate of change (3.2) is negative definite in $(\mathbf{v}, \tau)$. Considering E as a Lyapunov function, we can thus conclude that the trivial solution is asymptotically stable in the sense of Lyapunov. □

Theorem 3.1 suggests that when the heat source q is sufficiently small we should find steady solutions, that is Hadley regimes, which are close to the trivial state and which are themselves stable. For the axisymmetric case this is indeed true and its verification constitutes our main result. Our method of proof is based on that developed by Ladyzhenskaya [8] for similar problems involving the Navier-Stokes equations for incompressible flow. To proceed we need to introduce some terminology. We denote by $L_p(\mathscr{Q})$ the Banach space of p-summable axisymmetric functions on $\mathscr{Q}$ with weighted norm

$$\|f\|_p = \left(\int \rho_0 |f|^p \, dA\right)^{1/p},$$

where the integral is over the semi-annular region $\mathscr{Q}$ and dA is an element of area, and by $W^{k,p}(\mathscr{Q})$ the Sobolev space of functions defined on $\mathscr{Q}$ with p-summable derivatives $D^\alpha f$ of orders $|\alpha| \leqslant k$ and with norm

$$\|f\|_{k,p} = \left(\sum_{|\alpha|=0}^{k} \int \rho_0 |D^\alpha f|^p \, dA\right)^{1/p}.$$

By $\mathring{J}(\mathscr{Q})$ we denote the space of all smooth axisymmetric 4-dimensional vector fields (ψ, σ) on $\mathscr{Q}$, with $\psi = (\psi_1, \psi_2, \psi_3)$ having compact support in $\mathscr{Q}$ and satisfying the solenoidal-like condition $\nabla \cdot \rho_0 \psi = 0$ where ∇ is the spherical coordinate gradient (note that axisymmetry means no λ dependence), and with σ continuous on $\overline{\mathscr{Q}}$ and vanishing in an open neighborhood in $\mathscr{Q}$ of the rim $z = Z$, but being arbitrary on the rim $z = 0$. We define on $\mathring{J}(\mathscr{Q})$ the weighted inner product

$$[(\psi', \sigma'), (\psi'', \sigma'')] = \int \rho_0 \{\nabla \psi' \cdot \nabla \psi'' + \nabla \sigma' \cdot \nabla \sigma''\} \, dA, \tag{3.3}$$

where $\nabla \psi' \cdot \nabla \psi'' = \sum_{i=1}^{3} (\nabla \psi_i') \cdot (\nabla \psi_i'')$, with the second dot product being the usual one. We then define $H(\mathscr{Q})$ to be the completion of $\mathring{J}(\mathscr{Q})$ in the norm associated with the inner product (3.3), which we denote by $\|(\psi, \sigma)\|_H$.

By a *classical solution* $(\mathbf{v}, \tau)$ of the steady system (2.3) with boundary conditions (2.2) we mean a solution for which all derivatives appearing in (2.2) and (2.3) exist, are continuous, and satisfy the equations pointwise on $\partial\mathscr{Q}$ and in $\mathscr{Q}$. Finally

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by a *generalized solution* we mean a vector field $(\mathbf{v}, \tau) \in H(\mathcal{Q})$ such that

$$-\nu \int \rho_0 \{ \nabla \psi \cdot \nabla \mathbf{v} + \nabla \sigma \cdot \nabla \tau \} \, dA - \int \rho_0 \{ \psi \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) + P \sigma \mathbf{v} \cdot \nabla \tau \} \, dA$$
$$- \int 2 \rho_0 \psi \cdot (\omega \times \mathbf{v}) \, dA + \Gamma \int \rho_0 \{ \psi_3 \tau - \sigma w \} \, dA + P\Lambda \int \rho_0 \sigma q \, dA = 0, \quad (3.4)$$

for all $(\psi, \sigma) \in \mathring{J}(\mathcal{Q})$. Consequently a classical solution is also a generalized solution.

We can now summarize our results in the following theorems.

THEOREM 3.2. *For any axisymmetric heat source $q \in L_2(\mathcal{Q})$ there exists at least one axisymmetric generalized solution $(\mathbf{v}, \tau) \in H(\mathcal{Q})$ of the steady system* (2.3) *with boundary conditions* (2.2). *These solutions satisfy the bound*

$$\|(\mathbf{v}, \tau)\|_H \leqslant \nu^{-1} P \Lambda K^{1/2} \|q\|_2 \quad (3.5)$$

and are unique when the heating is sufficiently small, specifically when

$$\|q\|_2 \leqslant \nu^2 (4 \eta P \Lambda K L^{1/2})^{-1}, \quad (3.6)$$

where $\eta = \max\{1, P\}$ and K, L are constants depending only on ρ_0 and the domain $\mathcal{Q}$. Moreover the generalized solutions are classical solutions when $q \in W^{1,p}(\mathcal{Q})$ for some $p > 2$.

THEOREM 3.3. *A steady axisymmetric generalized solution $(\mathbf{v}, \tau)$ of system* (2.1) *with boundary conditions* (2.2), *which satisfies the bound*

$$\|(\mathbf{v}, \tau)\|_H < \nu (4 \eta K^{1/2} L^{1/2})^{-1} \quad (3.7)$$

is asymptotically stable in the sense of Lyapunov to both axisymmetric and asymmetric admissible generalized perturbations, provided the bound (3.7) *holds for constants K and L corresponding to the domain $\mathcal{S}$.*

We shall prove these theorems in the remaining sections of the paper. Our proofs there can be readily modified to the 3-dimensional asymmetric case to give the following corollary.

COROLLARY 3.3. *Theorems* 3.2 *and* 3.3 *remain valid for asymmetric steady solutions provided the constants K and L are those corresponding to the spherical shell $\mathcal{S}$.*

The constants K and L are indicated in Section 7, where numerical values for the bound (3.6) are defined for parameters appropriate to the atmosphere of the earth.

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In [9] Lorenz distinguishes between ideal and modified Hadley regimes. By the former he means axisymmetric regimes corresponding to axisymmetric heat sources and pertaining to idealized planets with surfaces all land or all ocean, or with shorelines of continents lying along latitude circles. He notes that among general circulation theorists, one school of thought was that ideal Hadley regimes were dynamically impossible, whereas another school of thought, which included V. Bjerknes, considered them to be possible, but always unstable. Our results here and the numerical computations of Henderson in [5] show that ideal Hadley regimes always exist for sufficiently smooth heat sources, and are in fact both stable and unique when the heating is sufficiently weak. When the intensity of heating becomes comparable to that of solar heating in Earth's atmosphere this stability may be lost, but this may be owing to a bifurcation to a stable flow which is Hadley-like in equatorial regions and Rossby-like in mid-latitudes. Our results are also applicable to Lorenz's modified Hadley regimes which possess longitudinal variations owing to such variation in the heat source and topography.

Lorenz commented that these issues had not been resolved because there was no proof of the existence of a Hadley regime. The present article thus contributes clarification to the theoretical question, and provides the first step in developing a mathematical theory of the hierarchy of flow regimes involved in atmospheric flow and climate.

4. Proof of existence

We prove the existence part of Theorem 3.2 by modifying to the present context the proof used by Ladyzhenskaya [8; Chapter 5] for the steady incompressible Navier-Stokes equations. For this we require some inequalities for the Hilbert space $H(\mathcal{Q})$. We note here that $\rho_0(z) = \rho_0(0)\exp(-z/H)$ is the equilibrium density profile, and we define $\rho_{\max} = \max \rho_0(z) = \rho_0(0)$ and $\rho_{\min} = \min \rho_0(z) = \rho_0(0)\exp(-Z/H)$.

LEMMA 4.1. *For any component u of a vector field $(\psi, \sigma) \in H(\mathcal{Q})$,*

$$\int \rho_0 u^2\, dA \leqslant K \int \rho_0 |\nabla u|^2\, dA \tag{4.1}$$

and

$$\int \rho_0 u^4\, dA \leqslant L \int \rho_0 u^2\, dA \int \rho_0 |\nabla u|^2\, dA, \tag{4.2}$$

where the constant K depends on ρ_0 and the domain $\mathcal{Q}$, and $L = 2\rho_{\max}\rho_{\min}^{-2}$.

PROOF. Let $(\psi, \sigma) \in \mathring{J}(\mathcal{Q})$. Thus

$$\int u^2\, dA \leqslant K_1 \int |\nabla u|^2\, dA, \tag{4.3}$$

and

$$\int u^4\, dA \leqslant 2 \int u^2\, dA \int |\nabla u|^2\, dA \tag{4.4}$$

for any component u of (ψ, σ), where the constant K_1 depends only on the domain $\mathcal{Q}$. Here (4.3) is the Poincaré inequality and (4.4) follows from Lemma 1 in section 1.1 of [8], even for the component σ which only vanishes in a neighborhood of the rim $z = Z$ of $\mathcal{Q}$. From (4.3),

$$\begin{aligned}\int \rho_0 u^2\, dA &\leqslant \rho_{\max} \int u^2\, dA \\ &\leqslant K_1 \rho_{\max} \int |\nabla u|^2\, dA \leqslant K_1 \rho_{\max} \rho_{\min}^{-1} \int \rho_0 |\nabla u|^2\, dA,\end{aligned}$$

and from (4.4)

$$\begin{aligned}\int \rho_0 u^4\, dA &\leqslant \rho_{\max} \int u^4\, dA \\ &\leqslant 2\rho_{\max} \int u^2\, dA \int |\nabla u|^2\, dA \\ &\leqslant 2\rho_{\max} \rho_{\min}^{-2} \int \rho_0 u^2\, dA \int \rho_0 |\nabla u|^2\, dA.\end{aligned}$$

Inequalities (4.1) and (4.2) then follow from these inequalities and the fact that $H(\mathcal{Q})$ is the completion of $\mathring{J}(\mathcal{Q})$. $\square$

Next we define the linear functionals

$$J_1(\psi, \sigma) = P\Lambda \int \rho_0 \sigma q\, dA, \tag{4.5}$$

$$J_2(\psi, \sigma) = \int 2\rho_0 \psi \cdot (\boldsymbol{\omega} \times \mathbf{v})\, dA, \tag{4.6}$$

$$J_3(\psi, \sigma) = \Gamma \int \rho_0 (\sigma w - \psi_3 \tau)\, dA, \tag{4.7}$$

and

$$J_4(\psi, \sigma) = \int \rho_0 [\psi \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) + P\sigma \mathbf{v} \cdot \nabla \tau]\, dA \tag{4.8}$$

on $\mathring{J}(\mathcal{Q})$ for fixed $q \in L_2(\mathcal{Q})$ and $(\mathbf{v}, \tau) \in H(\mathcal{Q})$. In the following three lemmas we show that these functionals are bounded on $\mathring{J}(\mathcal{Q})$.

Lemma 4.2. *The functional J_1 is bounded on $\mathring{J}(\mathscr{Q})$.*

Proof. Using the Cauchy-Schwarz and Poincaré inequalities, we have

$$\begin{aligned} |J_1(\psi, \sigma)| &\leqslant P\Lambda \int |\rho_0^{1/2}\sigma\rho_0^{1/2}q|\, dA \\ &\leqslant P\Lambda \left(\int \rho_0\sigma^2\, dA\right)^{1/2} \left(\int \rho_0 q^2\, dA\right)^{1/2} \\ &\leqslant P\Lambda K^{1/2} \left(\int \rho_0 |\nabla\sigma|^2\, dA\right)^{1/2} \|q\|_2 \\ &\leqslant P\Lambda K^{1/2} \|(\psi, \sigma)\|_H \|q\|_2, \end{aligned}$$

for any $(\psi, \sigma) \in \mathring{J}(\mathscr{Q})$. □

In the proof of the next two lemmas we use the elementary inequality

$$\sum_{i,j=1}^{N} \alpha_i^{1/2}\beta_j^{1/2} \leqslant 2^{N-1}\left(\sum_{i=1}^{N} \alpha_i\right)^{1/2} \left(\sum_{j=1}^{N} \beta_j\right)^{1/2} \tag{4.9}$$

for $N = 2$ and 3, where α_i and β_j are arbitrary nonnegative numbers. Also, for convenience, we sometimes write $\mathbf{v} = (v_1, v_2, v_3)$.

Lemma 4.3. *The functionals J_2 and J_3 are bounded on $\mathring{J}(\mathscr{Q})$.*

Proof. Since $\boldsymbol{\omega}$ is a unit vector, on using the Cauchy-Schwarz inequality, Poincaré inequality and inequality (4.9) with $N = 3$, we have

$$\begin{aligned} |J_2(\psi, \sigma)| &\leqslant \sum_{i,j=1}^{3} 2\int \rho_0 |\psi_i v_j|\, dA \\ &\leqslant \sum_{i,j=1}^{3} 2\left(\int \rho_0\psi_i^2\, dA\right)^{1/2} \left(\int \rho_0 v_j^2\, dA\right)^{1/2} \\ &\leqslant 2K \sum_{i,j=1}^{3} \left(\int \rho_0 |\nabla\psi_i|^2\, dA\right)^{1/2} \left(\int \rho_0 |\nabla v_j|^2\, dA\right)^{1/2} \\ &\leqslant 8K \left(\int \rho_0 \nabla\boldsymbol{\psi} \cdot \nabla\boldsymbol{\psi}\, dA\right)^{1/2} \left(\int \rho_0 \nabla\mathbf{v} \cdot \nabla\mathbf{v}\, dA\right)^{1/2} \\ &\leqslant 8K \|(\psi, \sigma)\|_H \|(\mathbf{v}, \tau)\|_H \end{aligned}$$

for any $(\psi, \sigma) \in \mathring{J}(\mathscr{Q})$. Similarly, but now with $N = 2$ in inequality (4.9), we obtain for any $(\psi, \sigma) \in \mathring{J}(\mathscr{Q})$

$$
\begin{aligned}
|J_3(\psi, \sigma)| &\leq \Gamma \int |\rho_0 \sigma w| \, dA + \Gamma \int |\rho_0 \psi_3 \tau| \, dA \\
&\leq \Gamma \left(\int \rho_0 \sigma^2 \, dA \right)^{1/2} \left(\int \rho_0 w^2 dA \right)^{1/2} + \Gamma \left(\int \rho_0 \psi_3^2 \, dA \right)^{1/2} \left(\int \rho_0 \tau^2 \, dA \right)^{1/2} \\
&\leq \Gamma K \left(\int \rho_0 |\nabla \sigma|^2 \, dA \right)^{1/2} \left(\int \rho_0 |\nabla w|^2 \, dA \right)^{1/2} \\
&\quad + \Gamma K \left(\int \rho_0 |\nabla \psi_3|^2 \, dA \right)^{1/2} \left(\int \rho_0 |\nabla \tau|^2 \, dA \right)^{1/2} \\
&\leq 2\Gamma K \left(\int \rho_0 \{ |\nabla \psi_3|^2 + |\nabla \sigma|^2 \} \, dA \right)^{1/2} \left(\int \rho_0 \{ |\nabla w|^2 + |\nabla \tau|^2 \} \, dA \right)^{1/2} \\
&\leq 2\Gamma K \|(\psi, \sigma)\|_H \|(\mathbf{v}, \tau)\|_H. \quad \square
\end{aligned}
$$

When integrating by parts in the proof of the following lemma we use the fact that for $(\mathbf{v}, \tau) \in H(\mathscr{Q})$ the velocity $\mathbf{v}$ vanishes almost everywhere on $\partial\mathscr{Q}$, in particular on the rim $z = 0$. This in turn follows from the facts that such a $\mathbf{v}$ is the limit in the Sobolev space $W^{1,2}(\mathscr{Q})$ of a sequence of vector fields (ψ_n), where the ψ_n have compact support in $\mathscr{Q}$, and that the Sobolev space $W^{1,2}(\mathscr{Q})$ is continuously embedded in $L_2(\partial\mathscr{Q})$ [1; Theorem 5.4B] with embedding constant K_2, and so

$$
\|\mathbf{v}\|_{L_2(\partial\mathscr{Q})} = \|\mathbf{v} - \psi_n\|_{L_2(\partial\mathscr{Q})} \leq K_2 \|\mathbf{v} - \psi_n\|_{1,2} \to 0
$$

as $n \to \infty$.

Lemma 4.4. *The functional J_4 is bounded on $\mathring{J}(\mathscr{Q})$.*

Proof. First we integrate by parts, using the continuity equation (2.3c) and $\mathbf{v} = \mathbf{0}$ almost everywhere on $\partial\mathscr{Q}$ to obtain

$$
\begin{aligned}
J_4(\psi, \sigma) &= \sum_{i=1}^{3} \int \rho_0 \psi_i \mathbf{v} \cdot \nabla v_i \, dA + P \int \rho_0 \sigma \mathbf{v} \cdot \nabla \tau \, dA \\
&= - \sum_{i=1}^{3} \int \rho_0 v_i \mathbf{v} \cdot \nabla \psi_i \, dA - P \int \rho_0 \tau \mathbf{v} \cdot \nabla \sigma \, dA
\end{aligned}
$$

for any $(\psi, \sigma) \in \mathring{J}(\mathscr{Q})$. Consequently with $\eta = \max(1, P)$, the Cauchy-Schwarz inequality, the Poincaré inequality, inequality (4.2) and the elementary inequality for nonnegative real numbers $(\Sigma_{i=1}^4 \alpha_i)^2 \leqslant 4\Sigma_{i=1}^4 \alpha_i^2$, we obtain

$$
\begin{aligned}
|J_4(\psi,\sigma)| &\leqslant \sum_{i=1}^{3} \int \left|\rho_0^{1/4} v_i \rho_0^{1/4} \mathbf{v} \cdot \rho_0^{1/2} \nabla \psi_i\right| dA + P \int \left|\rho_0^{1/4} \tau \rho_0^{1/4} \mathbf{v} \cdot \rho_0^{1/2} \nabla \sigma\right| dA \\
&\leqslant \sum_{j=2}^{3} \left\{ \sum_{i=1}^{3} \left(\int \rho_0 v_i^4\, dA \right)^{1/4} \left(\int \rho_0 v_j^4\, dA \right)^{1/4} \left(\int \rho_0 |\nabla \psi_i|^2\, dA \right)^{1/2} \right. \\
&\qquad \left. + P \left(\int \rho_0 \tau^4\, dA \right)^{1/4} \left(\int \rho_0 v_j^4\, dA \right)^{1/4} \left(\int \rho_0 |\nabla \psi_i|^2\, dA \right)^{1/2} \right\} \\
&\leqslant \eta \sum_{j=2}^{3} \left(\int \rho_0 v_j^4\, dA \right)^{1/4} \left\{ \sum_{i=1}^{3} \left(\int \rho_0 v_i^4\, dA \right)^{1/4} + \left(\int \rho_0 \tau^4\, dA \right)^{1/4} \right\} \|(\psi,\sigma)\|_H \\
&\leqslant \eta \left\{ \sum_{i=1}^{3} \left(\int \rho_0 v_i^4\, dA \right)^{1/4} + \left(\int \rho_0 \tau^4\, dA \right)^{1/4} \right\}^2 \|(\psi,\sigma)\|_H \\
&\leqslant \eta L^{1/2} \left\{ \sum_{i=1}^{3} \left(\int \rho_0 v_i^2\, dA \right)^{1/4} \left(\int \rho_0 |\nabla v_i|^2\, dA \right)^{1/4} \right. \\
&\qquad \left. + \left(\int \rho_0 \tau^2\, dA \right)^{1/4} \left(\int \rho_0 |\nabla \tau|^2\, dA \right)^{1/4} \right\}^2 \|(\psi,\sigma)\|_H \\
&\leqslant \eta L^{1/2} K^{1/2} \left\{ \sum_{i=1}^{3} \left(\int \rho_0 |\nabla v_i|^2\, dA \right)^{1/2} + \left(\int \rho_0 |\nabla \tau|^2\, dA \right)^{1/2} \right\}^2 \|(\psi,\sigma)\|_H \\
&\leqslant 4\eta L^{1/2} K^{1/2} \|(\mathbf{v},\tau)\|_H^2 \|(\psi,\sigma)\|_H. \quad \square
\end{aligned}
$$

We can apply the Riesz representation theorem [e.g. 8; page 31], on the Hilbert space $H(\mathcal{Q})$ for each of the bounded linear functionals (4.5)–(4.8) to obtain

LEMMA 4.5. *For each $q \in L_2(\mathcal{Q})$ and $(\mathbf{v}, \tau) \in H(\mathcal{Q})$ there exist unique elements $A_1(q)$, $A_2(\mathbf{v},\tau)$, $A_3(\mathbf{v},\tau)$ and $A_4(\mathbf{v},\tau)$ of $H(\mathcal{Q})$ such that*

$$J_1(\psi,\sigma) = [(\psi,\sigma), A_1(q)], \tag{4.10}$$

$$J_2(\psi,\sigma) = [(\psi,\sigma), A_2(\mathbf{v},\tau)], \tag{4.11}$$

$$J_3(\psi,\sigma) = [(\psi,\sigma), A_3(\mathbf{v},\tau)], \tag{4.12}$$

and

$$J_4(\psi,\sigma) = [(\psi,\sigma), A_4(\mathbf{v},\tau)] \tag{4.13}$$

for all $(\psi,\sigma) \in \mathring{J}(\mathcal{Q})$.

We need to show that A_2, A_3 and A_4 are compact operators on the Hilbert space $H(\mathscr{Q})$, that is, they map bounded sequences in $H(\mathscr{Q})$ into sequences in $H(\mathscr{Q})$ that have convergent subsequences. For this we note that a closed ball $B(R)$ of radius R in $H(\mathscr{Q})$ is a compact subset of the spaces $L_2(\mathscr{Q})$ and $L_4(\mathscr{Q})$, because $H(\mathscr{Q})$ is (componentwise) a closed subspace of the Sobolev space $W^{1,2}(\mathscr{Q})$, which is compactly embedded in the spaces $L_2(\mathscr{Q})$ and $L_4(\mathscr{Q})$ [1; Theorem 5.4B]. Many of the steps in the following proof are analogous to those in the proofs of Lemmas 4.3 and 4.4, and so are done without further justification.

LEMMA 4.6. *The operators* A_2, A_3 *and* A_4 *are compact on* $H(\mathscr{Q})$.

PROOF. Let $\{\mathbf{v}_n, \tau_n\}$ be a sequence in $B(R)$ for some fixed radius R. Because $\boldsymbol{\omega}$ is a unit vector we have

$$\begin{aligned}
|[(\psi, \sigma), A_2(\mathbf{v}_n, \tau) - A_2(\mathbf{v}_m, \tau_m)]| &\leq \sum_{i,j=1}^{3} 2\int |\rho_0 \psi_i (v_{n,j} - v_{m,j})| dA \\
&\leq \sum_{i,j=1}^{3} 2\left(\int \rho_0 \psi_i^2\, dA\right)^{1/2} \left(\int \rho_0 (v_{n,j} - v_{m,j})^2\, dA\right)^{1/2} \\
&\leq K^{1/2} \sum_{i,j=1}^{3} 2\left(\int \rho_0 |\nabla \psi_i|^2\, dA\right)^{1/2} \left(\int \rho_0 (v_{n,j} - v_{m,j})^2\, dA\right)^{1/2} \\
&\leq 8K^{1/2}\left(\int \rho_0 |\nabla \psi|^2\, dA\right)^{1/2} \left(\int \rho_0 |\mathbf{v}_n - \mathbf{v}_m|^2\, dA\right)^{1/2} \\
&\leq 8K^{1/2} \|(\psi, \sigma)\|_H \|(\mathbf{v}_n, \tau_n) - (\mathbf{v}_m, \tau_m)\|_2
\end{aligned}$$

so with $(\psi, \sigma) = A_2(\mathbf{v}_n, \tau_n) - A_2(\mathbf{v}_m, \tau_m)$ we have

$$\|A_2(\mathbf{v}_n, \tau_n) - A_2(\mathbf{v}_m, \tau_m)\|_H \leq 8K^{1/2} \|(\mathbf{v}_n, \tau_n) - (\mathbf{v}_m, \tau_m)\|_2. \tag{4.14}$$

From this and the remark preceding the statement of the lemma we conclude that A_2 is a compact operator on $H(\mathscr{Q})$. Similarly for operator A_3,

$$\begin{aligned}
&|[(\psi, \sigma), A_3(\mathbf{v}_n, \tau_n) - A_3(\mathbf{v}_m, \tau_m)]| \\
&\quad\leq \Gamma \int |\rho_0 \sigma (w_n - w_m)| dA + \Gamma \int |\rho_0 \psi_3 (\tau_n - \tau_m)| dA \\
&\quad\leq \Gamma \left(\int \rho_0 \sigma^2\, dA\right)^{1/2} \left(\int \rho_0 (w_n - w_m)^2\, dA\right)^{1/2} \\
&\qquad + \Gamma \left(\int \rho_0 \psi_3^2\, dA\right)^{1/2} \left(\int \rho_0 (\tau_n - \tau_m)^2\, dA\right)^{1/2} \\
&\quad\leq \Gamma K^{1/2} \left\{ \left(\int \rho_0 |\nabla \sigma|^2\, dA\right)^{1/2} + \left(\int \rho_0 |\nabla \psi_3|^2\, dA\right)^{1/2} \right\} \|(\mathbf{v}_n, \tau_n) - (\mathbf{v}_m, \tau_m)\|_2 \\
&\quad\leq 2\Gamma K^{1/2} \|(\psi, \sigma)\|_H \|(\mathbf{v}_n, \tau_n) - (\mathbf{v}_m, \tau_m)\|_2,
\end{aligned}$$

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which leads to an inequality similar to (4.14) and in turn to the compactness on $H(\mathscr{Q})$ of the operator A_3.

Finally for operator A_4

$$
\begin{aligned}
&|[(\psi, \sigma), A_4(\mathbf{v}_n, \tau) - A_4(\mathbf{v}_m, \tau_m)]| \\
&\quad \leqslant \sum_{i=1}^{3} \int |\rho_0(v_{n,i}\mathbf{v}_n - v_{m,i}\mathbf{v}_m) \cdot \nabla\psi_i| dA + P\int |\rho_0(\tau_n\mathbf{v}_n - \tau_m\mathbf{v}_m) \cdot \nabla\sigma| dA \\
&\quad \leqslant \sum_{i=1}^{3} \left\{ \int |\rho_0(v_{n,i} - v_{m,i})\mathbf{v}_n \cdot \nabla\psi_i| dA + \int |\rho_0 v_{m,i}(\mathbf{v}_n - \mathbf{v}_m) \cdot \nabla\psi_i| dA \right\} \\
&\qquad + P\int |\rho_0(\tau_n - \tau_m)\mathbf{v}_n \cdot \nabla\sigma| dA + P\int |\rho_0\tau_m(\mathbf{v}_n - \mathbf{v}_m) \cdot \nabla\sigma| dA \\
&\quad \leqslant \sum_{i=1}^{3} \left\{ \|v_{n,i} - v_{m,i}\|_4 \|\mathbf{v}_n\|_4 \|\nabla\psi_i\|_2 + \|v_{m,i}\|_4 \|\mathbf{v}_n - \mathbf{v}_m\|_r \|\nabla\psi_i\|_2 \right\} \\
&\qquad + P\left\{ \|\tau_n - \tau_m\|_4 \|\mathbf{v}_n\|_4 \|\nabla\sigma\|_2 + \|\tau_m\|_4 \|\mathbf{v}_n - \mathbf{v}_m\|_4 \|\nabla\sigma\|_2 \right\} \\
&\quad \leqslant M_1\left\{ \|\mathbf{v}_n - \mathbf{v}_m\|_4 \|\nabla\psi\|_2 + \|\tau_n - \tau_m\|_4 \|\nabla\sigma\|_2 \right\} \\
&\quad \leqslant M_2 \|(\psi, \sigma)\|_H \|(\mathbf{v}_n, \tau_n) - (\mathbf{v}_m, \tau_m)\|_4
\end{aligned}
$$

for appropriate constant M_1 and M_2. The compactness on $H(\mathscr{Q})$ of operator A_4 then follows from an inequality analogous to (4.19) and the remark preceding the lemma. □

We now note from (3.4), (4.5)–(4.8) and (4.10)–(4.13) that a generalized solution $(\mathbf{v}, \tau) \in H(\mathscr{Q})$ of the steady system (2.3) with boundary conditions (2.2) satisfies

$$[(\psi, \sigma), (-\nu - A_2 + A_3 - A_4)(\mathbf{v}, \tau) + A_1(q)] = 0 \tag{4.15}$$

for all $(\psi, \sigma) \in \mathring{J}(\mathscr{Q})$ or equivalently

$$(\mathbf{v}, \tau) = \nu^{-1}T(\mathbf{v}, \tau). \tag{4.16}$$

Here T is defined for fixed $q \in L_2(\mathscr{Q})$ and all $(\mathbf{v}, \tau) \in H(\mathscr{Q})$ as

$$T(\mathbf{v}, \tau) = (-A_2 + A_3 - A_4)(\mathbf{v}, \tau) + A_1(q), \tag{4.17}$$

and from Lemma 4.6 is a compact operator on $H(\mathscr{Q})$. Moreover for any $0 \leqslant \lambda \leqslant \nu^{-1}$ any solution $(\mathbf{v}, \tau)$ in $H(\mathscr{Q})$ of

$$(\mathbf{v}, \tau) = \lambda T(\mathbf{v}, \tau) \tag{4.18}$$

satisfies

$$\begin{aligned}
0 &= [(\mathbf{v}, \tau), (\mathbf{v}, \tau) - \lambda(\mathbf{v}, \tau)] \\
&= [(\mathbf{v}, \tau), (\mathbf{v}, \tau) - \lambda(-A_2 + A_3 - A_4)(\mathbf{v}, \tau) - \lambda A_1(q)] \\
&= [(\mathbf{v}, \tau), (\mathbf{v}, \tau)] + \lambda \sum_{i=2}^{4} (-1)^i [(\mathbf{v}, \tau), A_i(\mathbf{v}, \tau)] - \lambda[(\mathbf{v}, \tau), A_1(q)] \\
&= \|(\mathbf{v}, \tau)\|_H^2 - \lambda P\Lambda \int \rho_0 \tau q \, dA
\end{aligned}$$

because

$$[(\mathbf{v}, \tau), A_2(\mathbf{v}, \tau)] = \int \rho_0 \mathbf{v} \cdot (\boldsymbol{\omega} \times \mathbf{v}) \, dA = 0,$$

$$[(\mathbf{v}, \tau), A_3(\mathbf{v}, \tau)] = \Gamma \int \rho_0 (\tau w - w\tau) \, dA = 0$$

and

$$\begin{aligned}
[(\mathbf{v}, \tau), A_4(\mathbf{v}, \tau)] &= \int \rho_0 (\mathbf{v} \cdot (\mathbf{v} \nabla \mathbf{v}) + P\tau \mathbf{v} \cdot \nabla \tau) \, dA \\
&= \tfrac{1}{2} \int \rho_0 \mathbf{v} \cdot \nabla (\mathbf{v} \cdot \mathbf{v} + P\tau^2) \, dA = 0.
\end{aligned}$$

Hence the solutions of (4.18) satisfy

$$\begin{aligned}
\|(\mathbf{v}, \tau)\|_H^2 &= \lambda P\Lambda \int \rho_0 \tau q \, dA \\
&\leqslant \lambda P\Lambda \left(\int \rho_0 \tau^2 \, dA \right)^{1/2} \left(\int \rho_0 q^2 \, dA \right)^{1/2} \\
&\leqslant \lambda P\Lambda K^{1/2} \left(\int \rho_0 |\nabla \tau|^2 \, dA \right)^{1/2} \left(\int \rho_0 q^2 \, dA \right)^{1/2} \\
&\leqslant \nu^{-1} P\Lambda K^{1/2} \|(\mathbf{v}, \tau)\|_H \|q\|_2,
\end{aligned}$$

that is,

$$\|(\mathbf{v}, \tau)\|_H \leqslant \nu^{-1} P\Lambda K^{1/2} \|q\|_2 \tag{4.19}$$

unformly in $0 \leqslant \lambda \leqslant \nu^{-1}$. As T is a compact operator on $H(\mathscr{C})$ we can thus apply the Leray-Schauder principle [8; page 32], which says that if all possible solutions of equation (4.18) for $0 \leqslant \lambda \leqslant \nu^{-1}$ lie within some closed ball in $H(\mathscr{C})$, then equation (4.18) with $\lambda = \nu^{-1}$ has at least one solution in this ball. We conclude that (4.16), or equivalently (4.15), has at least one solution in $H(\mathscr{C})$. Hence the steady system (2.3) with boundary conditions (2.2) has at least one generalized solution satisfying the bound (4.19).

This completes the existence part of the proof for Theorem 3.2.

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5. Proof of uniqueness and smoothness

Let $(\mathbf{v}', \tau')$ and $(\mathbf{v}'', \tau'')$ be any two generalized solutions of system (2.3) with boundary conditions (2.2) corresponding to the same heat source $q \in L_2(\mathscr{Q})$ and let $V = \mathbf{v}'' - \mathbf{v}'$, $T = \tau'' - \tau'$. Then on using $(\mathbf{V}, T)$ for (ψ, σ) in (3.4) for each solution, rearranging, and integrating by parts, we obtain

$$\nu \int \rho_0(\nabla \mathbf{V} \cdot \nabla \mathbf{V} + \nabla T \cdot \nabla T)\, dA = \int \rho_0 \mathbf{v}' \cdot (\mathbf{V} \cdot \nabla \mathbf{V})\, dA + P \int \rho_0 \tau' \mathbf{V} \cdot \nabla T\, dA.$$

Hence by the same reasoning as in the proof of Lemma 4.4, we find that

$$\begin{aligned} \nu \|(\mathbf{V}, T)\|_H^2 &\leq 4\eta K^{1/2} L^{1/2} \|(\mathbf{v}', \tau')\|_H \|(\mathbf{V}, T)\|_H^2 \\ &\leq 4\eta P \Lambda K L^{1/2} \nu^{-1} \|q\|_2 \|(\mathbf{V}, T)\|_H^2, \end{aligned} \tag{5.1}$$

where $\eta = \max(1, P)$, on using the bound (4.19) for the generalized solution $(\mathbf{v}', \tau')$. Hence from (5.1), it follows that if

$$4\eta P \Lambda K L^{1/2} \nu^{-1} \|q\|_2 < \nu$$

then $\|(\mathbf{V}, T)\|_H = 0$, that is, two generalized solutions coincide in $H(\mathscr{Q})$ when the heat source q is bounded by

$$\|q\|_2 \leq \nu^2 (4\eta P \Lambda K L^{1/2})^{-1}. \tag{5.2}$$

This gives a sufficient condition on the heat source q for the uniqueness of generalized solutions of the steady system (2.3) with boundary conditions (2.2).

We shall now show that if the heat source q belongs to the Sobolev space $W^{1,p}(\mathscr{Q})$ for some $p > 2$ rather than just to $L_2(\mathscr{Q})$, then the corresponding generalized solution $(\mathbf{v}, \tau) \in H(\mathscr{Q})$ of system (2.3) with boundary conditions (2.2) is in fact a classical solution. For this we note that if τ is regarded as a given forcing term, then the velocity $\mathbf{v}$ satisfies (in the generalized sense) the steady compressible Navier-Stokes equations

$$\alpha_0 \nabla \cdot \nu \rho_0 \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} - 2\omega \times \mathbf{v} - \nabla p = -\Gamma \tau \mathbf{k}, \tag{5.3a}$$

$$\nabla \cdot \rho_0 \mathbf{v} = 0, \tag{5.3b}$$

with $\mathbf{v}$ vanishing on both rims $z = 0$ and $z = Z$ of $\mathscr{Q}$. Also with $\mathbf{v}$ regarded as given, the temperature τ satisfies the second order elliptic partial differential equation

$$\alpha_0 \nabla \cdot \nu \rho_0 \nabla \tau - P \mathbf{v} \cdot \nabla \tau = \Gamma w - P \Lambda q, \tag{5.4}$$

with the mixed Neumann-Dirichlet boundary conditions (2.2b) and (2.2c).

We recall that $H(\mathscr{Q})$ is (componentwise) a closed subspace of the Sobolev space $W^{1,2}(\mathscr{Q})$, which is compactly embedded in the space $L_p(\mathscr{Q})$ for the same p as above [1; Theorem 5.4], and so the generalized solution $(\mathbf{v}, \tau)$ is in $L_p(\mathscr{Q})$. Hence by Theorem 4.2 of Kaniel and Shinbrot [7] with $k = 1$ and $a = 0$, we have $\mathbf{v} \in W^{2,p}(\mathscr{Q})$. (Kaniel and Shinbrot prove their theorem for $\rho_0 = 1$ and $\boldsymbol{\omega} = \mathbf{0}$, but their proof is easily seen to carry over to our case.) The elliptic equation (5.4) thus has coefficients in $W^{2,p}(\mathscr{Q})$ and forcing term $\Gamma w - P\Lambda q$ in $W^{1,p}(\mathscr{Q})$. Hence by the results of Miranda [10], its solution τ is in the Sobolev space $W^{3,p}(\mathscr{Q})$. Now using Theorem 4.2 of Kaniel and Shinbrot once again with $k = 4$ we have $\mathbf{v}$ in $W^{5,p}(\mathscr{Q})$.

Our generalized solution $(\mathbf{v}, \tau)$ is thus in the Sobolev space $W^{3,p}(\mathscr{Q})$, which is compactly embedded in the Hölder space $C^{2+\lambda}(\overline{\mathscr{Q}})$ for any $0 \leqslant \lambda < 1 - 2/p$ [1; Theorem 5.4C]. Hence the generalized solution $(\mathbf{v}, \tau) \in C^{2+\lambda}(\overline{\mathscr{Q}})$ when the heat source $q \in W^{1,p}(\mathscr{Q})$, that is, the generalized solution is a classical solution. (We note that by the above argument the generalized solution $(\mathbf{v}, \tau)$ belongs to the Sobolev space $W^{k+2,p}(\mathscr{Q})$ and hence is in $C^{k+\lambda}(\overline{\mathscr{Q}})$ when the heat source $q \in W^{k,p}(\mathscr{Q})$ for any $k \geqslant 1$.)

This completes the proof of Theorem 3.2.

6. A stability criterion

Let $(\mathbf{v}_s, \tau_s) \in H(\mathscr{Q})$ be an axisymmetric generalized solution of the steady system (2.3) with boundary conditions (2.2) and let $(\mathbf{v}, \tau)$ be an axisymmetric generalized solution of the nonsteady system (2.1) with boundary conditions (2.2), which we can define analogously to (3.4) with $(\mathbf{v}, \tau) \in H(\mathscr{Q})$ for all $t \geqslant 0$. Then the perturbation $\mathbf{V} = \mathbf{v} - \mathbf{v}_s$, $T = \tau - \tau_s$, with pressure p_1, satisfies in the generalized sense the system of equations

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\mathbf{v}_s \cdot \nabla \mathbf{V} - \mathbf{V} \cdot \nabla \mathbf{v}_s - \nabla p_1 + \Gamma T \mathbf{k} - 2\boldsymbol{\omega} \times \mathbf{V} + \alpha_0 \nabla \cdot \nu \rho_0 \nabla \mathbf{V}, \tag{6.1a}$$

$$P\left(\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T\right) = -P\mathbf{v}_s \cdot \nabla T - P\mathbf{V} \cdot \nabla \tau_s - \Gamma W + \alpha_0 \nabla \cdot \rho_0 \nabla T, \tag{6.1b}$$

$$\nabla \cdot \rho_0 \mathbf{V} = 0, \tag{6.1c}$$

with boundary conditions (2.2).

We define the energy of such a perturbation $(\mathbf{V}, \mathbf{T})$ by

$$E = \tfrac{1}{2} \int \rho_0 (\mathbf{V} \cdot \mathbf{V} + PT^2)\, dA. \tag{6.2}$$

On differentiating E with respect to time t, substituting for the total derivatives of $\mathbf{V}$ and T from equations (6.1a) and (6.1b), integrating by parts and using the

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continuity equation and the vanishing of $\mathbf{V}$ almost everywhere on $\partial\mathcal{Q}$, we obtain

$$\frac{dE}{dt} = -\nu\int\rho_0(\nabla\mathbf{V}\cdot\nabla\mathbf{V} + \nabla T\cdot\nabla T)\,dA$$
$$+\int\rho_0\mathbf{v}_s\cdot(\mathbf{V}\cdot\nabla\mathbf{V})\,dA + P\int\rho_0\tau_s\mathbf{V}\cdot\nabla T\,dA. \tag{6.3}$$

By the same reasoning as in the proof of Lemma 4.4 we obtain

$$\left|\int\rho_0\mathbf{v}_s\cdot(\mathbf{V}\cdot\nabla\mathbf{V})\,dA + P\int\rho_0\tau_s\mathbf{V}\cdot\nabla T\,dA\right| \leqslant 4\eta K^{1/2}L^{1/2}\|(\mathbf{v}_s,\tau_s)\|_H\|(\mathbf{V},T)\|_H^2,$$

where $\eta = \max(1, P)$, which we insert into (6.3) to obtain

$$\frac{dE}{dt} \leqslant \left(-\nu + 4\eta K^{1/2}L^{1/2}\|(\mathbf{v}_s,\tau_s)\|_H\right)\|(\mathbf{V},T)\|_H^2. \tag{6.4}$$

The energy E is positive definite in $(\mathbf{V}, T)$, and its time derivative (6.3) is negative definite provided the steady solution $(\mathbf{v}_s, \tau_s)$ satisfies the bound

$$\|(\mathbf{v}_s,\tau_s)\|_H < \nu\left(4\eta K^{1/2}L^{1/2}\right)^{-1}. \tag{6.5}$$

Hence considering E as a Lyapunov function, we can conclude that steady solutions $(\mathbf{v}_s, \tau_s)$ satisfying (6.5) are asymptotically stable in the sense of Lyapunov for axisymmetric generalized perturbations belonging to $H(\mathcal{Q})$ for all $t \geqslant 0$. We call such perturbations in $H(\mathcal{Q})$ admissible.

The above argument is also valid for asymmetric generalized perturbations, but with all of the norms and integrals now over the spherical shell $\mathbb{S}$ rather than the semi-annular region $\mathcal{Q}$. The same bound (6.6) holds here, but the constants K and L are different because they now correspond to the analogues of inequalities (4.1) and (4.2) over $\mathbb{S}$.

7. Evaluation of the bounds on heating rates

The basic inequalities ensuring the existence, uniqueness, and stability of the axisymmetric solution were given in nondimensional form in (3.5), (3.6), and (3.7). In this section we provide numerical estimates of the relevant bound on the heating rate and show that eddy viscosity concepts suggest that our results are of some climatological significance.

To begin, we note that combination of the uniqueness criterion (3.6) with the existence criterion (3.5) produces the inequality (3.7) that ensures asymptotic stability. Hence the crucial inequality is

$$\|q\|_2 \leqslant \nu^2\left(4\eta P\Lambda KL^{1/2}\right)^{-1}. \tag{7.1}$$

To provide numerical estimates, we wish to express this inequality in dimensional form. To do so, we henceforth denote nondimensional quantities with an overbar.

The heating rate in units deg/sec can be seen from [4] to be $q = T_0\Omega\bar{q}$; with $\overline{\| \|_2} = \| \|_2/l$ we have $\|_q\|_2 = T_0\Omega l\overline{\|q\|}_2$. The constant K is $(\rho_M/\rho_m)K_1$ where $\rho_M = \rho_{\max}$ and $\rho_m = \rho_{\min}$ and K_1 is the constant in Poincaré's inequality (4.3). It can be estimated from Rayleigh's principle by choosing $u = \sin(\pi z/Z)$ which gives $K_1 = (Z/\pi)^2$ or $\bar{K}_1 = (Z/\pi l)^2$, where Z is the top of the model atmosphere. For L we have $L = 2\rho_M/\rho_m^2$.

Using definitions from Table 1 we may now write (7.1) as

$$\begin{aligned}\|q\|_2 &\leqslant \frac{T_0\Omega^2 l^4 \nu^{-2}}{4\eta P^{1/2}(g/N)(Z/\pi)^2(\rho_M/\rho_m)(2\rho_M/\rho_m^2)^{1/2}}\\ &= \frac{T_0\nu^2}{4\eta P^{1/2}(gH_\theta)^{1/2}(Z/\pi)^2(\rho_M/\rho_m)(2\rho_M/\rho_m^2)^{1/2}},\end{aligned} \tag{7.2}$$

where $H_\theta = (\partial\theta/\theta\partial z)^{-1} = c_p T_0/g = 25\,\text{km}$ for $T_0 = 250°K$. We define

$$\text{RMS}[q] = \|q\|_2/M^{1/2},$$

where

$$M^{1/2} = \left(\int \rho_0\, dA\right)^{1/2} = \left[2\pi l \int_0^Z \rho_M e^{-z/H}\, dz\right]^{1/2} = [2\pi l H \rho_M]^{1/2}$$

for Z/H large enough. Here $H = RT_0/g = 7\,\text{km}$.

Now we have as the final form

$$\text{RMS}[q] \leqslant \frac{\nu^2}{\eta P^{1/2}}\left(\frac{\rho_m}{\rho_M}\right)^2 \frac{T_0}{8(glHH_\theta\pi)^{1/2}(Z/\pi)^2}. \tag{7.3}$$

We choose $Z = 17\,\text{km}$, which includes 90 per cent of the mass in its present configuration and gives $\rho_m/\rho_M = 0.10$.

For the molecular values $\nu = 0.15\,\text{cm}^2/\text{sec}$, $P = 0.7$, and $\eta = 1$, we arrive at $\text{RMS}[q] = 1.3 \times 10^{-21}$ deg/day in comparison with present atmosphere values on the order of $\text{RMG}[q] = 1$ deg/day.

Two problems are easily identified. The first is that small-scale turbulent motions can be expected to be present and so an eddy viscosity ν_e should be used along with $\eta = P \sim 4$. The second is that we used a geometric estimate for $\|u\|_2^2 \leqslant K\|\nabla u\|_2^2$. If we let u be the vector velocity, then for kinetic energy E and dissipation rate D we have

$$K \geqslant \frac{\|u\|_2^2}{\|\nabla u\|_2^2} = \frac{2\nu E}{D} = K_2,$$

which gives $K/K_2 = (Z/\pi)^2(\rho_M/\rho_m)/(2\nu E/D) = 7 \times 10^7$; we have used $E = 8 \times 10^5 J/m^2$ and $D = 6W/m^2$, values appropriate to the present Rossby regime in the atmosphere.

With these two changes, (7.3) becomes

$$\mathrm{RMS}[q] \leqslant \frac{\nu_3^2}{\eta P^{1/2}}\left(\frac{\rho_m}{\rho_M}\right)\left(\frac{D}{\nu_e^2}\right)\frac{T_0}{8(glH_e H\pi)^{1/2}}, \tag{7.4}$$

and now $\nu_e = 1.5 \times 10^6\,\mathrm{cm}^2/\mathrm{sec}$ implies that RMS[q] = 1 deg/day. This value of ν_e is comparable to estimates that $\nu_e = 10^5\,\mathrm{cm}^2/\mathrm{sec}$ for large-scale flow resembling present atmospheric conditions [4].

These estimates demonstrate that our mathematical results are of significance in climate theory if eddy diffusion processes are considered and if geometric estimates of the Poincaré constant can be replaced by dynamic ones.

However, the stability bound in either form is a sufficient condition based on energetics and does not take account of the rate of planetary rotation. As pointed out in [4], other estimates based on more subtle considerations will presumably give wider ranges of heating in which stability of the axisymmetric solutions prevails.

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