PERTURBATION OF THE CONTINUOUS SPECTRUM OF SYSTEMS OF ORDINARY DIFFERENTIAL OPERATORS

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1. Introduction. Let

$$L_0 = \sum_{j=0}^{h} P_j(x)\left(\frac{d}{dx}\right)^j$$

be an ordinary differential operator of order h whose coefficients are (η, η) matrices defined on the interval $0 \leqslant x < \infty$, $h\eta = n = 2\nu$. Let the operator L_0 be formally self adjoint and let ν boundary conditions be given at $x = 0$ such that the eigenvalue problem

$$(1.1) \qquad L_0 u = \lambda u, \qquad [\phi_{0j}, u](0) = 0, \qquad j = 1, \ldots, \nu, \qquad \Im(\lambda) \neq 0,$$

has no non-trivial square integrable solution. This paper deals with the perturbed operator $L^\epsilon = L_0 + \epsilon q$ where ϵ is a real parameter and $q(x)$ is a bounded positive (η, η) matrix operator with piecewise continuous elements $0 \leqslant x < \infty$. Sufficient conditions involving L_0, q are given such that L^ϵ determines a self-adjoint operator H^ϵ and such that the spectral measure $E^\epsilon(\Delta')$ corresponding to H^ϵ is an analytic function of ϵ, where Δ' is a subset of a fixed bounded interval $\Delta = [\alpha, \beta]$. The results include and improve results obtained for scalar differential operators in an earlier paper **(3)**.

The theory leads to a perturbation series for the spectral measure $E^\epsilon(\Delta')$, $\Delta' \subseteq \Delta$ in terms of the Greens' function of the eigenvalue problem (1.1) and the operator q (cf. formula (5.1)). Knowledge of the series for the spectral measure is useful in obtaining approximate solutions of vibration problems involving the operator L^ϵ. As an application an approximate perturbation series solution of the inhomogeneous equation

$$(1.2) \qquad (L_0 + \epsilon q)u + u_{tt} = + P(x, t), \qquad u(x, 0) = f(x), \qquad u_t(x, 0) = 0$$

is constructed using the series for the spectral measure. The perturbation series is approximate because in general it does not represent the complete solution of (1.2) but only that part of the solution which consists of a superposition of waves with frequencies ν in the interval $\sqrt{\alpha}/2\pi \leqslant \nu \leqslant \sqrt{\beta}/2\pi$. This is due to the fact that the series for the spectral measure is known only over the interval $\Delta = [\alpha, \beta]$. In the case that the functions $P(x, t)$, $f(x)$ have

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a spectral decomposition involving only frequencies in the range $\sqrt{\alpha}/2\pi \leqslant \nu \leqslant \sqrt{\beta}/2\pi$ the perturbation series leads to an exact solution. Perturbed equations of the type (1.2) arise frequently in the approximate solution of vibration problems. A well-known scalar example is the equation for the deflection of a bar on an elastic foundation in which the factor (ϵq) represents the coefficient of rigidity of the foundation (cf. **9**).

Sufficient conditions for analyticity of the spectral measure are stated in detail in § 2. These conditions involve the resolvent $\mathfrak{G}^0(\lambda)$ of the operator H^0 determined by L_0, whose kernel is the Greens' function, and the operator q. Briefly summarized, the conditions placed on $\mathfrak{G}^0(\lambda)$ and q are that the operators $q^{\frac{1}{2}}\mathfrak{G}^0(\lambda)q^{\frac{1}{2}}$ and $\int_\Delta \mathfrak{G}^0(\lambda)q\mathfrak{G}^0(\lambda)dl$ are uniformly bounded for λ in a neighbourhood of Δ, $\lambda = l + i\delta$, $l \in \Delta$, $0 < \delta < \delta_0$. These boundedness conditions are only possible in case that $\mathfrak{G}^0(\lambda)$ has no pole in the interval Δ, which implies that the interval contains only the continuous spectrum of H^0. Therefore the results of the paper deal only with perturbation of the continuous part of the spectrum. The above conditions may be weakened by employing limiting arguments and explicit properties of the Greens' function. Weakened assumptions are discussed in § 5. In particular, the assumption that the operator q is bounded can be removed under altered conditions stated in § 5.

The convergence of the perturbation series for the spectral measure $E^\epsilon(\Delta)$ is proved in §§ 4 and 5. The results proved in § 4 are valid generally for operators with Carleman kernels while the results in § 5 deal specifically with ordinary differential operators. In § 6 the perturbation series solution of the vibration equation (1.2) is derived in terms of the series for the "spectral measure"

The conditions for analyticity of the spectral measure given here reduce to ones of Moser if $h = 2$, $\eta = 1$ (**10**). The conditions are not necessary as Brownell has demonstrated analyticity of the spectral measure for $h = 2$, $\eta = 1$ under different conditions (**2**).

2. Notation, assumptions, and preliminary facts. Repeated Latin indices should be summed from 1 to n and repeated Greek indices should be summed from 1 to η unless the contrary is explicitly stated.

Denote by "$\pi = L_{2,\eta}$" the product space

$$\pi = \prod_{i=1}^{\eta} L_2[0, \infty)$$

of vector functions $u = (u_1(x), \ldots, u_\eta(x))$, $0 \leqslant x < \infty$, whose components $u_\alpha(x)$ are functions in $L_2[0, \infty)$, $\alpha = 1, \ldots, \eta$. Given two functions u, v in π, $u \cdot v$ will denote the scalar product of u and v and (u, v) the inner product of u and v, $u \cdot v = u_\alpha(x)v_\alpha(x)$ and $(u, v) = \int_0^\infty u \cdot \bar{v} dx$. The norm of an element u of π is defined as $||u|| = \sqrt{((u, u))}$. For u, v in π the Schwarz inequality, $|(u, v)| \leqslant ||u|| \cdot ||v||$, holds.

If H is a linear operator on π to π then $\mathfrak{D}(H)$ denotes the domain of H. The norm of a bounded operator H is $||H||$,

$$||H|| = \sup_{||u||=1} ||Hu||.$$

H^* is the adjoint of H. If H is self-adjoint then E_l will be the spectral resolution of H and $\mathfrak{G}(\lambda)$ the resolvent of H, $\mathfrak{G}(\lambda) = (H - \lambda)^{-1}$. An integral operator H on π is said to have a Carleman kernel $H_{\alpha\beta}(x, \xi)$ if $H_{\alpha\beta}$ is a matrix such that, as a function of ξ, $(H_{\alpha 1}, H_{\alpha 2}, \ldots, H_{\alpha\eta}) \in \pi$, $\alpha = 1, \ldots, \eta$ for almost all x and

$$(2.1) \qquad (Hu)_\alpha = \int_0^\infty H_{\alpha\beta}(x, \xi) u_\beta(\xi) d\xi, \alpha = 1, \ldots, \eta, u \in \mathfrak{D}(H).$$

The kernel $H_{\alpha\beta}$ is called a Hilbert-Schmidt kernel if it satisfies

$$(2.2) \qquad \int_0^\infty \int_0^\infty \sum_{\alpha,\beta=1}^{\eta} |H_{\alpha\beta}(x, \xi)|^2 d\xi\, dx < \infty.$$

A number of facts concerning the spectral resolution E_l and the resolvent $\mathfrak{G}(\lambda)$ will be stated for later reference. (These are proved in **(11)** and **(12)**.) The operator H is represented in terms of the spectral measure E_l by

$$Hu = \int_{-\infty}^\infty l dE_l u,$$

$u \in \mathfrak{D}(H)$. $\mathfrak{D}(H)$ consists of those functions u in π such that

$$\int_{-\infty}^\infty l^2 d||E_l u||^2 < \infty.$$

For any bounded continuous function $g(l)$, $-\infty < l < \infty$ the integral

$$\int_{-\infty}^\infty g(l) dE_l u$$

exists and defines a bounded operator $g(H)$ and

$$||g(H)|| \leqslant \sup_{-\infty < l < \infty} |g(l)|.$$

The spectral resolution E_l is self adjoint and $E_l^2 = E_l$. Given an interval $\Delta = [\alpha, \beta]$ the spectral measure $E(\Delta)$ is defined by $E(\Delta) = E_\beta - E_\alpha$. The spectral measure may be written as a contour integral in terms of the resolvent $\mathfrak{G}(\lambda)$ by the formula (cf. **12**, p. 183):

$$(2.3)^* \qquad (E(\Delta)u, v) = \lim_{\delta \to 0+} \frac{1}{\pi} \int_\Delta (\mathfrak{J}\{\mathfrak{G}(\lambda)\}u, v)\, dl, \qquad u, v \in L_{2,\eta}$$

where

$$\mathfrak{J}(\mathfrak{G}) = \frac{1}{2i}(\mathfrak{G} - \mathfrak{G}^*), \qquad \lambda = l + i\delta.$$

*This formula holds provided the endpoints of Δ are not in the point spectrum of H.

Next notation and facts concerning the spectral theory for the matrix differential operator L_0 will be summarized (following Kodaira **(7)**). It is assumed that the elements of the coefficient matrices $P_j(x)$ of

$$L_0 = \sum_{j=0}^{h} P_j(x)\left(\frac{d}{dx}\right)^j$$

have continuous derivatives up to jth order $j = 0, \ldots, h$, $0 \leqslant x < \infty$, and det. $P_h(x) > 0$, $0 \leqslant x < \infty$. Let $[u, v](x)$ be the bilinear boundary form associated with the differential operator L_0 such that

$$\int_0^x (L_0 u \cdot \bar{v} - u \cdot \overline{L_0 v})dx = [u, v](x) - [u, v](0). \tag{2.4}$$

It will be assumed that there exist functions ϕ_{0j}, $j = 1, \ldots, \nu$ such that $[\phi_{0j}, \phi_{0k}](0) = 0$, $j, k = 1, \ldots, \nu$ and such that the eigenvalue problem

$$L_0 u = \lambda u, \qquad [\phi_{0j}, u](0) = 0, \qquad j = 1, \ldots, \nu \tag{2.5}$$

has no non-trivial square integrable solution $u(x)$ in π for $\mathfrak{J}(\lambda) \neq 0$.

Under these assumptions the operator L_0 together with the boundary conditions $[\phi_{0j}, u](0) = 0$, $j = 1, \ldots, \nu$ determines a self-adjoint operator H^0. The domain of H^0, $\mathfrak{D}(H^0)$, consists of the set of functions u in π such that $u^{(j)}$ are continuous $j = 1, \ldots, h - 1$, $0 \leqslant x < \infty$, $u^{(h-1)}$ is absolutely continuous in every bounded subinterval of $[0, \infty)$, $L_0 u \in \pi$, and $[\phi_{0j}, u](0) = 0$, $j = 1, \ldots, \nu$. The values of H^0 are given by $H^0 u = L_0 u$, $u \in \mathfrak{D}(H^0)$.

Let $\mathfrak{D}_{00}$ be that subset of $\mathfrak{D}(H^0)$ whose elements are functions which vanish outside a compact set.

If H^0 is the self-adjoint operator determined by the differential operator L_0 then the resolvent $\mathfrak{G}^0(\lambda) = (H^0 - \lambda)^{-1}$ has a Carleman kernel $G_{\alpha\beta}(x, \xi, \lambda)$, called Greens' function, with the representation

$$G_{\alpha\beta}(x, \xi, \lambda) = \begin{cases} M^{jk}(\lambda) s_{j\alpha}(x, \lambda) s_{k\beta}(\xi, \lambda), & x \geqslant \xi \\ M^{jk}(\lambda) s_{k\alpha}(x, \lambda) s_{j\beta}(\xi, \lambda), & x < \xi \end{cases} \tag{2.6}$$

where $s_j(x, \lambda) = (s_{j1}, \ldots, s_{j\eta})$, $j = 1, \ldots, n$ is a fundamental system of solutions of the eigenvalue problem (2.5) and $M^{jk}(\lambda)$ are analytic functions of λ, $\mathfrak{J}(\lambda) \neq 0$. If (2.6) is inserted into (2.3) one obtains for differential operators

$$(E(\Delta)u, v) = \int_\Delta (s_j, v)(s_k, u) d\rho^{jk}(l) \tag{2.7}$$

where the spectral density function $\rho^{jk}(l)$ is defined as

$$\rho^{jk}(l) = \lim_{\delta \to 0+} \frac{1}{\pi} \int_{0+}^{l} \mathfrak{J}\{M^{jk}(l + i\delta)\} dl \tag{2.8}$$

(cf. **7**, pp. 542-543).

A number of assumptions regarding the operators H^0, $\mathfrak{G}^0(\lambda)$, and q will now be introduced relative to a fixed interval $\Delta = [\alpha, \beta]$ which are crucial for the later argument. It is assumed that H^0, $\mathfrak{G}^0(\lambda)$, q satisfy:

(i) *H^0 is self-adjoint and q is positive, symmetric, and bounded.*

(ii) *The operator $q^{\frac{1}{2}}\mathfrak{G}^0(\lambda)q^{\frac{1}{2}}$ has a Hilbert-Schmidt kernel and*

$$\overline{\lim_{\delta\to 0+}} \,||q^{\frac{1}{2}}\mathfrak{G}^0(\lambda)q^{\frac{1}{2}}|| \leqslant K_0, \qquad \lambda = l + i\delta, \qquad l \in \Delta, \qquad 0 < \delta < \delta_0.$$

(iii) *The operator $q^{\frac{1}{2}}\mathfrak{G}^0(\lambda)$ has a Carleman kernel $\mathfrak{B}_{\alpha\beta}(x, \xi, \lambda)$ which satisfies $|\mathfrak{B}_{\alpha\gamma}(x, \xi, \lambda)| \leqslant C_{\alpha\beta}(x)D_{\beta\gamma}(\xi)$ where $C_{\alpha\beta}$, $D_{\alpha\beta}$ are independent of λ, $\lambda = l + i\delta$, $l \in \Delta$, $0 < \delta < \delta_0$ and $\int_0^\infty \sum_{\alpha\beta}|C_{\alpha\beta}(x)|^2dx < \infty$.*

(iv) *The operator $\int_\Delta \mathfrak{G}^0(\lambda)q\mathfrak{G}^0(\lambda)dl$ is bounded and*

$$\overline{\lim_{\delta\to 0+}} \left(\int_\Delta \mathfrak{G}^0(\lambda)q\mathfrak{G}^0(\lambda)dl\, u, u\right) \leqslant \overline{\lim_{\delta\to 0+}} \int_\Delta ||q^{\frac{1}{2}}\mathfrak{G}^0(\lambda)u||^2\, dl \leqslant P_0||u||^2.$$

Note that these assumptions make sense for operators with Carleman kernels which are not necessarily differential operators. In § 5 assumptions are made specifically concerning differential operators which are more explicit than these assumptions. The assumptions arise in a natural way in the construction of the spectral measure $E^\epsilon(\Delta)$ corresponding to H^ϵ. (ii) is similar to assumptions introduced by Kuroda, Agudo, and Wolf in their work on perturbation of the essential spectrum. Kuroda assumed that for some λ_0, $\mathfrak{I}(\lambda_0) \neq 0$, the operator $q^{\frac{1}{2}}\mathfrak{G}^0(\lambda_0)$ has a kernel of Hilbert-Schmidt type **(8)**, while Agudo and Wolf assumed that $q\mathfrak{G}^0(\lambda_0)$ has a kernel of Hilbert-Schmidt type **(1)** for some λ_0, $\mathfrak{I}(\lambda_0) \neq 0$. Because we wish to derive results relative to the interval Δ we have imposed assumptions, relative to this interval, which are of a stronger character than those of the above authors. In particular, note that as a consequence of (iii) the resolvent $\mathfrak{G}^0(\lambda)$ does not have a pole along Δ, so that it is this assumption which implies that Δ does not contain any element of the point spectrum of H^0.

3. Preliminary Theorems. Two theorems will be required regarding existence of limits of integrals of the form (2.3). First a theorem from classical analysis:

THEOREM 1. *Let $\psi(\lambda)$ be a regular function of λ, $\lambda = l + i\delta$, $l \in \Delta$, $0 < \delta < \delta_0$. Let*

$$\int_\alpha^\beta |\mathfrak{I}\{\psi(l + i\delta)\}|\, dl < M_0, \qquad 0 < \delta < \delta_0 \tag{3.1}$$

and

$$\int_0^{\delta_0} |\psi(\alpha + i\nu)|d\nu < \infty \qquad \int_0^{\delta_0} |\psi(\beta + i\nu)|d\nu < \infty. \tag{3.2}$$

Then

$$\lim_{\delta\to 0+} \int_{\Delta'} \mathfrak{I}\{\psi(\lambda)\}dl \tag{3.3}$$

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exists for all intervals Δ', $\Delta' \subseteq \Delta$. *The function*

$$p(l) = \lim_{\delta \to 0} \int_{\alpha}^{l} \Im\{\psi(\lambda)\} dl$$

has bounded variation on Δ, $p(l) = \frac{1}{2}(p(l+) + p(l-))$, *and the set function* (3.3) *is countably additive.*

A proof of Theorem 1 is given in (**13**, p. 346). Note that the discontinuities of $p(l)$ form a countable set and we may renormalize $p(l)$ so as to be right continuous at points of discontinuity.

Theorem 1 leads directly to a theorem concerning bilinear forms:

THEOREM 2. *Let* $B(\lambda)$ *be a bounded operator regular in* λ, $\lambda = l + i\delta$, $l \in \Delta$, $0 < \delta < \delta_0$. *Let* $B^*(\lambda) = B(\bar{\lambda})$,

$$\Im(B) = \frac{1}{2i}(B - B^*),$$

and let $B(\lambda)$ *satisfy*

$$(3.4) \qquad \int_0^{\delta_0} |(B(l + i\nu)u, v)| d\nu < \infty, \qquad l \in \Delta, \qquad u, v \in \mathfrak{D}_{00}$$

and

$$(3.5) \qquad \overline{\lim_{\delta \to 0+}} \left| \int_{\Delta} (\Im\{B(\lambda)\}u, v) dl \right| \leqslant M_0 ||u|| \, ||v||, \qquad u, v \in \mathfrak{D}_{00}$$

Then

$$(3.6) \qquad \lim_{\delta \to 0+} \int_{\Delta'} (\Im\{B(\lambda)\}u, v) dl$$

exists for all intervals $\Delta' \subseteq \Delta$ *and is a bounded bilinear form on* π. *There exists a bounded operator* $C(\Delta')$ *such that for* $\Delta' \subseteq \Delta$

$$(3.7) \qquad (C(\Delta')u, v) = \lim_{\delta \to 0+} \int_{\Delta'} (\Im(B)u, v) dl, \qquad u, v \in \pi$$

and $||C(\Delta')|| \leqslant M_0$.

Proof. Given $u \in \mathfrak{D}_{00}$ define $\psi(\lambda) = (B(\lambda)u, u)$. Since $B^*(\lambda) = B(\bar{\lambda})$ it follows that $\Im\{\psi(\lambda)\} = (\Im\{B(\lambda)\}u, u)$. The integrals $\int_0^{\delta_0} |\psi(\alpha + i\nu)| d\nu$, $\int_0^{\delta_0} |\psi(\beta + i\nu)| d\nu$ exist by (3.4). By (3.5), given any number $k > 0$ for some $\delta(k)$, $0 < \delta(k) < \delta_0$

$$(3.8) \qquad \int_{\Delta} |\Im\{\psi(l + i\nu)\}| dl \leqslant (M_0 + k) ||u||^2, \qquad 0 < \nu < \delta(k).$$

This shows the function $\psi(\lambda)$ satisfies the hypothesis of Theorem 1 so, by Theorem 1, we conclude that

$$\lim_{\delta \to 0+} \int_{\Delta'} (\Im(B)u, u) dl$$

exists for $\Delta' \subseteq \Delta$.

Since k is arbitrary (3.8) implies

$$\lim_{\delta\to 0+}\left|\int_{\Delta}(\mathfrak{J}(B)u, u)dl\right| \leqslant M_0||u||^2, \qquad u, \in \mathfrak{D}_{00}. \tag{3.9}$$

The bilinear form $(\mathfrak{J}\{B\}u, v)$ may be decomposed by the polarization identity into

$$(\mathfrak{J}\{B\}u, v) = (\mathfrak{J}\{B\}u_1, u_1) + i(\mathfrak{J}\{B\}u_2, u_2) - i(\mathfrak{J}\{B\}u_3, u_3) - (\mathfrak{J}\{B\}u_4, u_4) \tag{3.10}$$

where

$$u_1 = \frac{u+v}{2}, u_2 = \frac{u+iv}{2i}, u_3 = \frac{u-iv}{2i}, u_4 = \frac{u-v}{2}.$$

By what was shown above the quadratic forms

$$\lim_{\delta\to 0+}\int_{\Delta'}(\mathfrak{J}(B)u_i, u_i)dl$$

$i = 1, \ldots, 4$ exist and satisfy (3.9). Using (3.10) it follows that

$$\lim_{\delta\to 0+}\int_{\Delta'}(\mathfrak{J}(B)u, v)dl$$

exists. The integral

$$\int_{\Delta'}(\mathfrak{J}(B)u, v)dl$$

is a bounded bilinear form so the limit

$$\left(\lim_{\delta\to 0+}\int_{\Delta'}(\mathfrak{J}(B)u, v)dl\right)$$

is also a bilinear form. Since in general a bilinear form has the same bound as the associated quadratic form, (3.9) implies

$$\lim_{\delta\to 0+}\left|\int_{\Delta'}(\mathfrak{J}(B)u, v)dl\right| \leqslant M_0||u||\,||v||, \qquad u, v \in \mathfrak{D}_{00}. \tag{3.11}$$

The set $\mathfrak{D}_{00}$ is dense in π. Therefore the bilinear form

$$\lim_{\delta\to 0+}\int_{\Delta'}(\mathfrak{J}(B)u, v)dl$$

determines a bounded operator $C(\Delta')$ by a theorem of Frechet (cf. **12**, p. 63). The bound of the operator $C(\Delta')$ is less than M_0 by (3.11).

4. The perturbation series for the spectral measure. If H^0, q are operators on π satisfying the assumptions (i), . . . , (iv) given in § 2 then q is bounded $H^\epsilon = H^0 + \epsilon q$ is a self-adjoint operator, $\mathfrak{D}(H^\epsilon) = \mathfrak{D}(H^0)$, and consequently there exists a spectral measure $E^\epsilon(\Delta')$, $\Delta' \subseteq \Delta$. In this section it will be shown that the spectral measure $E^\epsilon(\Delta')$ may be expanded into a

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convergent series in powers of ϵ. The results are valid for general operators with Carleman kernels. Section 5 contains more explicit results for differential operators.

Let $\mathfrak{G}^\epsilon(\lambda)$ be the resolvent of H^ϵ. The resolvents $\mathfrak{G}^0(\lambda)$, $\mathfrak{G}^\epsilon(\lambda)$ satisfy the resolvent equation

$$\mathfrak{G}^\epsilon(\lambda) = \mathfrak{G}^0(\lambda) - \epsilon\mathfrak{G}^0(\lambda)q\mathfrak{G}^\epsilon(\lambda). \tag{4.1}$$

If (4.1) is solved formally by iteration one obtains the Neumann series for the resolvent:

$$\mathfrak{G}^\epsilon(\lambda) = \sum_{\nu=0}^{\infty} (-\epsilon)^\nu \mathfrak{G}^0(q\mathfrak{G}^0)^\nu. \tag{4.2}$$

The substitution of (4.2) into (2.3) results in a formal series for the spectral measure

$$E^\epsilon(\Delta') = \sum_{\nu=0}^{\infty} (\epsilon)^\nu E^{(\nu)}(\Delta') \tag{4.3}$$

where the operators $E^{(\nu)}$ are defined by

$$E^{(\nu)} = \frac{1}{\pi} \lim_{\delta\to 0+} (-1)^\nu \int_{\Delta'} \mathfrak{J}\{\mathfrak{G}^0(q\mathfrak{G}^0)^\nu\}dl. \tag{4.4}$$

The series (4.2), (4.3) are not in general convergent and (4.4) does not in general define a bounded operator. However, under the special assumptions (i), . . . , (iv) this is the case. This is shown in the next three theorems.

THEOREM 3. *If H^0 is a self-adjoint operator and q is a positive bounded symmetric operator and if H^0, q satisfy assumption* (ii) *then for $|\epsilon| < K_0^{-1}$ the resolvent $\mathfrak{G}^\epsilon(\lambda)$ of the operator $H^\epsilon = H^0 + \epsilon q$ is given by*

$$\mathfrak{G}^\epsilon(\lambda) = \sum_{\nu=0}^{\infty} (-\epsilon)^\nu \mathfrak{G}^0(\lambda)(q\mathfrak{G}^0(\lambda))^\nu \tag{4.5}$$

for $\lambda = l + i\delta$, $l \in \Delta$, $0 < \delta < \delta(\epsilon)$, $0 < \delta(\epsilon) < \delta_0$.

Proof. Given ϵ, $|\epsilon| < K_0^{-1}$, by assumption (ii) choose $l_1(\epsilon)$ such that

$$l_1(\epsilon) < \left(\frac{1}{|\epsilon|} - K_0\right)$$

and choose $\delta(\epsilon)$ such that

$$||q^{\frac{1}{2}}\mathfrak{G}^0(\lambda)q^{\frac{1}{2}}|| \leqslant (K_0 + l_1), \qquad l \in \Delta, \qquad 0 < \delta < \delta(\epsilon). \tag{4.6}$$

Then for $\nu \geqslant 1$, $l \in \Delta$, $0 < \delta < \delta(\epsilon)$,

$$\begin{aligned} ||\mathfrak{G}^0(\lambda)(q\mathfrak{G}^0(\lambda))^\nu|| &\leqslant \mathfrak{G}^0 q^{\frac{1}{2}}(q^{\frac{1}{2}}\mathfrak{G}^0 q^{\frac{1}{2}})^{\nu-1}q^{\frac{1}{2}}\mathfrak{G}^0|| \\ &\leqslant ||\mathfrak{G}^0||^2||q^{\frac{1}{2}}||^2||q^{\frac{1}{2}}\mathfrak{G}^0 q^{\frac{1}{2}}||^{\nu-1} \leqslant ||\mathfrak{G}^0||^2||q^{\frac{1}{2}}||^2(K_0 + l_1)^{\nu-1}. \end{aligned} \tag{4.7}$$

Formula (4.7) implies that the series (4.5) is absolutely convergent in norm for $\lambda = l + i\delta$, $l \in \Delta$, $0 < \delta < \delta(\epsilon)$ since $||\mathfrak{G}^0||$ and $||q^{\frac{1}{2}}||$ are finite.

To show that the series (4.5) represents the resolvent we note that the resolvent is the unique operator which satisfies the equations

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$$\text{(4.8)} \qquad \begin{cases} \mathfrak{G}^\epsilon(\lambda)(H^0 + \epsilon q - \lambda)u = u, & u \in \mathfrak{D}(H^0) \\ (H^0 + \epsilon q - \lambda)\mathfrak{G}^\epsilon(\lambda)y = y & y \in \pi. \end{cases}$$

It is easily seen by substitution that the series (4.5) satisfies the equations (4.8). Therefore the series (4.5) is equal to the resolvent operator.

THEOREM 4. *If H^0 is a self-adjoint operator and q is a positive bounded symmetric operator and if H^0, q satisfy assumptions* (ii), (iii), (iv), *then for $\Delta' \leqslant \Delta$*

$$\text{(4.9)} \qquad \lim_{\delta\to 0+} \frac{1}{\pi}\int_{\Delta'} (\mathfrak{J}\{\mathfrak{G}^0(q\mathfrak{G}^0)^\nu\}u, v)dl$$

exists and determines a bounded operator $E^{(\nu)}$ such that for $u, v \in \pi = L_{2,\eta}$

$$\text{(4.10)} \qquad (E^{(\nu)}(\Delta')u, v) = \frac{1}{\pi}\lim_{\delta\to 0+}\int_{\Delta'}(\mathfrak{J}\{\mathfrak{G}^0(q\mathfrak{G}^0)^\nu\}u, v)dl.$$

$$\text{(4.11)} \qquad ||E^{(\nu)}|| \leqslant \frac{1}{\pi} P_0 K_0^{\nu-1}, \ \nu \geqslant 1.$$

Proof. Let $B(\lambda) = \mathfrak{G}^0(\lambda)(q\mathfrak{G}^0(\lambda))^\nu$. $B(\lambda)$ is regular for $\lambda = l + i\delta$, $l \in \Delta$, $0 < \delta < \delta_0$ because $\mathfrak{G}^0(\lambda)$ is regular. From $(\mathfrak{G}^0(\lambda))^* = \mathfrak{G}^0(\bar\lambda)$ it follows $B^*(\lambda) = B(\bar\lambda)$. The operator $B(\lambda)$ may be written, $\nu \geqslant 1$,

$$\text{(4.12)} \qquad B(\lambda) = (\mathfrak{G}^0 q^{\frac{1}{2}})(q^{\frac{1}{2}}\mathfrak{G}^0 q^{\frac{1}{2}})^{\nu-1}(q^{\frac{1}{2}}\mathfrak{G}^0).$$

Next, using assumption (ii), for some δ_1, l_1

$$\begin{aligned} \text{(4.13)} \qquad |(B(\lambda)u, v)| &\leqslant |((q^{\frac{1}{2}}\mathfrak{G}^0 q^{\frac{1}{2}})^{\nu-1}(q^{\frac{1}{2}}\mathfrak{G}^0)u,\ (q^{\frac{1}{2}}\mathfrak{G}^0(\bar\lambda))v)| \\ &\leqslant ||q^{\frac{1}{2}}\mathfrak{G}^0 q^{\frac{1}{2}}||^{\nu-1}||q^{\frac{1}{2}}\mathfrak{G}^0 u||\ ||q^{\frac{1}{2}}\mathfrak{G}^0 v|| \\ &\leqslant (K_0 + l_1)^{\nu-1}||q^{\frac{1}{2}}\mathfrak{G}^0 u||\ ||q^{\frac{1}{2}}\mathfrak{G}^0 v||, \qquad 0 < \delta < \delta_1. \end{aligned}$$

The quantity

$$\sup_{0<\nu<\delta_1} ||q^{\frac{1}{2}}\mathfrak{G}^0(l + i\nu)u||$$

is finite for $l \in \Delta$, $u \in \mathfrak{D}_{00}$ by assumption (iii). For $u, v \in \mathfrak{D}_{00}$, using (4.13),

$$\begin{aligned} \text{(4.14)} \qquad \int_0^{\delta_1} |(B(\lambda)u, v)|d\nu &\leqslant (K_0 + l_1)^{\nu-1}\int_0^{\delta_1} ||q^{\frac{1}{2}}\mathfrak{G}^0 u||\,||q^{\frac{1}{2}}\mathfrak{G}^0 v||d\nu \\ &\leqslant \delta_1(K_0 + l_1)^{\nu-1}\sup_{0<\nu<\delta_1}||q^1\mathfrak{G}^0 u||\,||q^1\mathfrak{G}^0 v|| < \infty. \end{aligned}$$

By assumptions (ii), (iv), and (4.13), for some l_2, δ_2

$$\begin{aligned} \text{(4.15)} \qquad \int_\Delta |(\mathfrak{J}\{B(\lambda)\}u, v)|dl &\leqslant \frac{1}{2}\int_\Delta |(B(\lambda)u, v)|dl + \frac{1}{2}\int_\Delta |(B(\lambda)u, v)|dl \\ &\leqslant (K_0 + l_1)^{\nu-1}\int_\Delta ||q^{\frac{1}{2}}\mathfrak{G}^0 u||\,||q^{\frac{1}{2}}\mathfrak{G}^0 v||dl \\ &\leqslant (K_0 + l_1)^{\nu-1}\left\{\int_\Delta ||q^{\frac{1}{2}}\mathfrak{G}^0 u||^2 dl \int_\Delta ||q^{\frac{1}{2}}\mathfrak{G}^0 v||^2 dl\right\}^{\frac{1}{2}} \\ &\leqslant (K_0 + l_1)^{\nu-1}(P_0 + l_2)||u||\,||v|| \end{aligned}$$

for $\lambda = l + i\delta$, $l \in \Delta$, $0 < \delta < \delta_2 < \delta_1$.

The inequalities (4.14), (4.15), show that $B(\lambda)$ satisfies the hypothesis of Theorem 2. Applying Theorem 2 we conclude that the bilinear form (4.9) exists and determines the operator $E^{(\nu)}(\Delta')$. Since the constants l_1, l_2 are arbitrarily small the operator $E^{(\nu)}(\Delta')$ satisfies (4.11).

As a consequence of Theorem 4 the series

$$\sum_{\nu=0}^{\infty} (\epsilon)^{\nu} E^{(\nu)}$$

is a convergent series of bounded operators $|\epsilon| < K_0^{-1}$. It remains to verify (4.3).

THEOREM 5. *If H^0 is a self-adjoint operator and q is a positive bounded symmetric operator and if H^0, q satisfy* (ii), (iii), (iv) *then for $|\epsilon| < K_0^{-1}$ the operator $H^{\epsilon} = H^0 + \epsilon q$ is self adjoint and the spectral measure $E^{\epsilon}(\Delta')$, $\Delta' \subseteq \Delta$ corresponding to H^{ϵ}, is analytic in ϵ. For $u, v \in \pi$*

$$(E(\Delta')u, v) = \sum_{0}^{\infty} \epsilon^{\nu} (E^{(\nu)}(\Delta')u, v). \tag{4.16}$$

Proof. H^{ϵ} is self adjoint, $\mathfrak{D}(H^{\epsilon}) = \mathfrak{D}(H^0)$, since q is assumed bounded. By (2.3) the spectral measure is given by the integral

$$(E^{\epsilon}(\Delta')u, v) = \lim_{\delta \to 0+} \frac{1}{\pi} \int_{\Delta'} (\mathfrak{J}\{\mathfrak{G}^{\epsilon}(\lambda)\}u, v) dl. \tag{4.17*}$$

By Theorems 3 and 4, for $u, v \in \mathfrak{D}_{00}$

$$\begin{aligned} (E^{\epsilon}(\Delta')u, v) &= \lim_{\delta \to 0+} \frac{1}{\pi} \int_{\Delta} (\mathfrak{J}\{\mathfrak{G}^{\epsilon}(\lambda)\}u, v) dl \\ &= \lim_{\delta \to 0+} \frac{1}{\pi} \sum (-\epsilon)^{\nu} \int_{\Delta'} (\mathfrak{J}\{\mathfrak{G}^0(q\mathfrak{G}^0)^{\nu}\}u, v) dl \\ &= \sum (-\epsilon)^{\nu} \lim_{\delta \to 0+} \frac{1}{\pi} \int_{\Delta'} (\mathfrak{J}\{\mathfrak{G}^0(q\mathfrak{G}^0)^{\nu}\}u, v) dl \\ &= \sum \epsilon^{\nu} (E^{(\nu)}(\Delta')u, v). \end{aligned} \tag{4.18}$$

The interchange of summation with integration and limit operations in (4.18) is permissible because of the uniform convergence of the series involved. Because the set $\mathfrak{D}_{00}$ is dense in π the equation (4.17) extends to all elements of π by a theorem of Frechet (**12**, p. 63).

In § 6 the following extension of Theorem 5 is used:

THEOREM 6. *If H^0 is a self-adjoint operator and q is a positive bounded symmetric operator, if H^0, q satisfy* (ii), (iii), (iv), *and if $g(l)$ is a bounded function analytic on Δ then for $|\epsilon| < K_0^{-1}$ $H^{\epsilon} = H^0 + \epsilon q$ is self adjoint with spectral measure $E^{\epsilon}(\Delta')$ and the integral*

*This formula is valid since the endpoints of the interval Δ' are not in the point spectrum of H^{ϵ}. This is because the interval Δ does not contain any elements of the point spectrum since $\mathfrak{G}^{\epsilon}(\lambda)$ does not have a pole in Δ. That $\mathfrak{G}^{\epsilon}(\lambda)$ dces not have a pole in Δ follows from assumption (iii) and Theorem 3.

$$\int_{\Delta'} g(l)d(E_l^\epsilon u, v)$$

is analytic in ϵ,

$$\int_{\Delta'} g(l)d(E_l^\epsilon u, v) = \sum \epsilon^\nu \int_{\Delta'} g(l)d(E_l^{(\nu)} u, v) \tag{4.19}$$

$$\left| \int_{\Delta'} g(l)d(E_l^\epsilon u, v) \right| \leqslant \frac{1}{\pi} \sup_{\Delta'} |g(l)| P_0 K_0^{\nu-1} ||u|| \; ||v||, \nu \geqslant 1. \tag{4.20}$$

The proof is obtained by a modification of the proof of Theorem 5. Note that in case $h = 2$, $\eta = 1$, the theorem was stated by Moser (**10**, p. 385).

5. The perturbation series for the spectral measure of differential operators. The results derived in §4 are valid, in general, for operators with Carleman kernels. In this section more explicit results are obtained for differential operators using the representation formula (2.6) for the Green's function. It will be assumed that H^0 is a self-adjoint operator determined by a differential operator L_0 satisfying the conditions given in § 2. q will be assumed to be a positive symmetric matrix multiplication operator such that the elements $q_{\alpha\beta}^{\frac{1}{2}}(x)$ of the matrix $q^{\frac{1}{2}}$ are piecewise continuous on $0 \leqslant x < \infty$. The elements M^{jk} of the characteristic matrix (cf. (2.6)) will be assumed to have limiting values

$$\lim_{\delta \to 0+} M^{jk}(l + i\delta) = M_+^{jk}(l).$$

When the formula (2.6) for the Greens' function and the matrix $q_{\alpha\beta}^{\frac{1}{2}}(x)$ are substituted into (4.4) one obtains a series for the spectral measure corresponding to the differential operator $L^\epsilon = L_0 + \epsilon q$:

$$E^\epsilon(\Delta')u = \sum_0^\infty \epsilon^\nu E^{(\nu)}(\Delta')u \tag{5.1}$$

where

$$E^{(\nu)}(\Delta')u = \frac{1}{\pi} \int_{\Delta'} \sum_{\mu+\chi=\nu} \psi_j^{(\mu)}(x, l)(\psi_k^{(\chi)}, u)\, d\rho^{jk}(l) \tag{5.2}$$

and the vectors $\psi_j^{(\mu)}$ are defined by

$$\psi_j^{(\mu)} = (\overline{\mathfrak{G}^0(l + i0)q})^\mu s_j(x, l). \tag{5.3}$$

(Formula 5.2) is derived from (2.6) and (4.4) noting that

$$\mathfrak{J}\{\mathfrak{G}^0(\lambda)(q\mathfrak{G}^0(\lambda))^\nu\} = \sum_{\mu+\chi=\nu} (\overline{\mathfrak{G}^0(\lambda)q})^\mu \mathfrak{J}(\mathfrak{G}^0(\lambda))(q\mathfrak{G}^0(\lambda))^\chi.$$

Explicitly written out the components of the vectors $\psi_j^{(0)}$, $\psi_j^{(1)}$ are by (2.6), (5.3)

$$\psi_{j\alpha}^{(0)} = s_{j\alpha}(x, l). \tag{5.4}$$

$$\psi_{j\alpha}^{(1)} = M_{+}^{\tau p}(l)\Big(s_{\tau\alpha}(x, l)\int_0^x s_{p\beta}(\xi, l)q_{\beta\gamma}(\xi)s_{j\gamma}(\xi, l)d\xi \\ s_{p\alpha}(x, l)\int_x^\infty s_{\tau\beta}(\xi, l)q_{\beta\gamma}(\xi)s_{j\gamma}(\xi, l)d\xi\Big). \tag{5.5}$$

The series for the spectral measure (5.1) involves the quantities M^{jk}, $s_{j\alpha}$, and $q_{\alpha\beta}(x)$. In our main theorem, which follows, we shall prove convergence of the series (5.1) under assumptions involving these quantities:

THEOREM 7. *Let the matrix differential operator L_0 and the matrix multiplication operator q be such that for a fixed finite interval $\Delta = [\alpha\beta]$:*

(v) $$\lim_{\delta\to 0+}\int_0^\infty |M^{jk}(\lambda)|^2|s_j(x, \lambda)|^2|q(x)|\int_0^x |s_k(\xi, \lambda)|^2|q(\xi)|d\xi\, dx \leqslant \gamma,^2|M^{jk}| \leqslant K_1,$$
$j, k = 1, \ldots, n$

(vi) $$\begin{cases} \overline{\lim_{\delta\to 0+}}\int_\Delta\int_0^\infty |M^{jk}(\lambda)|^2|s_j(x, \lambda)|^2|q(x)|\left|\int_0^x (s_k\cdot u)d\xi\right|^2 dx\, dl \leqslant P_1||u||^2 \\ \overline{\lim_{\delta\to 0+}}\int_\Delta\int_0^\infty |M^{jk}(\lambda)|^2|s_k(x, \lambda)|^2|q(x)|\left|\int_x^\infty (s_j\cdot u)d\xi\right|^2 dx\, dl \leqslant P_1||u||^2 \end{cases}$$
$j, k = 1, \ldots, n$

where

$$|s_j|^2 = s_j\,.\,s_j\,,\ |q| = \max_\beta\sum_\gamma |q_{\beta\gamma}^{\frac{1}{2}}(x)|^2,\ \lambda = l + i\delta,\ l\in\Delta,\ 0<\delta<\delta_0.$$

Then for $|\epsilon| < (2n^2\eta\gamma)^{-1}$ the operator $L^\epsilon = L_0 + \epsilon q$ determines a self-adjoint operator H^ϵ and the corresponding spectral measure $E^\epsilon(\Delta')$ is analytic in ϵ, $\Delta' \subseteq \Delta$.

Proof. By definition the differential operator L_0 determines a self-adjoint operator H^0. Since q is bounded $H^\epsilon = H^0 + \epsilon q$ is a self-adjoint operator and $L^\epsilon u = H^\epsilon u$, $u \in \mathfrak{D}(H^0)$.

We shall first carry out the proof that the spectral measure $E^\epsilon(\Delta')$ corresponding to H^ϵ is analytic in ϵ under the additional assumptions that the columns of the matrix $q_{\alpha\beta}^{\frac{1}{2}}$ are vectors in $\mathfrak{D}_{00}$ and that the endpoints of the interval Δ' are points of continuity of the spectral measure $E^\epsilon(\Delta')$.

By (2.6) and Minkowski's inequality

$$\begin{aligned} &\int_0^\infty\int_0^\infty |q_{\alpha\beta}^{\frac{1}{2}}(x)G_{\beta\gamma}^0(x, \xi, \lambda)q_{\gamma\delta}^{\frac{1}{2}}(\xi)|^2 d\xi\, dx \\ &\leqslant \int_0^\infty\int_0^x |M^{jk}(\lambda)s_{j\beta}(x, \lambda)q_{\alpha\beta}^{\frac{1}{2}}(x)s_{k\gamma}(\xi, \lambda)q_{\gamma\delta}^{\frac{1}{2}}(\xi)|^2 d\xi\, dx \\ &\qquad + \int_0^\infty\int_x^\infty |M^{jk}(\lambda)s_{k\beta}(x, \lambda)q_{\alpha\beta}^{\frac{1}{2}}(x)s_{j\gamma}(\xi, \lambda)q_{\gamma\delta}^{\frac{1}{2}}(\xi)|^2 d\xi\, dx \\ &\leqslant \Big(\sum_{j,k}\Big\{\int_0^\infty |M^{jk}|^2|s_j|^2|q|\int_0^x |s_k|^2|q|d\xi\, dx\Big\}^{\frac{1}{2}}\Big)^2 \\ &\qquad + \Big(\sum_{j,k}\Big\{\int_0^\infty |M^{jk}|^2|s_k|^2|q|\int_x^\infty |s_j|^2|q|d\xi\, dx\Big\}^{\frac{1}{2}}\Big)^2. \end{aligned} \tag{5.6}$$

Inequality (v) and (5.6) lead to

$$(5.7) \qquad \overline{\lim_{\delta\to 0+}} \int_0^\infty \int_0^\infty \sum_{\alpha,\delta} |q^{\frac{1}{2}}_{\alpha\beta} G^0_{\beta\gamma} q^{\frac{1}{2}}_{\gamma\delta}|^2 d\xi\, dx \leqslant 2n^4\eta^2\gamma^2.$$

Since $q_{\alpha\beta}{}^{\frac{1}{2}} G_{\beta\gamma}{}^0 q_{\gamma\delta}{}^{\frac{1}{2}}$ is the kernel of the operator $q^{\frac{1}{2}}\mathfrak{G}^0(\lambda)q^{\frac{1}{2}}$ it follows from (5.7) that assumption (ii) holds with $K_0 = 2n^2\eta\gamma$. The kernel $q_{\alpha\beta}{}^{\frac{1}{2}} G_{\beta\gamma}{}^0$ of the operator $q^{\frac{1}{2}}\mathfrak{G}^0(\lambda)$ is a Carleman kernel since G^0 is a Carleman kernel and $q^{\frac{1}{2}}$ is bounded. Again by (2.6) and Minkowski's inequality

$$(5.8) \quad \int_\Delta ||q^{\frac{1}{2}}\mathfrak{G}^0(\lambda)u||^2 dl = \int_\Delta \int_0^\infty \sum_\alpha \left| \int_0^\infty q^{\frac{1}{2}}_{\alpha\beta}(x) G^0_{\beta\gamma}(x, \xi, \lambda) u_\gamma(\xi) d\xi \right|^2 dx\, dl$$

$$= \sum_\alpha \int_\Delta \int_0^\infty \left| M^{jk} q^{\frac{1}{2}}_{\alpha\beta}(x) \left(s_{j\beta}(x) \int_0^x s_{k\gamma}(\xi) u_\gamma(\xi) d\xi \right.\right.$$

$$\left.\left. + s_{k\beta}(x) \int_x^\infty s_{j\gamma}(\xi) u_\gamma(\xi) d\xi \right) \right|^2 dx\, dl$$

$$\leqslant \sum_\alpha \left(\sum_{j,k} \left\{ \int_\Delta \int_0^\infty |M^{jk}|^2 |q^{\frac{1}{2}}_{\alpha\beta} s_{j\beta}|^2 \left| \int_0^x s_{k\gamma} u_\gamma d\xi \right|^2 dx\, dl \right\}^{\frac{1}{2}} \right.$$

$$\left. + \left\{ \int_\Delta \int_0^\infty |M^{jk}|^2 |q^{\frac{1}{2}}_{\alpha\beta} s_{k\beta}|^2 \left| \int_x^\infty s_{j\gamma} u_\gamma d\xi \right|^2 dx\, dl \right\}^{\frac{1}{2}} \right)^2.$$

Inequality (vi) and (5.8) imply

$$(5.9) \qquad \overline{\lim_{\delta\to 0+}} \int_\Delta ||q^{\frac{1}{2}}\mathfrak{G}^0(\lambda)u||^2 dl \leqslant 8n^4\eta P_1 ||u||^2.$$

Inequality (5.9) implies that assumption (iv) of § 2 holds with $P_0 = 8n^4\eta P_1$. By (v) the elements of the characteristic matrix $|M^{jk}(1 + i\delta)|$ are uniformly bounded $j, k = 1, \ldots, n, l \in \Delta, 0 < \delta < \delta_0$. Since $|M^{jk}|$ are uniformly bounded and since $s_j(x, \lambda)$ are entire functions of λ and the columns of the matrix $q_{\alpha\beta}{}^{\frac{1}{2}}$ are vectors in $\mathfrak{D}_{00}$, it follows that

$$(5.10) \qquad |q^{\frac{1}{2}}_{\alpha\beta}(x) G^0_{\beta\gamma}(x, \xi, \lambda)| \leqslant C_{\alpha\beta}(x) D_{\beta\gamma}(\xi)$$

where

$$\int_0^\infty \sum_{\alpha,\beta} |C_{\alpha\beta}(x)|^2 dx < \infty$$

for some functions $C_{\alpha\beta}(x)$, $D_{\alpha\beta}(\xi)$. This shows that assumption (iii) of § 2 holds. Since the assumptions (ii), (iii), (iv) hold, the conclusion of the theorem follows from Theorem 5.

The theorem is extended to the case where the columns of the matrix $q_{\alpha\beta}{}^{\frac{1}{2}}$ are not in $\mathfrak{D}_{00}$ by a limiting argument. Let $(q_b{}^{\frac{1}{2}})_{\alpha\beta}$ be a sequence of matrices whose columns are vectors in $\mathfrak{D}_{00}$ with the property that

$$||q_b^{\frac{1}{2}}|| \leqslant ||q^{\frac{1}{2}}||, \lim_{b\to\infty} ||q_b^{\frac{1}{2}} - q^{\frac{1}{2}}|| = 0.$$

By what has been proved above the operator $L_b{}^\epsilon = L_0 + \epsilon q_b$ determines a

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self-adjoint operator $H_b{}^\epsilon = H^0 + \epsilon q_b$ with analytic spectral measure $E_b{}^\epsilon(\Delta')$. Also for $|\epsilon| < K_0{}^{-1}$, $K_0 = 2n^2\eta\gamma$,

$$(5.11) \qquad (E_b^\epsilon(\Delta')u, v) = \sum_0^\infty \epsilon^\nu (E_b^{(\nu)}(\Delta')u, v)$$

where

$$(5.12) \qquad (E_b^{(\nu)}(\Delta')u, v) = \lim_{\delta\to 0+} \frac{1}{\pi}\int_{\Delta'} (\Im\{\mathfrak{G}^0(q_b\mathfrak{G}^0)^\nu\}u, v)dl.$$

$$(5.13) \qquad ||E_b^{(\nu)}(\Delta')|| \leqslant \frac{1}{\pi} P_0 K_0^{\nu-1}, \nu \geqslant 1.$$

Let $\mathfrak{G}_b{}^\epsilon(\lambda)$ be the resolvent of $H_b{}^\epsilon$. The resolvents $\mathfrak{G}_b{}^\epsilon(\lambda)$, $\mathfrak{G}^\epsilon(\lambda)$ are represented by the series $\sum_0{}^\infty(-\epsilon)^\nu\mathfrak{G}^0(q_b\mathfrak{G}^0)^\nu$, $\sum_0{}^\infty(-\epsilon)^\nu\mathfrak{G}^0(q\mathfrak{G}^0)^\nu$ for $|\epsilon| < K_0{}^{-1}$, $\lambda = l + i\delta$, $l \in \Delta$, $0 < \delta < \delta(\epsilon)$, by Theorem 3.

Given ϵ, $|\epsilon| < K_0{}^{-1}$, by assumption (ii) for some $l_1(\epsilon)$ $\nu \geqslant 3$,

$$(5.14) \quad ||(\mathfrak{G}^0(q_b\mathfrak{G}^0)^\nu - \mathfrak{G}^0(q\mathfrak{G}^0)^\nu)u||$$
$$\leqslant ||q_b^{\frac{1}{2}} - q^{\frac{1}{2}}||(2)||\mathfrak{G}^0||^3||q^{\frac{1}{2}}||^3(K_0 + l_1)^{\nu-3}[2(K_0 + l_1) + (\nu - 2)||q^{\frac{1}{2}}||^2||\mathfrak{G}^0||]\,||u||$$

for $\lambda = l + i\delta$, $l \in \Delta$, $0 < \delta < \delta_1(\epsilon) < \delta(\epsilon)$. From (5.14) it follows that

$$(5.15) \quad \lim_{b\to\infty} ||(\mathfrak{G}_b^\epsilon(\lambda) - \mathfrak{G}^\epsilon(\lambda))u|| = \lim_{b\to\infty} \left|\left| \sum_{\nu=1}^\infty (-\epsilon)^\nu(\mathfrak{G}^0(q_b\mathfrak{G}^0)^\nu - \mathfrak{G}^0(q\mathfrak{G}^0)^\nu)u \right|\right| = 0.$$

This shows the resolvent $\mathfrak{G}_b{}^\epsilon(\lambda)$ converges strongly to the resolvent $\mathfrak{G}^\epsilon(\lambda)$, $b \to \infty$. Applying a theorem of Rellich strong convergence of $\mathfrak{G}_b{}^\epsilon(\lambda)$ to $\mathfrak{G}^\epsilon(\lambda)$ implies that the bilinear form $(E_b{}^\epsilon(\Delta')u, v)$ converges to $(E^\epsilon(\Delta')u, v)$, $b \to \infty$. On the other hand the right side of (5.12) converges to the bounded bilinear form

$$(5.16) \qquad \lim_{\delta\to 0+} \frac{1}{\pi}\int_{\Delta'} (\Im\{\mathfrak{G}^0(q\mathfrak{G}^0)^\nu\}u, v)dl.$$

The bilinear form (5.16) determines a bounded operator $E^{(\nu)}(\Delta')$ and by (5.13) $||E^{(\nu)}(\Delta')|| \leqslant 1/\pi P_0 K_0{}^{\nu-1}$. Since

$$\lim_{b\to\infty} (E_b^\epsilon(\Delta')u, v) = (E^\epsilon(\Delta')u, v),$$

formula (5.11) implies

$$(5.17) \qquad (E^\epsilon(\Delta')u, v) = \sum_0^\infty \epsilon^\nu (E^{(\nu)}(\Delta')u, v).$$

Therefore $E^\epsilon(\Delta')$ is analytic in ϵ, $|\epsilon| < K_0{}^{-1}$.

The restriction that the endpoints of the interval Δ' are points of continuity of the spectral measure $E^\epsilon(\Delta')$ is removed by observing that, by the definition of $E^{(\nu)}(\Delta')$ the series on the right side of (5.17) varies continuously with the endpoints of the interval Δ'. Therefore every point of the interval Δ is a point of continuity of the spectral measure $E^{\epsilon}(\Delta')$.

COROLLARY. *If the operators L_0, q satisfy the inequalities* (v), (vi) *and if $g(l)$ is analytic on the interval Δ then for $|\epsilon| < (2\eta n^2\gamma)^{-1}$ the integral $\int_{\Delta'} g(l)d(E_l^{\epsilon}u, v)$ is analytic in ϵ and satisfies the equations* (4.19), (4.20).

The corollary is a direct consequence of Theorem 6.

The remainder of this section is devoted to noting some weaker assumptions under which the conclusions of Theorem 7 hold. Detailed proofs will be omitted. First the conclusion of Theorem 7 holds if inequalities (v), (vi) are replaced by the inequalities:

$$\text{(v)}' \quad \overline{\lim_{\delta\to 0+}} \int_0^\infty |\tilde{M}^{jk}(l+i\delta)|^2 |\tilde{s}_j(x,l)|^2 |q(x)| \int_0^x |\tilde{s}_k(\xi,l)|^2 |q(\xi)| d\xi\, dx \leqslant \gamma^2,$$

$$|\tilde{M}^{jk}(\lambda)| \leqslant K_1, \qquad j, k = 1, \ldots, n$$

$$\text{(vi)}' \quad \overline{\lim_{\delta\to 0+}} \int_\Delta \int_0^\infty |\tilde{M}^{jk}(l+i\delta)|^2 |\tilde{s}_j(x,l)|^2 |q(x)| \left| \int_0^x (\tilde{s}_k \cdot u) d\xi \right|^2 dx\, dl \leqslant P_1 ||u||^2$$

$$\overline{\lim_{\delta\to 0+}} \int_\Delta \int_0^\infty |\tilde{M}^{jk}(l+i\delta)|^2 |\tilde{s}_k(x,l)|^2 |q(x)| \left| \int_x^\infty (\tilde{s}_j \cdot u) d\xi \right|^2 dx\, dl \leqslant P_1 ||u||^2$$

where $\tilde{M}^{jk}$, $\tilde{s}_j$ are any functions with the property that $\tilde{s}_j(x, \lambda)$ are analytic in λ, $\lambda = l + i\delta$, $l \in \Delta$, $0 \leqslant \delta < \delta_0$, and

$$G^0_{\alpha\beta}(x,\xi,\lambda) = \tilde{M}^{jk}(\lambda)\tilde{s}_{j\alpha}(x,\lambda)\tilde{s}_{k\beta}(\xi,\lambda), \qquad x \geqslant \xi,$$

$$G^0_{\alpha\beta}(x,\xi,\lambda) = \tilde{M}^{jk}(\lambda)\tilde{s}_{k\alpha}(x,\lambda)\tilde{s}_{j\beta}(\xi,\lambda), \qquad \xi > x.$$

The method for proving this is to introduce an approximate Greens' function $\tilde{G}^0$ defined by

$$\tag{5.18} \tilde{G}^0_{\alpha\beta}(x,\xi,\lambda) = \begin{cases} \tilde{M}^{jk}(\lambda)\tilde{s}_{j\alpha}(x,l)\tilde{s}_{k\beta}(\xi,l), & x \geqslant \xi \\ \tilde{M}^{jk}(\lambda)\tilde{s}_{k\alpha}(x,l)\tilde{s}_{j\beta}(\xi,l), & \xi > x. \end{cases}$$

Since the functions $\tilde{s}_j\ (x, \lambda)$ are analytic in λ, $\lambda = l + i\delta$, $l \in \Delta$, $0 < \delta < \delta_0$ we have

$$\tag{5.19} |\tilde{G}^0_{\alpha\beta}(x,\xi,\lambda) - G^0_{\alpha\beta}(x,\xi,\lambda)| \leqslant M\delta$$

where M may depend on x, ξ but is independent of δ. Let $\tilde{\mathfrak{G}}^0(\lambda)$ be the integral operator with kernel $\tilde{G}_{\alpha\beta}{}^0(x, \xi, \lambda)$. The conclusion of Theorem 7 can now be obtained under the assumptions (v)′, (vi)′ by repeating the argument of Theorem 5 and Theorem 7 with $\mathfrak{G}^0(\lambda)$ replaced by $\tilde{\mathfrak{G}}^0(\lambda)$, using (5.19) and the assumption that the limit $M_+{}^{jk}(l)$ exists.

Theorem 7 may also be extended to the case where q is an unbounded matrix multiplication operator if the inequality (v) (or (v)′) is replaced by the stronger inequalities

$$\text{(v)}'' \qquad \begin{cases} |M^{jk}(l+i\delta)| \leqslant K_1, & l \in \Delta, \quad 0 < \delta < \delta_0 \\ \displaystyle\int_0^\infty |s_j(x,l)|^2 |q(x)|^i dx \leqslant \gamma_1, & i = 1, 2. \end{cases}$$

where $j, k = 1, \ldots, n$. The argument, which will be omitted here, consists in approximating q with a matrix operator q_b with the property that the columns of the matrix $q_{\alpha\beta}^{\frac{1}{2}}$ are vectors in $\mathfrak{D}_{00}$. (The argument is essentially the same as that given for the scalar case in (**3**, p. 321).)

6. Approximate weak solutions of the vibration equation. Let L_0 be a matrix differential operator and let q be a matrix multiplication operator satisfying the same conditions as in § 5, and let H^0 be the self-adjoint operator determined by L_0. The results of the earlier sections concerning analyticity of the spectral measure $E^\epsilon(\Delta')$ corresponding to the operator $L^\epsilon = L_0 + \epsilon q$ are directly applicable to the problem of finding weak series solutions of the vibration equation (1.2). In addition to the earlier assumptions it will be assumed that the differential operator L_0 is positive.* Furthermore it will be assumed that the vectors $f(x)$, $P(x, t)$ are in the domain of H^0, $0 \leqslant t < T$, and that $||P(x, t)||$ is an integrable function of t, $0 \leqslant t \leqslant T$.

The solution of the wave equation (1.2) may be written formally using the operator calculus as

$$u(t) = \cos(\sqrt{H^\epsilon}\, t) f + \int_0^t \frac{\sin(\sqrt{H^\epsilon}\, t)(t-\tau)}{\sqrt{H}} P(\tau) d\tau. \tag{6.1}$$

The definition of the operators

$$\cos(\sqrt{H^\epsilon}\, t), \frac{\sin(\sqrt{H^\epsilon}\, t)}{\sqrt{H}}$$

appearing in (6.1) is

$$\cos(\sqrt{H^\epsilon}\, t) = \int_0^\infty \cos\sqrt{l}\, t dE_l^\epsilon, \frac{\sin(\sqrt{H^\epsilon}\, t)}{\sqrt{H}} = \int_0^\infty \frac{\sin(\sqrt{l}\, t)}{\sqrt{l}} dE_l^\epsilon. \tag{6.2}$$

Formula (6.1) is known to represent a weak solution of equation (1.2) under the above assumptions regarding f and $P(t)$. The proof is a direct consequence of (6.2) and the definition of weak solution. (For a definition of weak solution and discussion of when weak solutions are regular solutions cf. (**5**).)

We shall consider approximate solutions of (1.2) in the form

$$u_\Delta(t) = \int_\Delta \cos\sqrt{l}\, t dE_l^\epsilon f + \int_0^t \int_\Delta \frac{\sin\sqrt{l}\,(t-\tau)}{\sqrt{l}} dE_l^\epsilon P(\tau) d\tau \tag{6.3}$$

*Here it is meant that L_0 has a positive spectrum. Criteria for ordinary scalar differential operators to be positive have been given by Heinz (**6**).

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where $\Delta = [\alpha, \beta]$ is a finite interval $0 < \alpha < \beta < \infty$. The function $u_\Delta(t)$ will be called a finite wave packet. The finite wave packet $u_\Delta(t)$ approximates the solution $u(t)$ in the weak sense as the size of the interval Δ increases, $\alpha \to 0$, $\beta \to \infty$, by the definition of $u_\Delta(t)$ and $u(t)$. (6.3) may also be written

$$(6.4)\qquad u_\Delta(t) = \cos(\sqrt{H^\epsilon}\, t) f_\Delta + \int_0^t \frac{\sin \sqrt{H^\epsilon}\,(t-\tau)}{\sqrt{H}} P_\Delta(\tau) d\tau$$

where $f_\Delta = \int_\Delta dE_l f$, $P_\Delta = \int_\Delta dE_l P(\tau)$. It is apparent by (6.4) that the finite wave packet $u_\Delta(t)$ represents an exact (weak) solution of the vibration equation (1.2) in the case $f_\Delta = f$ and $P_\Delta = P$. That is, when the initial function f and the forcing function $P(t)$ have a spectral representation which is a superposition of "waves" with frequencies ν in the interval $\sqrt{\alpha}/2\pi \leqslant \nu \leqslant \sqrt{\beta}/2\pi$. (In the case that $L_0 = d^2/dx^2$ and $q = 0$ formula (6.1) reduces to the classical formula for the wave packet for one-dimensional wave motion (**4**, p. 135).)

The object of this section is to obtain the perturbation series for the finite wave packet $u_\Delta(t)$ and prove its convergence. The series is obtained at once from (4.3), (6.3):

$$(6.5)\qquad u_\Delta(t) = \sum_0^\infty \epsilon^\nu u^{(\nu)}(t)$$

where

$$(6.6)\qquad u^{(\nu)}(t) = \int_\Delta \cos \sqrt{l}\, t dE_l^{(\mu)} f + \int_0^t \int_\Delta \frac{\sin \sqrt{l}\,(t-\tau)}{\sqrt{l}} dE_l^{(\nu)} P(\tau) d\tau.$$

Convergence of the series for the finite wave packet $u_\Delta(t)$ is dealt with next:

THEOREM 8. *If the differential operator L_0 and the matrix multiplication operator q satisfy the inequalities* (v), (vi) (*or* (v)′, (vi)′) *for some finite real positive interval $\Delta = [\alpha, \beta]$ $0 < \alpha < \beta < \infty$ then for $|\epsilon| < (2n^2\eta\gamma)^{-1}$ the finite wave packet $u_\Delta(t)$ may be expanded into a convergent perturbation series.*

Proof. By the corollary to Theorem 7 for all $v \in \pi$, $0 \leqslant \tau \leqslant t \leqslant T$,

$$(6.7)\qquad \int_\Delta \cos \sqrt{l}\, t d(E_l^\epsilon f, v) = \sum \epsilon^\nu \int_\Delta \cos \sqrt{l}\, t d(E_l^{(\nu)} f, v)$$

$$(6.8)\qquad \int_\Delta \frac{\sin \sqrt{l}\,(t-\tau)}{\sqrt{l}} d(E_l^\epsilon P, v) = \sum \epsilon^\nu \int_\Delta \frac{\sin \sqrt{l}\,(t-\tau)}{\sqrt{l}} d(E_l^{(\nu)} P, v)$$

since the functions $\cos \sqrt{l}\, t$, $(\sin \sqrt{l}\, t)/\sqrt{l}$ are analytic for $l \in \Delta$. Also if $K_0 = 2n^2\gamma\eta$, then

$$(6.9)\qquad \left| \int_\Delta \cos \sqrt{l}\, t d(E_1^{(\nu)} f, v) \right| \leqslant \frac{1}{\pi} 2K_0^{\nu-1} P_0 ||f|| \, ||v||, \qquad \nu \geqslant 1.$$

$$(6.10)\qquad \left| \int_\Delta \frac{\sin \sqrt{l}\,(t-\tau)}{\sqrt{l}} d(E_1^{(\nu)} P(\tau), v) \right| \leqslant \frac{1}{\pi} 2K_0^{\nu-1} P_0 ||P(\tau)|| \, ||v||, \quad \nu \geqslant 1.$$

The series (6.7), (6.8) are absolutely convergent for $|\epsilon| < K_0^{-1}$ by (6.9), (6.10). Also the series (6.8) may be integrated termwise with respect to τ by (6.10) since $||P(\tau)||$ is integrable in $0 \leqslant \tau \leqslant T$, and

$$(6.11) \quad \int_0^t \int_\Delta \frac{\sin \sqrt{l}\,(t-\tau)}{\sqrt{l}} d(E_l^\epsilon P(\tau), v)d\tau = \sum \epsilon^\nu \int_0^t \int_\Delta \frac{\sin \sqrt{l}\,(t-\tau)}{\sqrt{l}} d(E_l^{(\nu)} P(\tau), v)d\tau.$$

The conclusion of the theorem follows from (6.7), (6.11).

7. An example. The theory leads to results concerning fourth order differential operators which are analogous to results which are known to hold for second order differential operators. Here we consider the (2, 2) matrix differential operator

$$(7.1) \qquad L^\epsilon = L_0 + \epsilon q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{d^4}{dx^4} + \epsilon \begin{pmatrix} q_{11}(x) & q_{12}(x) \\ q_{21}(x) & q_{22}(x) \end{pmatrix}$$

where the matrix q is the square of a symmetric matrix $q^{\frac{1}{2}}$ whose elements $q_{\alpha\beta}^{\frac{1}{2}}(x)$ are piecewise continuous $0 \leqslant x < \infty$. The boundary conditions which will be associated with L_0 are $u(0) = u^{(2)}(0) = 0$. (These correspond to $[\phi_j, u](0) = 0, j = 1, 2$ where

$$\phi_1 = (x, x), \; \phi_2 = \left(\frac{x^3}{3!}, \frac{x^3}{3!}\right)$$

in the bracket notation.)

The Greens' function $G^0(x, \xi, \lambda)$ corresponding to the one-dimensional eigenvalue problem $(d^4/dx^4)u - \lambda u = 0$, $u(0) = u^{(2)}(0) = 0$, is known to be

$$(7.2) \qquad G^0(x, \xi, \lambda) = \begin{cases} \dfrac{1}{2w^3}\,(e^{iwx} \sin w\xi - e^{-wx} \sinh w\xi), & x \geqslant \xi \\ \dfrac{1}{2w^3}\,(e^{iw\xi} \sin wx - e^{-w\xi} \sinh wx), & \xi > x \end{cases}$$

where $w = \sqrt[4]{r}\, e^{i\theta/4}$, $\lambda = re^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$, (cf. **14**, p. 279). The Greens' function $G_{\alpha\beta}{}^0(x, \xi, \lambda)$ corresponding to the matrix eigenvalue problem $L_0 u = \lambda u$, $u(0) = u^{(2)}(0) = 0$, is seen to be $G_{\alpha\beta}{}^0(x, \xi, \lambda) = \delta_{\alpha\beta} G^0(x, \xi, \lambda)$, $\alpha, \beta = 1, 2$ where G^0 is given by (7.2). Using this explicit Greens' function one may calculate the explicit terms of the perturbation series for the spectral measure and the finite wave packet corresponding to the operator (7.1). Concerning convergence of these perturbation series we have:

THEOREM 9. *Let L_0 be the matrix differential operator*

$$(L_0 u)_\alpha = \delta_{\alpha\beta} \frac{d^4}{dx^4} u_\beta$$

and let q be a positive matrix which is the square of the matrix $q^{\frac{1}{2}}$ whose elements

$q_{\alpha\beta}^{\frac{1}{2}}(x)$ *are piecewise continuous functions of* x, $0 \leqslant x < \infty$, *and which satisfy the condition*

$$\int_0^\infty \sum_{\alpha,\beta} |q_{\alpha\beta}^{\frac{1}{2}}(x)|^2 dx < \infty. \tag{7.3}$$

Let $\Delta = [\alpha, \beta]$, $0 < \alpha < \beta < \infty$ *be a fixed finite interval. Then, for sufficiently small* ϵ, *the operator*

$$\delta_{\alpha\beta}\frac{d^4}{dx^4} + \epsilon q_{\alpha\beta}$$

determines a self-adjoint operator H^ϵ *and the corresponding spectral measure* $E^\epsilon(\Delta')$ *is analytic in* ϵ. *Also the finite wave packet* $u_\Delta(t)$ *corresponding to the interval* Δ *is analytic in* ϵ.

Proof. Define functions $\tilde{s}_j$, and $\tilde{M}^{jk}$ by

$$\begin{cases} \tilde{s}_{1\alpha} = \delta_{1\alpha}e^{iwx}, \tilde{s}_{2\alpha} = \delta_{1\alpha}e^{-iwx}, \tilde{s}_{3\alpha} = \delta_{1\alpha}e^{-wx}, \tilde{s}_{4\alpha} = \delta_{1\alpha}e^{wx} \\ \tilde{s}_{5\alpha} = \delta_{2\alpha}e^{iwx}, \tilde{s}_{6\alpha} = \delta_{2\alpha}e^{-iwx}, \tilde{s}_{7\alpha} = \delta_{2\alpha}e^{-wx}, \tilde{s}_{8\alpha} = \delta_{2\alpha}e^{wx}. \end{cases} \tag{7.4}$$

$$\tilde{M}^{jk} = \frac{1}{4w^3}\begin{pmatrix} -i & +i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & +i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.5}$$

By (7.2), (7.4), (7.5) $G_{\alpha\beta}{}^0(x, \xi, \lambda) = \tilde{M}^{jk}\tilde{s}_{j\alpha}\tilde{s}_{k\beta}$, $x \geqslant \xi$, $G_{\alpha\beta}{}^0 = \tilde{M}^{jk}\tilde{s}_{k\alpha}\tilde{s}_{j\alpha}$, $\xi > x$. By direct calculation one verifies that inequalities (v)′, (vi)′ hold for the values of $\tilde{M}^{jk}$, $\tilde{s}_j$ defined by (7.4), (7.5). Since inequalities (v), (vi)′ hold, the conclusion of the theorem follows from Theorem 7, the remarks after Theorem 7, and Theorem 8.

Note that for second order scalar operators, $L_0 = -(d^2/dx^2)$, the condition given by Moser for convergence of the series for the spectral measure corresponding to the operator $-(d^2/dx^2) + {}_zq(x)$ is that $q(x)$ is piecewise continuous and satisfy $\int_0^\infty |q(x)|dx < \infty$. The condition $\int_0^\infty \sum_{\alpha\beta} |q_{\alpha\beta}^{\frac{1}{2}}(x)|^2 dx < \infty$ of Theorem 9 is a generalization of the latter condition (cf. **10**, p. 391).

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