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Generalized Reductive Lie Algebras: Connections With Extended Affine Lie Algebras and Lie Tori

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Abstract. We investigate a class of Lie algebras which we call *generalized reductive Lie algebras*. These are generalizations of semi-simple, reductive, and affine Kac–Moody Lie algebras. A generalized reductive Lie algebra which has an irreducible root system is said to be *irreducible* and we note that this class of algebras has been under intensive investigation in recent years. They have also been called *extended affine Lie algebras*. The larger class of generalized reductive Lie algebras has not been so intensively investigated. We study them in this paper and note that one way they arise is as fixed point subalgebras of finite order automorphisms. We show that the core modulo the center of a generalized reductive Lie algebra is a direct sum of centerless Lie tori. Therefore one can use the results known about the classification of centerless Lie tori to classify the cores modulo centers of generalized reductive Lie algebras.

Introduction

In 1990 Høegh–Krohn and B. Torresani [HK-T] introduced a new interesting class of Lie algebras over field of complex numbers, called *quasi simple Lie algebras* by proposing a system of fairly natural and not very restrictive axioms. These Lie algebras are characterized by the existence of a symmetric nondegenerate invariant bilinear form, a finite dimensional Cartan subalgebra, a discrete root system which contains some nonisotropic roots, and the ad-nilpotency of the root spaces attached to non-isotropic roots. As it will appear from the sequel, these algebras are natural generalizations of reductive Lie algebras, and affine Kac–Moody Lie algebras. For this reason and other reasons indicated in the introduction of the paper [AABGP] we call this class of Lie algebras *generalized reductive Lie algebras* (GRLA for short).

In [HK-T], the authors extract some basic properties of GRLAs from the axioms, but for the further study of such Lie algebras they assume the irreducibility of the corresponding root systems. Namely, a GRLA is called *an extended affine Lie algebra* (EALA for short) if the set of non-isotropic roots is indecomposable and isotropic roots are non-isolated (see Definition 1.1 for terminology). We note that EALAs have been under intensive investigation in recent years, however the more general class of generalized reductive Lie algebras has not been so intensively investigated.

In [AABGP] the axioms for an EALA are introduced in steps in such a way that the power of each axiom is clearly shown before introducing the next one. This in

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particular provides a good framework for the study of Lie algebras which satisfy only a part of the axioms. In Section 1, we follow the same steps as in [AABGP, Chapter I] to obtain the basic structural properties of a GRLA. However, it turns out that some of the results in [AABGP, Chapter I] can be proved using fewer axioms than those in [AABGP](see Proposition 1.4(i)).

In Section 2, we get more information about the structure of a GRLA $\mathcal{G}$ by decomposing its root system R into a finite union of indecomposable subroot systems, and then corresponding to each subroot system we construct an indecomposable generalized reductive subalgebra of $\mathcal{G}$. More precisely, we show that up to some isolated spaces, a GRLA is a finite sum of certain indecomposable generalized reductive subalgebras and an abelian subalgebra, with trivial Lie bracket between distinct summands (on the level of core). In particular, if there are no isolated root spaces (that is if $\mathcal{G}$ satisfies part (b) of axiom GR6 of a GRLA), the structure of $\mathcal{G}$ can be thought of a generalization of a reductive Lie algebra. In fact when the nullity is zero, $\mathcal{G}$ is nothing but a reductive Lie algebra (see Corollary 2.11 for details). The main result of this section is that the core modulo center of a GRLA $\mathcal{G}$ is isomorphic to a direct sum of the cores modulo centers of some indecomposable generalized reductive ideals of $\mathcal{G}$. When the nullity is less than or equal two, this result can be read as: the core of $\mathcal{G}$ modulo its center is a direct some of the cores modulo centers of some extended affine Lie algebras (see Theorem 2.10(iv) and Corollary 2.11(i)).

In Section 3, the main section, we show that the core modulo center of an indecomposable GRLA is a centerless Lie torus, and therefore the core modulo center of a GRLA is a direct sum of centerless Lie tori (Theorem 3.1 and Corollary 3.2). Therefore, one can use the results of [BGK], [BGKN], [AG], [Y], [ABG] and [AFY] regarding the classification of centerless Lie tori to classify the cores modulo centers of GRLAs for types which the classification is achieved. In principle, the classification of centerless Lie tori is done for all types except type BC_2. See also [A2, Proposition 1.28], [AG, Proposition 1.28] and [N2, Theorem 6] for the relation between an EALA and its core modulo center. For a deep study of EALAs and their root systems we refer the reader to [S], [BGK], [BGKN], [AABGP], [AG], [ABG], [A1], [A3]. Also see [N1-2] and [AKY] for some new classes of Lie algebras which are closely related to EALAs.

In Section 4, we give several examples of GRLAs and we show some methods of constructing new GRLAs from old ones. In particular, it is shown that GRLAs arise as the fixed point subalgebras of finite order automorphisms.

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1 Generalized Reductive Lie Algebras

Let $\mathcal{G}$ be a Lie algebra over the field of complex numbers, let $\mathcal{H}$ be a subalgebra of $\mathcal{G}$ and $(\cdot\,,\cdot)\colon \mathcal{G}\times\mathcal{G}\to\mathbb{C}$ be a bilinear form on $\mathcal{G}$. Consider the following axioms for the triple $\big(\mathcal{G}, (\cdot\,,\,\cdot), \mathcal{H}\big)$:

GR1 The form $(\cdot\,,\,\cdot)$ is symmetric, nondegenerate and invariant on $\mathcal{G}$.

GR2 $\mathcal{H}$ is a nontrivial finite dimensional abelian subalgebra which is self-centralizing and $\mathrm{ad}(h)$ is diagonalizable for all $h \in \mathcal{H}$.

According to GR2 we have a vector space decomposition $\mathcal{G} = \bigoplus_{\alpha \in \mathcal{H}^\star} \mathcal{G}_\alpha$, where

$$\mathcal{G}_\alpha = \{x \in \mathcal{G} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathcal{H}\}.$$

The set $R = \big\{\, \alpha \in \mathcal{H}^\star \mid \mathcal{G}_\alpha \neq \{0\} \big\}$ is called the *root system* of $\mathcal{G}$.

From GR1–GR2 it follows that

$$\mathcal{G}_0 = \mathcal{H}, \quad 0 \in R, \tag{1.1}$$

and

$$(\mathcal{G}_\alpha, \mathcal{G}_\beta) = \{0\} \text{ unless } \alpha + \beta = 0. \tag{1.2}$$

In particular,

$$R = -R, \tag{1.3}$$

and the form restricted to $\mathcal{H}$ is nondegenerate. For $\alpha \in \mathcal{H}^\star$ let t_α be the unique element in $\mathcal{H}$ which represents α via the form. Then for any $\alpha \in R$,

$$[\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}] = \mathbb{C}t_\alpha. \tag{1.4}$$

Transfer the form to $\mathcal{H}^\star$ through

$$(\alpha, \beta) := (t_\alpha, t_\beta) \quad \text{for } \alpha, \beta \in \mathcal{H}^\star. \tag{1.5}$$

Let

$$R^\times = \{\alpha \in R \mid (\alpha, \alpha) \neq 0\} \text{ and } R^0 = \{\alpha \in R \mid (\alpha, \alpha) = 0\}.$$

Elements of $R^\times$ (*resp.*, R^0) are called non-isotropic (isotropic) roots of R. The next axioms are as follows:

GR3 For any $\alpha \in R^\times$ and $x \in \mathcal{G}_\alpha$, $\mathrm{ad}_{\mathcal{G}}(x)$ acts locally nilpotently on $\mathcal{G}$.

GR4 R is a discrete subset of $\mathcal{H}^\star$.

GR5 $R^\times \neq \varnothing$.

Definition 1.1 A triple $(\mathcal{G}, (\cdot\,,\,\cdot), \mathcal{H})$ satisfying axioms GR1–GR5 is called a *generalized reductive Lie algebra* (GRLA for short). We call a generalized reductive Lie algebra *indecomposable* if it satisfies

GR6a $R^\times$ is indecomposable, that is $R^\times$ is not a disjoint union of two of its nonempty subsets which are orthogonal with respect to the form.

We call a GRLA *non-singular* if it satisfies

GR6b For $\sigma \in R^0$, there exists $\alpha \in R^\times$ such that $\alpha + \sigma \in R$, that is isotropic roots of R are non-isolated.

Finally, a GRLA is called an *extended affine Lie algebra* (EALA for short) if it satisfies GR1–GR6. When there is no confusion, we simply write $\mathcal{G}$ instead of $(\mathcal{G}, (\cdot\,,\cdot), \mathcal{H})$. The *core* of a GRLA $\mathcal{G}$ is by definition, the subalgebra $\mathcal{G}_c$ of $\mathcal{G}$ generated by root spaces corresponding to non-isotropic roots. It follows that $\mathcal{G}_c$ is a perfect ideal of $\mathcal{G}$. A GRLA is called *tame* if $\mathcal{G}_c$ contains its centralizer in $\mathcal{G}$.

Remark 1.2 (i) It follows from axioms GR1–GR2 and GR6b that $R^\times \neq \varnothing$, so the axiom GR5 is redundant for a non-singular GRLA.

(ii) It is easy to see that a GRLA $\mathcal{G}$ is tame if and only if $C_{\mathcal{G}}(\mathcal{G}_c) = \{x \in \mathcal{G} \mid (x, \mathcal{G}_c) = \{0\}\}$. The proof of [ABP, Lemma 3.62] shows that a tame GRLA is non-singular.

(iii) Semisimple Lie algebras, finite dimensional reductive Lie algebras and a direct sum of EALAs are examples of non-singular GRLAs. Heisenberg Lie algebras (with derivation added) satisfy axioms GR1–GR4, however they are not GRLAs as $R^\times = \varnothing$. It is shown in [ABY, Section 3] that the fixed point subalgebra of an EALA under a finite order automorphism satisfies GR1–GR4.

From now on we assume that $(\mathcal{G}, (\,\cdot\,,\,\cdot\,), \mathcal{H})$ *is a* GRLA *with the corresponding root system* R. So we may use all the results in [AABGP] which are obtained by axioms GR1–GR5. Let us state from [AABGP] some of the important properties of $\mathcal{G}$ which will be of use in the sequel. We emphasize in particular the axioms which are used in the proof of each result.

It is shown in [AABGP, I.(1.18)] that if $\mathcal{G}$ satisfies GR1–GR2, then for $\alpha \in R^\times$ there exist $e_a \in \mathcal{G}_\alpha$ and $f_\alpha \in \mathcal{G}_{-\alpha}$ such that

$$(e_\alpha, h_\alpha := [e_\alpha, f_\alpha], f_\alpha) \text{ is a } \mathfrak{sl}_2\text{-triple,} \tag{1.6}$$

that is the $\mathbb{C}$-span of $\{e_a, h_\alpha, f_\alpha\}$ is a Lie subalgebra of $\mathcal{G}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Note that

$$h_\alpha = \frac{2t_\alpha}{(\alpha, \alpha)}.$$

If $\mathcal{U}$ is a vector space equipped with a bilinear form, let us write

$$\alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)} \quad \text{for } \alpha \in \mathcal{U} \text{ with } (\alpha, \alpha) \neq 0.$$

Theorem 1.3 (AABGP, Theorem I.1.29) *Let* $\mathcal{G}$ *satisfy* GR1–GR3 *and* $\alpha \in R^\times$. *Then*

(a) *For* $\beta \in R$, *we have* $(\beta, \alpha^\vee) \in \mathbb{Z}$.
(b) *For* $\beta \in R$, $\beta - (\beta, \alpha^\vee)\alpha \in R$.
(c) *If* $k \in \mathbb{C}$ *and* $k\alpha \in R$, *then* $k = 0, \pm 1$.
(d) $\dim \mathcal{G}_\alpha = 1$.
(e) *For any* $\beta \in R$, *there exist two non-negative integers* u, d *such that for any* $n \in \mathbb{Z}$ *we have* $\beta + n\alpha \in R$ *if and only if* $-d \leq n \leq u$. *Moreover,* $d - u = (\beta, \alpha^\vee)$.

The statement of part (i) of the following proposition is the same as [AABGP, Proposition I.2.1], however the proof given here is different, as we are not allowed to use axiom GR6. We do not even use GR4 in the proof.

Proposition 1.4

(i) *Let* $\mathcal{G}$ *satisfy* GR1–GR3. *Then* $(R, R^0) = \{0\}$.
(ii) $\delta \in R^0$ *is isolated if and only if* $\mathcal{G}_\delta \subseteq C_{\mathcal{G}}(\mathcal{G}_c)$.

Proof (i) First let $\alpha \in R^\times$ and $\delta \in R^0$. Suppose to the contrary that $(\alpha, \delta) \neq 0$. By [AABGP, Lemma I.1.30], $\alpha + n\delta \in R$ for sufficiently large n, and it is clear that $\alpha + n\delta \in R^\times$ except at most for one n. But then for suitable n we have

$$\frac{2(\alpha + n\delta, \delta)}{(\alpha + n\delta, \alpha + n\delta)} = \frac{2(\alpha, \delta)}{(\alpha, \alpha) + 2n(\alpha, \delta)} \notin \mathbb{Z}$$

which contradicts Theorem 1.3(a).

Next let $\delta, \eta \in R^0$. We must show $(\delta, \eta) = 0$. If not, then $\eta + \delta$ and $\eta - \delta$ are non-isotropic and are not orthogonal to δ, η. Therefore by the previous step, $\eta \pm \delta \notin R$. So we get a contradiction if we show that

$$(\eta, \delta) \neq 0 \Rightarrow \eta + \delta \in R \text{ or } \eta - \delta \in R.$$

Suppose $(\eta, \delta) \neq 0$ and $\eta - \delta \notin R$. Choose $x_\delta \in \mathcal{G}_\delta$ and $x_{-\delta} \in \mathcal{G}_{-\delta}$ such that $[x_{-\delta}, x_\delta] = t_\delta$. Take any $0 \neq x_\eta \in \mathcal{G}_\eta$. Then using the Jacobi identity, we have

$$\left[x_{-\delta}, [x_\delta, x_\eta]\right] = (\eta, \delta) x_\eta \neq 0.$$

Thus $[x_\delta, x_\eta] \neq 0$, and so $\eta + \delta \in R$.

(ii) Suppose first that $\delta \in R^0$ is not isolated, that is there exists $\alpha \in R^\times$ such that $\alpha + \delta \in R$. We must show that $[\mathcal{G}_\delta, \mathcal{G}_c] \neq \{0\}$. Since $\mathcal{G}_\alpha \subseteq \mathcal{G}_c$, it is enough to show that $[\mathcal{G}_\delta, \mathcal{G}_\alpha] \neq \{0\}$. Consider the non-negative integers u, d appearing in the α-string through δ, as in part (e) of Theorem 1.3. We have $d = u$ as by part (i), $(\delta, \alpha) = 0$. Let $0 \neq x \in \mathcal{G}_{\delta - d\alpha}$. Then $[x, \mathcal{G}_{-\alpha}] \subseteq \mathcal{G}_{\delta-(d+1)\alpha} = \{0\}$. By [AABGP, Lemma I.1.21] and part (d) of Theorem 1.3, for any $0 \neq z \in \mathcal{G}_\alpha$, we have

$$(\operatorname{ad} z)^N(x) \neq 0 \text{ but } (\operatorname{ad} z)^{N+1}(x) = 0,$$

where $N = 2(\delta - d\alpha, -\alpha)/(\alpha, \alpha) = 2d$. Since $\alpha + \delta \in R$, we have $d = u \geq 1$, so $d + 1 \leq 2d = N$. Thus $[z, (\operatorname{ad} z)^d(x)] = (\operatorname{ad} z)^{d+1}(x) \neq 0$. But $(\operatorname{ad} z)^d(x) \in \mathcal{G}_\delta$ and so $[\mathcal{G}_\delta, \mathcal{G}_\alpha] \neq \{0\}$. Conversely, if δ is isolated then $\alpha + \delta \notin R$ for all $\alpha \in R^\times$. Thus $[\mathcal{G}_\delta, \mathcal{G}_\alpha] = \{0\}$. ■

Remark 1.5 (i) According to part (ii) of Proposition 1.4, axiom GR6b is equivalent to

GR6b′ For any $\delta \in R^0$, the root space $\mathcal{G}_\delta$ is not contained in the centralizer of the core.

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(ii) The proof of part (ii) of Proposition 1.4 and [AG, Lemma 1.3] show that for any $\alpha \in R^\times$ and $\beta \in R$, $\alpha + \beta$ is a root if and only if $[\mathcal{G}_\alpha, \mathcal{G}_\beta] \neq \{0\}$. In particular, if $\alpha + \beta \in R^\times$, then $[\mathcal{G}_\alpha, \mathcal{G}_\beta] = \mathcal{G}_{\alpha+\beta}$. The proof of Proposition 1.4(ii) is in fact a modified version of a standard $\mathfrak{sl}_2$-argument.

Define an equivalence relation on $R^\times$ by saying that two roots α and β in $R^\times$ are related if and only if there is a sequence of roots $\alpha_0 = \alpha, \alpha_1, \dots, \alpha_t = \beta$ in $R^\times$ such that $(\alpha_i, \alpha_{i+1}) \neq 0$ for $0 \leq i \leq t-1$. This determines a partition $R^\times = \bigcup_{i\in I} R_i^\times$ where $R_i^\times$'s are indecomposable and $(R_i^\times, R_j^\times) = 0$ for $i \neq j$. In particular, if $\mathcal{V}$ is the real span of R and $\mathcal{V}_i$ is the real span of $R_i^\times$ then

$$(\mathcal{V}_i, \mathcal{V}_j) = 0 \quad \text{for } i \neq j, \tag{1.7}$$

and

$$\mathcal{V} = \sum_{i\in I} \mathcal{V}_i + \operatorname{span}_{\mathbb{R}} R^0. \tag{1.8}$$

Fix $i \in I$. It follows from part (a) of Theorem 1.3 and indecomposability of $R_i^\times$ (see [AABGP, I. Section 2]) that there exist nonzero scalars $c_i \in \mathbb{C}$ such that the form on $\mathcal{V}_i$ defined by

$$(\cdot\,,\cdot)_i := c_i(\cdot\,,\cdot), \tag{1.9}$$

is real valued and

$$(\gamma, \gamma)_i > 0 \quad \text{for some } \gamma \in R_i^\times. \tag{1.10}$$

If (1.10) does not hold for all $\gamma \in R_i^\times$, then it follows from indecomposability of $R_i^\times$ that there are roots $\alpha, \beta \in R_i^\times$ such that $(\alpha, \alpha)_i > 0$, $(\beta, \beta)_i < 0$ and $(\alpha, \beta)_i \neq 0$. But as the proof of [AABGP, Lemma I.2.3] suggests this leads to the existence of a complex simple Lie algebra of dimension 6 which is absurd. Thus

$$(\alpha, \alpha)_i > 0 \quad \text{for all } \alpha \in R_i^\times. \tag{1.11}$$

Also note that for $\alpha, \beta \in \mathcal{V}_i$ with $(\alpha, \alpha)_i \neq 0$, we have

$$(\beta, \alpha^\vee)_i := \frac{2(\beta, \alpha)_i}{(\alpha, \alpha)_i} = (\beta, \alpha^\vee). \tag{1.12}$$

The proof of the following lemma is almost the same as [AABGP, Lemma I.2.6], however we have to be careful for one part in which the α-string through β is used. We provide the proof here with the necessary modifications.

Lemma 1.6 *For $\alpha, \beta \in R^\times$, we have $-4 \leq (\beta, \alpha^\vee) \leq 4$.*

Proof Since $(R_i^\times, R_j^\times) = \{0\}$ for $i \neq j$ and $(R, R^0) = \{0\}$ by Proposition 1.4, we may assume that $\alpha, \beta \in R_i^\times$ for some i. Since $(\beta, \alpha^\vee) = (\beta, \alpha^\vee)_i$ we may replace $(\cdot\,,\cdot)$ with $(\cdot\,,\cdot)_i$. So it is enough to show that if $(\alpha, \beta)_i < 0$ then $(\beta, \alpha^\vee)_i \geq -4$. Suppose to the contrary that $(\alpha, \beta)_i < 0$ but $a = (\beta, \alpha^\vee) \leq -5$. Let $b = (\alpha, \beta^\vee)$. From (1.11) we have $b \leq -1$. By Theorem 1.3(e) all elements of the string

$$\beta - d\alpha, \dots, \beta - \alpha, \beta, \beta + \alpha, \dots, \beta + u\alpha,$$

are elements of R, where $d - u = a \leq -5$. Thus $u \geq 5$. In particular $\beta + 2\alpha \in R$. If $(\beta + 2\alpha, \alpha)_i = 0$ then $(\beta, \alpha)_i = -2(\alpha, \alpha)_i$. So

$$-5 \geq a = \frac{2(\beta, \alpha)_i}{(\alpha, \alpha)_i} = \frac{-4(\alpha, \alpha)_i}{(\alpha, \alpha)_i} = -4$$

which is absurd. Thus $(\beta + 2\alpha, \alpha)_i \neq 0$ and so $\beta + 2\alpha \in R_i^\times$. Then

$$(\beta + 2\alpha, \beta + 2\alpha)_i = \frac{2(\beta, \alpha)_i}{a}(a + 4) < 0.$$

This contradicts (1.11). ∎

The restriction of the form $(\cdot\,,\cdot)$ to $\mathcal{V} \times \mathcal{V}$ defines a symmetric bilinear map

$$(\cdot\,,\cdot)\colon \mathcal{V} \times \mathcal{V} \to \mathbb{C}.$$

Let

$$\mathcal{V}^0 := \left\{ v \in \mathcal{V} \mid (v, \mathcal{V}) = \{0\} \right\},$$

be the radical of this map. Let $\overline{\mathcal{V}} = \mathcal{V}/\mathcal{V}^0$ and consider the canonical map $^-\colon \mathcal{V} \to \overline{\mathcal{V}}$. We have $\overline{\mathcal{V}} \neq \{0\}$ as we have assumed that $R^\times \neq \varnothing$. Define on the real vector space $\overline{\mathcal{V}} \times \overline{\mathcal{V}}$ a complex valued symmetric bilinear map

$$(\cdot\,,\cdot)\colon \overline{\mathcal{V}} \times \overline{\mathcal{V}} \to \mathbb{C} \tag{1.13}$$

by

$$(\overline{\alpha}, \overline{\beta}) := (\alpha, \beta) \quad \text{for } \alpha, \beta \in \mathcal{V}.$$

Then $(\cdot\,,\cdot)$ is nondegenerate on $\overline{\mathcal{V}}$. Moreover, if $\overline{\mathcal{V}}_i$ is the image of $\mathcal{V}_i$ under the map $^-$, then the restriction of $(\cdot\,,\cdot)$ to $\overline{\mathcal{V}}_i \times \overline{\mathcal{V}}_i$ is also nondegenerate and for $\overline{\alpha}, \overline{\beta} \in \overline{\mathcal{V}}_i \setminus \{0\}$,

$$(\overline{\beta}, \overline{\alpha}^\vee)_i := \frac{2(\overline{\beta}, \overline{\alpha})_i}{(\overline{\alpha}, \overline{\alpha})_i} = (\overline{\beta}, \overline{\alpha}^\vee) \in \mathbb{Z}. \tag{1.14}$$

Set

$$\overline{R} = \{\overline{\alpha} \mid \alpha \in R\} \text{ and } \overline{R}_i = \{\overline{\alpha} \mid \alpha \in R_i^\times\} \cup \{0\}.$$

Lemma 1.7 *$\overline{R}$ is finite.*

Proof Consider a basis $\{\overline{\alpha}_1, \ldots, \overline{\alpha}_\ell\} \subseteq \overline{R}$ of $\overline{\mathcal{V}}$. If $\overline{\alpha}_j \in \overline{R}_{i_j}$ for some $i_j \in I$, then by (1.11) the term $c_{i_j}(\overline{\alpha}_j, \overline{\alpha}_j)$ is a nonzero real number and so

$$\Big\{ \frac{2\overline{\alpha}_1}{c_{i_1}(\overline{\alpha}_1, \overline{\alpha}_1)}, \ldots, \frac{2\overline{\alpha}_\ell}{c_{i_\ell}(\overline{\alpha}_\ell, \overline{\alpha}_\ell)} \Big\}$$

is also a basis of $\overline{\mathcal{V}}$. Now for $\overline{\beta} \in \overline{\mathcal{V}}$ define

$$\varphi(\overline{\beta}) = \left(c_{i_1}\Big(\overline{\beta}, \frac{2\overline{\alpha}_1}{c_{i_1}(\overline{\alpha}_1, \overline{\alpha}_1)}\Big), \ldots, c_{i_\ell}\Big(\overline{\beta}, \frac{2\overline{\alpha}_\ell}{c_{i_\ell}(\overline{\alpha}_\ell, \overline{\alpha}_\ell)}\Big) \right).$$

Note that for each j, if $(\overline{\beta}, \overline{\alpha}_j) \neq 0$, then $\overline{\beta} \in \overline{R}_{i_j}$ and so by (1.14), $\varphi(\overline{\beta}) \in \mathbb{Z}^\ell$. Now from Lemma 1.6 we see that $\varphi(\overline{R})$ has at most 9^ℓ elements. Since the bilinear map (1.13) on $\overline{\mathcal{V}}$ is nondegenerate, φ is one to one and so $\overline{R}$ is finite. ∎

We have from Lemma 1.7 that the index set I is finite, say $I = \{1, \ldots, k\}$, and so

$$R^\times = \bigcup_{i=1}^{k} R_i^\times \quad \text{and} \quad \mathcal{V} = \sum_{i=1}^{k} \mathcal{V}_i + \operatorname{span}_{\mathbb{R}} R^0, \tag{1.15}$$

where $\operatorname{span}_{\mathbb{R}} R^0 \subseteq \mathcal{V}^0$.

For $\alpha \in R^\times$ define $w_\alpha \colon \mathcal{V} \to \mathcal{V}$ by

$$w_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha, \quad (\beta \in \mathcal{V}).$$

By Theorem 1.3(b), $w_\alpha(R) \subseteq R$. Since w_α preserves the form $(\cdot\,,\cdot)$, we have from (1.7) that for $\alpha \in R_i^\times$,

$$w_\alpha(R_i^\times) \subseteq R_i^\times \quad \text{and} \quad w_\alpha(\mathcal{V}_i) \subseteq \mathcal{V}_i.$$

In a similar manner, for $\overline{\alpha} \in \overline{R}_i \setminus \{0\}$, we can define $w_{\overline{\alpha}}$ on $\overline{\mathcal{V}}_i$. Then

$$w_{\overline{\alpha}}(\overline{\beta}) = \overline{w_\alpha(\beta)} \quad \text{and} \quad w_{\overline{\alpha}}(\overline{R}_i) \subseteq \overline{R}_i. \tag{1.16}$$

Lemma 1.8

(i) *The symmetric bilinear form $(\cdot\,,\cdot)_i$ on $\overline{\mathcal{V}}_i$ is positive definite.*
(ii) *The symmetric bilinear form $(\cdot\,,\cdot)_i$ on $\mathcal{V}_i$ is positive semidefinite.*

Proof Clearly (ii) is an immediate consequence of (i). By Lemma 1.7, $\overline{R}_i$ is finite, and $\overline{R}_i \setminus \{0\}$ is indecomposable with respect to the form $(\cdot\,,\cdot)_i$, as $R_i^\times$ is indecomposable. Now follow the proof of [AABGP, Theorem I.2.14] with $\overline{R}_i$ in place of $\overline{R}$ and $\overline{\mathcal{V}}_i$ in place of $\overline{\mathcal{V}}$. ∎

We now would like to put together the forms $(\cdot\,,\cdot)_i$ to obtain a positive semidefinite bilinear form on $\mathcal{V}$. Using (1.15) we can write each element $\alpha \in \mathcal{V}$ in the form

$$\alpha = \sum_{i=1}^{k} \alpha_i + \delta_\alpha \quad \text{where } \alpha_i \in \mathcal{V}_i,\ \delta_\alpha \in \operatorname{span}_{\mathbb{R}} R^0 \subseteq \mathcal{V}^0. \tag{1.17}$$

Lemma 1.9 *Let* $\alpha = \sum_{i=1}^{k} \alpha_i + \delta_\alpha$ *and* $\beta = \sum_{i=1}^{k} \beta_i + \delta_\beta$ *be two elements of* $\mathcal{V}$ *in the form* (1.17) *and define*

$$(\alpha, \beta)' = \sum_{i=1}^{k} (\alpha_i, \beta_i)_i. \tag{1.18}$$

Then $(\cdot\,,\,\cdot)'$ *is a well-defined real valued positive semidefinite symmetric bilinear form on* $\mathcal{V}$*. Moreover,*

$$(\beta, \alpha^\vee)' := \frac{2(\beta, \alpha)'}{(\alpha, \alpha)'} = (\beta, \alpha^\vee) \quad \textit{for } \alpha, \beta \in R^\times. \tag{1.19}$$

Proof To show that $(\cdot\,,\,\cdot)'$ is well defined consider α and β as in the statement and let $\alpha = \sum_{i=1}^{k} \alpha_i' + \delta_\alpha'$ and $\beta = \sum_{i=1}^{k} \beta_i' + \delta_\beta'$ be other expressions of α and β in the form (1.17). We must show that

$$\sum_{i=1}^{k} (\alpha_i, \beta_i)_i = \sum_{i=1}^{k} (\alpha_i', \beta_i')_i.$$

Now using (1.7) and the fact that δ_α, δ_β, δ_α', δ_β' are isotropic, we have

$$\begin{aligned}
(\alpha_i, \beta_i)_i &= c_i\big(\alpha - \sum_{j \neq i} \alpha_j - \delta_\alpha, \beta_i\big) \\
&= c_i\Big(\sum_{j=1}^{k} \alpha_j' + \delta_\alpha' - \sum_{j \neq i} \alpha_j - \delta_\alpha, \beta_i\Big) \\
&= c_i(\alpha_i', \beta_i) \\
&= c_i\big(\alpha_i', \beta - \sum_{j \neq i} \beta_j - \delta_\beta\big) \\
&= c_i\big(\alpha_i', \sum_{j=1}^{k} \beta_j' + \delta_\beta' - \sum_{j \neq i} \beta_j - \delta_\beta\big) \\
&= c_i(\alpha_i', \beta_i') = (\alpha_i', \beta_i')_i.
\end{aligned}$$

This proves that the form $(\cdot\,,\,\cdot)'$ is well-defined. Now since $(\cdot\,,\,\cdot)_i$ is positive semidefinite on $\mathcal{V}_i$, for each i, it is clear that $(\cdot\,,\,\cdot)'$ is positive semidefinite on $\mathcal{V}$ and that (1.19) holds. ∎

We have from Lemmas 1.8, 1.7, (1.14) and (1.16) that

$$\overline{R}_i \text{ is an irreducible finite root system in } \overline{\mathcal{V}}_i. \tag{1.20}$$

Lemma 1.9 together with other properties which we have seen about $\mathcal{V}$ and R lead us to state the following definition.

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Definition 1.10 Let $\mathcal{V}$ be a nontrivial finite dimensional real vector space with a nontrivial positive semidefinite symmetric bilinear form $(\cdot,\cdot)$ and let R be a subset of $\mathcal{V}$. We say R is a *generalized reductive root system* (GRRS for short) in $\mathcal{V}$ if R satisfies the following 5 axioms:

(R1) $R = -R$.
(R2) R spans $\mathcal{V}$.
(R3) R is discrete in $\mathcal{V}$.
(R4) If $\alpha \in R^\times$ and $\beta \in R$, there exist two non-negative integers u, d such that for any $n \in \mathbb{Z}$ we have $\beta + n\alpha \in R$ if and only if $-d \leq n \leq u$. Moreover, $d - u = (\beta, \alpha^\vee)$.
(R5) $\alpha \in R^\times \Rightarrow 2\alpha \notin R$.

We call the GRRS R *non-singular* if it satisfies:

(R6) for any $\delta \in R^0$, there exists $\alpha \in R^\times$ such that $\alpha + \delta \in R$.

We say a root satisfying this condition is *nonisolated* and call isotropic roots which do not satisfy this *isolated*.

The GRRS R is called *indecomposable* if it satisfies:

(R7) $R^\times$ cannot be decomposed into a disjoint union of two nonempty subsets which are orthogonal with respect to the form.

A non-singular indecomposable GRRS R is known at the literature as *an extended affine root system* (EARS for short). One may also call it an *irreducible* GRRS. The *nullity* of a GRRS R is defined to be the dimension of the real span of R^0.

Since the form in the definition of a GRRS is nontrivial we have from (R2) that $R^\times \neq \varnothing$. Then it follows from this, (R1) and (R4) that $0 \in R$. The root system R of a (non-singular) GRLA $\mathcal{G}$ is a (non-singular) GRRS. In fact the existence of a nontrivial positive semi-definite bilinear form was shown in Lemma 1.9, and by (1.1), (1.3), GR4 and Theorem 1.3(d)–(e) axioms (R1)–(R5) also hold. From Remark 1.2(ii) we know that the root system of a tame GRLA is a non-singular GRRS. We define the *nullity* of a GRLA to be the nullity of its root system.

For a GRRS R we set

$$R_{\mathrm{iso}} = \{\delta \in R^0 \mid \delta + \alpha \notin R \text{ for any } \alpha \in R^\times\}, \text{ and } R_{\mathrm{niso}} = R^0 \setminus R_{\mathrm{iso}}.$$

That is R_{iso} (resp., R_{niso}) is the set of isolated (non-isolated) roots of R. So R is non-singular if and only if $R_{\mathrm{iso}} = \varnothing$.

2 Intrinsic Decomposition of a GRLA

Let $\big(\mathcal{G}, (\cdot,\cdot), \mathcal{H}\big)$ be a GRLA with root system R. Let $\mathcal{V}$, $\overline{\mathcal{V}}$, $\mathcal{V}_i$ and $\overline{\mathcal{V}}_i$ be as in Section 1. Let $(\cdot,\cdot)$ be the form on $\mathcal{H}^\star$, defined by (1.5), restricted to $\mathcal{V}$. Fix a real valued positive semidefinite symmetric bilinear form $(\cdot,\cdot)_i$ on $\mathcal{V}_i$, as in (1.9). Let $(\cdot,\cdot)$ be the real positive semidefinite symmetric bilinear form on $\mathcal{V}$ defined by (1.18). Then

$$(\cdot,\cdot)'|_{\mathcal{V}_i} = (\cdot,\cdot)_i = c_i(\cdot,\cdot).$$

We also have from Lemma 1.8(i) that the form $(\cdot\,,\cdot)_i$ on $\overline{\mathcal{V}}_i$ is real-valued and positive definite, and $\overline{R}_i$ is an irreducible finite root system in $\overline{\mathcal{V}}_i$. Note also that the forms $(\cdot\,,\cdot)$ and $(\cdot\,,\cdot)'$ on $\mathcal{V}$ have the same radical $\mathcal{V}^0$.

Fix a basis $\overline{\Pi}_i = \{\overline{\alpha}_{i1},\dots,\overline{\alpha}_{i\ell_i}\}$ of $\overline{R}_i$ and choose a preimage $\alpha_{ij} \in R_i$ of $\overline{\alpha}_{ij}$, $1 \le j \le \ell_i$. Put $\dot{\Pi}_i = \{\alpha_{i1},\dots,\alpha_{i\ell_i}\}$.

Set

$$\dot{\mathcal{V}}_i = \operatorname{span}_{\mathbb{R}} \dot{\Pi}_i,\ \dot{R}_i = \{\dot{\alpha} \in \dot{\mathcal{V}}_i \mid \overline{\dot{\alpha}} \in \overline{R}_i\} \text{ and } \mathcal{V}_i^0 = \mathcal{V}^0 \cap \mathcal{V}_i.$$

Then

$$\mathcal{V}_i = \dot{\mathcal{V}}_i \oplus \mathcal{V}_i^0 \text{ and } \mathcal{V} = \Big(\bigoplus_{i=1}^{k} \dot{\mathcal{V}}_i\Big) \oplus \mathcal{V}^0. \tag{2.1}$$

(See (1.15) for this last equality.) Now the map $\bar{\ }$ restricted to $\dot{\mathcal{V}}_i$ is an isometry from $\dot{\mathcal{V}}_i$ onto $\overline{\mathcal{V}}_i$ with respect to the form $(\cdot\,,\cdot)' = c_i(\cdot\,,\cdot)$ which maps $\dot{R}_i$ bijectively onto $\overline{R}_i$, and so $\dot{R}_i$ is a finite root system in $\dot{\mathcal{V}}_i$ isomorphic to $\overline{R}_i$.

Moreover,

$$R_i^{\times} = \{\dot{\alpha} + \delta \in R \mid \dot{\alpha} \in \dot{R}_{\mathrm{sh}}\, i^{\times}, \delta \in \mathcal{V}^0\}.$$

Let $\mathcal{W}$ be the Weyl group of R and $\dot{\mathcal{W}}_i$ be the group generated by $\{w_{\dot{\alpha}} \mid \dot{\alpha} \in \dot{R}_i^{\times}\}$. Since $(\dot{R}_i, \mathcal{V}_j) = \{0\}$ for $i \neq j$ and $(R_i, \mathcal{V}^0) = \{0\}$, by restriction $\dot{\mathcal{W}}_i$ is isomorphic to the Weyl group of the finite root system $\dot{R}_i$. Moreover, since R contains all reduced roots of $\dot{R}_i$ (see [AABGP, Proposition II.2.11]) and $w_{r\dot{\alpha}} = w_{\dot{\alpha}}$, $\dot{\alpha} \in \dot{R}_i^{\times}$, $r \in \mathbb{R} \setminus \{0\}$, $\dot{\mathcal{W}}_i$ is a subgroup of $\mathcal{W}$. It follows from this that

$$R_i^{\times} = \{\dot{\alpha} + \delta \in R \mid \dot{\alpha} \in \dot{R}_i^{\times}, \delta \in R^0\}. \tag{2.2}$$

In fact if $\dot{\alpha} + \delta \in R$, $\dot{\alpha} \in \dot{R}_i^{\times}$, $\delta \in \mathcal{V}^0$, then $\dot{\alpha} = r w_{\dot{\beta}_1} \cdots w_{\dot{\beta}_m}(\dot{\gamma})$ for some $\dot{\beta}_1,\dots,\dot{\beta}_m$, $\dot{\gamma} \in \dot{\Pi}_i$ where $r = 2$ or 1, depending on either $\dot{\alpha}/2$ is a root in $\dot{R}_i$ or not. Since R is $\mathcal{W}$ invariant and isotropic elements are fixed under the action of the Weyl group we get $\pm r\dot{\gamma} + \delta \in R$. Now consider the $\dot{\gamma}$-string through $r\dot{\gamma} + \delta$ to conclude that $\delta \in R$ (see [AABGP, Proposition II.2.11(b)]). A similar argument shows that if we put

$$R_i^0 := \{\delta \in R^0 \mid \alpha + \delta \in R \text{ for some } \alpha \in R_i^{\times}\},$$

then

$$\{\delta \in R^0 \mid \dot{\alpha} + \delta \in R \text{ for some } \dot{\alpha} \in \dot{R}_i^{\times}\} \subseteq R_i^0. \tag{2.3}$$

Now let

$$R_i = R_i^{\times} \cup R_i^0.$$

Then

$$R = \Big(\bigcup_{i=1}^{k} R_i\Big) \cup R_{\mathrm{iso}}. \tag{2.4}$$

Since $\mathcal{V} = \operatorname{span}_{\mathbb{R}} R$, it follows from (2.4), (2.1) and (2.2) that

$$\mathcal{V}^0 = \operatorname{span}_{\mathbb{R}} R^0.$$

Set

$$R_i' = R_i \cup (\langle R_i \rangle \cap R^0).$$

By (2.3),

$$(R_i')^0 = \langle R_i \rangle \cap R^0 = \langle R_i^0 \rangle \cap R^0. \tag{2.5}$$

The proof of the following lemma is essentially the same as [ABY, Lemma 1.2], however for the reader's convenience we give the details. In what follows we denote by $\langle S \rangle$, the $\mathbb{Z}$-span of a subset S of a vector space.

Lemma 2.1 *Let R be a* GRRS *and R_1 be a subset of R with $R_1^\times := R_1 \cap R^\times \neq \varnothing$. Suppose that*

(a) $R_1 = -R_1$,
(b) $\{\delta \in R^0 \mid \alpha' + \delta \in R_1$ *for some* $\alpha' \in R_1^\times\} \subseteq R_1$,
(c) $\alpha' \in R_1, \beta \in R, (\alpha', \beta) \neq 0 \Rightarrow \beta \in R_1$.

Then R_1 is a GRRS *in its real span. Moreover, if we set*

$$R_1' = R_1^\times \cup (\langle R_1 \rangle \cap R^0),$$

then R_1' is also a GRRS *in the real span of R_1.*

Proof Since $R_1^\times \neq \varnothing$, it is enough to show that axioms (R1)–(R5) hold for R_1. Clearly (R1)–(R3) and (R5) hold for R_1. We now check (R4). Let $\alpha' \in R_1^\times$ and $\beta' \in R_1$. Since (R4) holds for R, it is enough to show that for $n \in \mathbb{Z}$,

$$\beta' + n\alpha' \in R \Rightarrow \beta' + n\alpha' \in R_1.$$

Since $\beta' \in R_1$, we may assume that $n \neq 0$. Assume first that $n > 0$. So let $\beta' + n\alpha' \in R$, $n > 0$. If $\beta' + n\alpha' \in R^\times$, then $(\beta' + n\alpha', \beta') \neq 0$ or $(\beta' + n\alpha', \alpha') \neq 0$. In either case, we get from (c) that $\beta' + n\alpha' \in R_1$. Next, let $\beta' + n\alpha' \in R^0$. Since (R4) holds for R and $n > 0$, we have $\beta' + (n-1)\alpha' \in R^\times$. So repeating our previous argument we get $\beta' + (n-1)\alpha' \in R_1$. Since

$$\beta' + n\alpha' + (-\alpha') = \beta' + (n-1)\alpha' \in R_1^\times$$

it follows from (a) and (b) that $\beta' + n\alpha' \in R_1$. If $n < 0$ and $\beta' + n\alpha' \in R$, then $-\beta' - n\alpha' \in R$. Now by the previous step $-\beta' - n\alpha' \in R_1$ and so by (a), $\beta' + n\alpha' \in R_1$. This completes the proof of the first assertion.

Next let R_1' be as in the statement. Clearly R_1 and R_1' have the same real span. Since $R_1^\times = (R_1')^\times$, it is easy to check that R_1' satisfies conditions (a)–(c), and so is a GRRS. ∎

Corollary 2.2 *R_i is an* EARS *(an irreducible* GRRS*) and R_i' is an indecomposable* GRRS *in $\mathcal{V}_i$.*

Proof It is clear from the way R_i is defined that conditions (a) and (b) of Lemma 2.1 hold for R_i. Condition (c) also holds as R_i is indecomposable. ■

We define the *rank* of R_i to be the *rank* of finite root system $\overline{R}_i$.

Remark 2.3 (i) From [AABGP, Theorem II.2.37] we know that the set of isotropic roots of an EARS is of the form $S+S$ where S is a semilattice in the radical of the form. If the nullity ν is 1, then S is a lattice ([AABGP, Corollary II.1.7]). If $\nu = 2$ then from [AABGP, II. Section 1] we know that the $\mathbb{Z}$-span Λ of S is of the form $\Lambda = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2$ where $\sigma_1, \sigma_2 \in S$ and $\{\sigma_1, \sigma_2\}$ is a basis of $\mathcal{V}^0$. Then S is one of the semilattices

$$\Lambda,\ S' := 2\Lambda \cup (\sigma_1 + 2\Lambda) \cup (\sigma_2 + 2\Lambda), \quad S' + \sigma_1,\ S' + \sigma_2.$$

Thus $S + S = \Lambda$. Therefore if $\nu \le 2$, the set of isotropic roots is a lattice. Now according to Corollary 2.2 each R_i is an EARS and so R_i^0 is a lattice if $\nu \le 2$. Thus by (2.5), $(R_i')^0 = R_i^0$ and so $R_i' = R_i$. In particular if R is non-singular, all R_i''s are also non-singular.

(ii) Even when R is non-singular, the root systems R_i' might be singular. According to part (i), this only can happen if $\nu \ge 3$. To see an example let $\nu = 3$, $\mathcal{V} = \mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_2 \oplus \mathbb{R}\sigma_1 \oplus \mathbb{R}\sigma_2 \oplus \mathbb{R}\sigma_3$. Define a positive semi-definite bilinear form on $\mathcal{V}$ by letting σ_i's to be isotropic and $(\alpha_i, \alpha_j) = 2\delta_{ij}$. Set

$$R_1 = (S + S) \cup (\pm\alpha_1 + S) \text{ and } R_2 = \Lambda \cup (\pm\alpha_2 + \Lambda),$$

where

$$\Lambda = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2 \oplus \mathbb{Z}\sigma_3 \text{ and } S = 2\Lambda \cup (\sigma_1 + 2\Lambda) \cup (\sigma_2 + 2\Lambda) \cup (\sigma_3 + 2\Lambda).$$

By [AABGP, Theorem II.2.37], R_1 and R_2 are GRRS of type A_1 in the real span of R_1 and R_2, respectively. Set $R = R_1 \cup R_2$. Then R is a non-singular GRRS in $\mathcal{V}$. However $R_1' = \Lambda \cup (\pm\alpha_1 + S)$ is a singular GRRS.

We now return to the GRLA $\mathcal{G}$ and the corresponding GRRS R. Set

$$\mathcal{H}(R) = \operatorname{span}_{\mathbb{C}}\{t_\alpha \mid \alpha \in R\},$$
$$\mathcal{H}^0(R) = \big\{ h \in \mathcal{H}(R) \mid \big(h, \mathcal{H}(R)\big) = \{0\}\big\} \quad \text{and}$$
$$\dot{\mathcal{H}}_i = \operatorname{span}_{\mathbb{C}}\{t_\alpha \mid \alpha \in \dot{R}_i\} = \operatorname{span}_{\mathbb{C}}\{t_\alpha \mid \alpha \in \dot{R}_i^{\times}\}.$$

Lemma 2.4

(i) *The form $(\cdot\,,\cdot)$ on $\mathcal{H}$ restricted to $\dot{\mathcal{H}}_i$ is nondegenerate.*
(ii) $\mathcal{H}(R) = \mathcal{H}^0(R) \oplus (\bigoplus_{i=1}^k \dot{\mathcal{H}}_i)$.
(iii) $\mathcal{H}^0(R) = \operatorname{span}_{\mathbb{C}}\{t_\delta \mid \delta \in R^0\}$.

Proof (i) Let $h = \sum (k_j + ik'_j) t_{\alpha_j} \in \dot{\mathcal{H}}_t$ be in the radical of the form $(\cdot\,,\,\cdot)$ restricted to $\dot{\mathcal{H}}_t$, where $\alpha_j \in \dot{R}_t$, $k_j, k'_j \in \mathbb{R}$ and i is the complex number with $i^2 = -1$. In particular

$$\Big(\sum k_j\alpha_j, \dot{\mathcal{V}}_t\Big) + i\Big(\sum k'_j\alpha_j, \dot{\mathcal{V}}_t\Big) = \{0\}.$$

Multiplying both sides by the scalar c_t (see (1.9)) we obtain

$$\Big(\sum k_j\alpha_j, \dot{\mathcal{V}}_t\Big)_t + i\Big(\sum k'_j\alpha_j, \dot{\mathcal{V}}_t\Big)_t = \{0\}.$$

Since $(\cdot\,,\,\cdot)_t$ is real valued and positive definite (in particular nondegenerate) on $\dot{\mathcal{V}}_t$ we get $\sum k_j\alpha_j = 0$, $\sum k'_j\alpha_j = 0$ and so $h = 0$.

(ii) Since $\dot{R}_i$ and $\dot{R}_j$ are orthogonal if $i \neq j$, we have $(\dot{\mathcal{H}}_i, \dot{\mathcal{H}}_j) = \{0\}$. So by part (i), $\dot{\mathcal{H}}_i \cap (\sum_{j\neq i} \dot{\mathcal{H}}_j) = \{0\}$. In particular, the sum $\sum_{i=1}^k \dot{\mathcal{H}}_i$ is direct and the form restricted to $\bigoplus_{i=1}^k \dot{\mathcal{H}}_i$ is nondegenerate. Thus $(\bigoplus_{i=1}^k \dot{\mathcal{H}}_i) \cap \mathcal{H}^0(R) = \{0\}$. Note that if $\alpha \in R$, then $\alpha = \dot{\alpha} + \delta$, where $\dot{\alpha} \in \dot{R}_i$ and $\delta \in \mathcal{V}^0$, for some $1 \leq i \leq k$. Now $t_{\dot{\alpha}} \in \dot{\mathcal{H}}_i$ and $t_\delta \in \mathcal{H}^0(R)$. So $t_\alpha \in \dot{\mathcal{H}}_i \oplus \mathcal{H}^0(R)$.

(iii) Let $\alpha \in R$. By (2.2), $\alpha = \dot{\alpha} + \delta$ where $\dot{\alpha} \in \dot{R}_i$ for some $1 \leq i \leq k$ and $\delta \in R^0$. Therefore t_α is in the span of $\{t_{\dot{\alpha}}, t_\delta \mid \dot{\alpha} \in \dot{\Pi}_i, \delta \in R^0\}$. It follows that if $h \in \mathcal{H}^0(R) \subseteq \mathcal{H}(R)$, then $h = \dot{h} + h^0$ where $\dot{h} \in \bigoplus \dot{\mathcal{H}}_i$ and $h^0 \in \operatorname{span}_{\mathbb{C}}\{t_\delta \mid \delta \in R^0\}$. But by part (ii), $\dot{h} = 0$ and $h = h^0$. ■

Set

$$\mathcal{H}_i^0 := \operatorname{span}_{\mathbb{C}}\{t_\delta \mid \delta \in R^0, \dot{\alpha} + \delta \in R \text{ for some } \dot{\alpha} \in \dot{R}_i^\times\} \subset \mathcal{H}(R)^0.$$

From (2.3) it follows that $\mathcal{H}_i^0 \subseteq \operatorname{span}_{\mathbb{C}}\{t_\delta \mid \delta \in R_i^0\}$. By Corollary 2.2, R_i is an EARS. So by [AABGP, Corollary II.2.31] if $\delta \in R_i^0$, then $\delta = \delta_1 + \delta_2$, where $t_{\delta_1}, t_{\delta_2} \in \mathcal{H}_i^0$. Thus $t_\delta \in \mathcal{H}_i^0$ and

$$\mathcal{H}_i^0 = \operatorname{span}_{\mathbb{C}}\{t_\delta \mid \delta \in R_i^0\} = \operatorname{span}_{\mathbb{C}}\{t_\delta \mid \delta \in (R'_i)^0\}. \tag{2.6}$$

Define

$$\dot{\mathcal{H}} = \bigoplus_{i=1}^k \dot{\mathcal{H}}_i \text{ and } \mathcal{H}^0 = \sum_{i=1}^k \mathcal{H}_i^0 \subseteq \mathcal{H}(R)^0.$$

Recall from Definition 1.1 that the subalgebra $\mathcal{G}_c$ of $\mathcal{G}$ is the subalgebra of $\mathcal{G}$ generated by root spaces $\mathcal{G}_\alpha$, $\alpha \in R^\times$.

Lemma 2.5 $\dot{\mathcal{H}} \oplus \mathcal{H}^0 = \sum_{\alpha\in R^\times} [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}] = \mathcal{G}_c \cap \mathcal{H}$.

Proof It follows immediately from the definition of $\mathcal{G}_c$ and (1.1) that the second equality holds. Since $\mathcal{G}$ satisfies axioms GR1–GR4 of an EALA, we have from (1.4) that if $\alpha \in R^\times$ then $[\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}] = \mathbb{C}t_\alpha$. So if $\dot{\alpha} \in \dot{\Pi}_i \subset R_i^\times \subset R^\times$, then $t_{\dot{\alpha}} \in [\mathcal{G}_{\dot{\alpha}}, \mathcal{G}_{-\dot{\alpha}}]$. Thus $t_{\dot{\alpha}} \in \sum_{\alpha\in R^\times} [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]$ for all $\dot{\alpha} \in \dot{R}_i$, so $\dot{\mathcal{H}}_i \subseteq \sum_{\alpha\in R^\times} [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]$. Next let $\delta \in R_i^0$. Then $\alpha + \delta \in R$ for some $\alpha \in R_i^\times$. Also by (2.2) and (2.3) $\alpha = \dot{\alpha} + \eta$

for some $\dot{\alpha} \in \dot{R}_i \setminus \{0\}$ and $\eta \in R_i^0$. Now $t_{\dot{\alpha}}, t_\alpha, t_{\alpha+\delta} \in \sum_{\alpha \in R^\times} [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]$. Thus the sum contains t_δ too, and so contains $\dot{\mathcal{H}} \oplus \mathcal{H}^0$.

Conversely, let $\alpha \in R^\times$, then $\alpha \in R_i$ for some i. By (2.2) and (2.3) $\alpha = \dot{\alpha} + \delta$ where $\dot{\alpha} \in \dot{R}_i \setminus \{0\}$ and $\delta \in R_i^0$. Then $t_\alpha = t_{\dot{\alpha}} + t_\delta \in \dot{\mathcal{H}} \oplus \mathcal{H}_i^0$. ■

It can be read from proof of Lemma 2.5 that

Corollary 2.6 $\dot{\mathcal{H}}_i \oplus \mathcal{H}_i^0 = \sum_{\alpha \in R_i^\times} [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]$.

Set

$$R_0^0 = R_{\text{iso}} \Big\backslash \Big(\bigcup_{i=1}^{k} R_i' \Big) = R_{\text{iso}} \Big\backslash \Big(\bigcup_{i=1}^{k} (R_i')_{\text{iso}} \Big). \tag{2.7}$$

Since $(\cdot\,,\,\cdot)$ is nondegenerate on $\mathcal{H}$ and $\dot{\mathcal{H}}$, we have $\mathcal{H} = \dot{\mathcal{H}} \oplus \dot{\mathcal{H}}^\perp$ where $\dot{\mathcal{H}}^\perp$ is the orthogonal complement of $\dot{\mathcal{H}}$ in $\mathcal{H}$. Since $(\dot{\mathcal{H}} \oplus \mathcal{H}^0, \mathcal{H}^0) = \{0\}$, there exists a subspace $\mathcal{D}$ of $\dot{\mathcal{H}}^\perp$ such that

$$\begin{gathered} \dim \mathcal{D} = \dim \mathcal{H}^0, \qquad (\mathcal{D}, \mathcal{D} \oplus \dot{\mathcal{H}}) = \{0\}, \\ (\cdot\,,\,\cdot) \text{ is nondegenerate on } \dot{\mathcal{H}} \oplus \mathcal{H}^0 \oplus \mathcal{D}. \end{gathered} \tag{2.8}$$

Next, let $\mathcal{W}$ be the orthogonal complement of $\dot{\mathcal{H}} \oplus \mathcal{H}^0 \oplus \mathcal{D}$ in $\mathcal{H}$, then we have

$$\begin{gathered} \mathcal{H} = \dot{\mathcal{H}} \oplus \mathcal{H}^0 \oplus \mathcal{D} \oplus \mathcal{W}, \qquad (\dot{\mathcal{H}} \oplus \mathcal{H}^0 \oplus \mathcal{D}, \mathcal{W}) = \{0\}, \\ (\cdot\,,\,\cdot) \text{ is nondegenerate on } \mathcal{W}. \end{gathered} \tag{2.9}$$

Now consider a basis $B = \{h_1, \ldots, h_m\}$ of $\mathcal{H}^0$ such that B contains a basis of $\mathcal{H}_i^0$, for each i. Using (2.8), we may pick a basis $B' = \{d_1, \ldots, d_m\}$ of $\mathcal{D}$ such that $(h_i, d_j) = \delta_{ij}$. Let

$$\mathcal{D}_i = \operatorname{span}_{\mathbb{C}}\{d_j \in B' \mid h_j \in \mathcal{H}_i^0\}.$$

Then $\mathcal{D} = \sum_{i=1}^k \mathcal{D}_i$ and

$$\begin{gathered} \dim \mathcal{D}_i = \dim \mathcal{H}_i^0, \qquad (\mathcal{D}_i, \mathcal{D} \oplus \dot{\mathcal{H}}) = \{0\}, \\ (\cdot\,,\,\cdot) \text{ is nondegenerate on } \dot{\mathcal{H}} \oplus \mathcal{H}_i^0 \oplus \mathcal{D}_i. \end{gathered} \tag{2.10}$$

Note that if $\alpha \in R_i'$, then $\alpha = \dot{\alpha} + \delta$ where $t_{\dot{\alpha}} \in \dot{\mathcal{H}}_i$ and $t_\delta \in \mathcal{H}_i^0$. Since$(\dot{\mathcal{H}}_i \oplus \mathcal{H}_i^0, \dot{\mathcal{H}}_j) = \{0\}$ for $i \neq j$, we have $\alpha(\dot{\mathcal{H}}) = \dot{\alpha}(\dot{\mathcal{H}}) = \dot{\alpha}(\dot{\mathcal{H}}_i)$. Now from (2.8), (2.9), (2.10) and the way the spaces $\mathcal{D}_i$'s are defined, we have $\delta(\mathcal{H}) = \delta(\mathcal{D}) = \delta(\mathcal{D}_i)$. Thus

$$\alpha(\mathcal{H}) = \alpha(\mathcal{H}_i) = \alpha(\dot{\mathcal{H}}_i \oplus \mathcal{D}_i) \quad \text{for } \alpha \in R_i'. \tag{2.11}$$

It follows from this that for $\alpha \in R_i' \setminus \{0\}$,

$$\mathcal{G}_\alpha = \Big\{ x \in \sum_{\beta \in R_i' \setminus \{0\}} \mathcal{G}_\beta \;\Big|\; [h, x] = \alpha(h)x \text{ for all } h \in \mathcal{H}^i \Big\}. \tag{2.12}$$

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In fact, it is clear that $\mathcal{G}_\alpha$ is a subset of the right hand side. To see the reverse inclusion let $x = x_{\beta_1} + \cdots + x_{\beta_t}$ be an element of the right hand side, where $\beta'_j \in R'_i \setminus \{0\}$ for all j, $x_\beta \in \mathcal{G}_\beta$. Let $h = \dot{h} + h^0 + d + h^0_0 + w^0_0 + w_0$ be an arbitrary element of $\mathcal{H}$ in the form (2.9), where $\dot{h} = \sum_{j=1}^k \dot{h}_j$, $\dot{h}_j \in \dot{\mathcal{H}}_j$. With respect to the basis B' of $\mathcal{D}$ we may write $d = d' + d''$ where $d' \in \mathcal{D}_i$ and $d'' \in \operatorname{span}_{\mathbb{C}}\{d_j \mid d_j \notin \mathcal{D}_i\}$. Now using (2.11) we have

$$[h, x] = \sum_{j=1}^t \beta_j(h) x_{\beta_j} = \sum_{j=1}^t \beta_j(\dot{h}_i + d') x_{\beta_j} = [\dot{h}_i + d', x] = \alpha(\dot{h}_i + d')x = \alpha(h)x.$$

So $x \in \mathcal{G}_\alpha$.

Starting from each R_i, we now would like to construct a generalized reductive subalgebra $\mathcal{G}^i$ of $\mathcal{G}$ which is indecomposable. For this set

$$\mathcal{H}_i = \dot{\mathcal{H}}_i \oplus \mathcal{H}^0_i \oplus \mathcal{D}_i \text{ and } \mathcal{G}^i = \mathcal{H}_i \oplus \sum_{\alpha \in R'_i \setminus \{0\}} \mathcal{G}_\alpha \tag{2.13}$$

Proposition 2.7

(i) $\big(\mathcal{G}^i, (\cdot\,,\cdot), \mathcal{H}_i\big)$ *is an indecomposable generalized reductive subalgebra of* $\mathcal{G}$.
(ii) $\mathcal{G}^i_c \cap \mathcal{H}_i = \dot{\mathcal{H}}_i \oplus \mathcal{H}^0_i$.

Proof First we must show that $\mathcal{G}^i$ is a subalgebra of $\mathcal{G}$. Note that R'_i, R_i and $R_i^\times$ have the same linear span. It then follows from (1.4) and Corollary 2.6 that

$$[\mathcal{G}^i, \mathcal{G}^i] \cap \mathcal{H} = \sum_{\alpha \in R'_i} [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}] = \sum_{\alpha \in R_i^\times} [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}] = \dot{\mathcal{H}}_i \oplus \mathcal{H}^0_i \subset \mathcal{G}^i.$$

Since $\mathcal{G}_0 = \mathcal{H}$ acts diagonally on $\mathcal{G}$, we have $[\mathcal{H}, \mathcal{G}^i] \subset \mathcal{G}^i$. Thus it only remains to show that if $\alpha, \beta \in R'_i \setminus \{0\}$ and $\alpha + \beta \in R \setminus \{0\}$, then $\alpha + \beta \in R'_i$. If $\alpha + \beta$ is isotropic, it is clear from definition of R'_i that $\alpha + \beta \in R'_i$. If $\alpha + \beta$ is non-isotropic, then it can not be orthogonal to both α and β and so $\alpha + \beta \in R_i \subset R'_i$, since R_i is indecomposable. Thus $\mathcal{G}^i$ is a subalgebra of $\mathcal{G}$. Moreover, from (2.11) and (2.12) we have $\mathcal{H}_i = C_{\mathcal{G}^i}(\mathcal{H}_i)$ and

$$\mathcal{G}_\alpha = \{x \in \mathcal{G}^i \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathcal{H}^i\} \quad (\alpha \in R'_i \setminus \{0\}).$$

Thus

$$\mathcal{G}^i = \sum_{\alpha \in R'_i} (\mathcal{G}^i)_\alpha, \quad \text{where } (\mathcal{G}^i)_0 = \mathcal{H}_i \text{ and } (\mathcal{G}^i)_\alpha = \mathcal{G}_\alpha \text{ for } \alpha \neq 0. \tag{2.14}$$

Next we must show that GR1–GR5 and GR6a hold for $\mathcal{G}^i$. By (2.10), (1.2) and Corollary 2.2, GR1 holds for $\mathcal{G}^i$. Considering R'_i as a subset of $\mathcal{H}_i^\star$, we see from (2.14) that elements from $\mathcal{H}_i$ act diagonally on $\mathcal{G}^i$ via the adjoint representation. So GR2 holds as $C_{\mathcal{G}^i}(\mathcal{H}_i) = \mathcal{H}_i$. Validity of GR3 and GR4 for $\mathcal{G}^i$ follows from the fact that these axioms hold for $\mathcal{G}$ and that $R_i' \subset R$, and it is clear that GR5 holds. Finally, GR6a holds, since R_i' is indecomposable. Part (ii) follows from Corollary 2.6. ■

Put

$$\mathcal{I} = \sum_{\delta \in R_0^0} \mathcal{G}_\delta. \tag{2.15}$$

From (1.1), (2.4), (2.9) and (2.15) we have

$$\mathcal{G} = \sum_{i=1}^{k} \mathcal{G}^i \oplus \mathcal{W} \oplus \mathcal{I}. \tag{2.16}$$

Here $\mathcal{W}$ and each $\mathcal{G}^i$'s are Lie subalgebras of $\mathcal{G}$ and $\mathcal{I}$ is a subspace of $\mathcal{G}$. The direct sums appearing in (2.16) are just sums of vector spaces. We now would like to investigate the Lie bracket between these spaces, at the level of cores.

Lemma 2.8

(i) *$\mathcal{G}_c = \sum_{i=1}^k \mathcal{G}_c^i$ where $[\mathcal{G}_c^i, \mathcal{G}_c^j] = \{0\}$ for $i \neq j$. In particular, $\mathcal{G}_c^i$ is an ideal of $\mathcal{G}$, for each i.*

(ii) *If $x = \sum_{i=1}^k x_i \in \mathcal{Z}(\mathcal{G}_c)$ where $x_i \in \mathcal{G}_c^i$, then $x_i \in \mathcal{Z}(\mathcal{G}_c^i)$ for each i. In particular, $\mathcal{Z}(\mathcal{G}_c) = \sum_{i=1}^k \mathcal{Z}(\mathcal{G}_c^i)$ and as Lie algebras*

$$\frac{\mathcal{G}_c}{\mathcal{Z}(\mathcal{G}_c)} \cong \bigoplus_{i=1}^{k} \frac{\mathcal{G}_c^i}{\mathcal{Z}(\mathcal{G}_c^i)}.$$

Proof (i) Let $i \neq j$, $\alpha \in R_i^\times$ and $\beta \in R_j^\times$. Then $(\alpha, \beta) = 0$ and so $\alpha + \beta$ is orthogonal to neither of α and β. Thus $\alpha + \beta$ is not a root of $\mathcal{G}$. This shows that $[\mathcal{G}_c^i, \mathcal{G}_c^j] = \{0\}$ for $i \neq j$, and that $\mathcal{G}_c^i$ is an ideal of $\mathcal{G}_c$. Since $\mathcal{G}_c^i$ is perfect, it follows from the Jacobi identity that $\mathcal{G}_c^i$ is an ideal of $\mathcal{G}$. In particular, $\sum_{i=1}^k \mathcal{G}_c^i$ is a subalgebra of $\mathcal{G}$ containing all non-isotropic root spaces. Clearly any subalgebra of $\mathcal{G}$ containing all non-isotropic root spaces must contain this sum. Thus $\mathcal{G}_c = \sum \mathcal{G}_c^i$.

(ii) Let x be as in the statement. Then by part (i) for each i, we have

$$\{0\} = [x, \mathcal{G}_c^i] = \sum_{j=1}^{k} [x_j, \mathcal{G}_c^i] = [x_i, \mathcal{G}_c^i].$$

So $x_i \in \mathcal{Z}(\mathcal{G}_c^i)$. It now follows that $\bigoplus_{i=1}^k \mathcal{Z}(\mathcal{G}_c^i)$ is the kernel of the epimorphism

$$\bigoplus_{i=1}^{k} \mathcal{G}_c^i \to \frac{\mathcal{G}_c}{\mathcal{Z}(\mathcal{G}_c)}, \qquad (x_i) \mapsto \sum x_i + \mathcal{Z}(\mathcal{G}_c).$$ ■

Lemma 2.9

(i) $\mathcal{W} \subseteq C_{\mathcal{G}}(\sum_{i=1}^{k} \mathcal{G}^i)$.

(ii) $\sum_{\delta \in R_{\rm iso}} \mathcal{G}_\delta \subseteq C_{\mathcal{G}}(\mathcal{G}_c)$. *In particular,* $\mathcal{W} \oplus \mathcal{I} \subseteq C_{\mathcal{G}}(\mathcal{G}_c)$.

Proof (i) Let $\alpha \in R_i'$. We have $\langle R_i' \rangle = \langle R_i^\times \rangle$. It follows from this that $t_\alpha \in \mathcal{H}_i$. Then from (2.9) we have

$$[\mathcal{W}, \mathcal{G}_\alpha] = \alpha(\mathcal{W})\mathcal{G}_\alpha = (t_\alpha, \mathcal{W})\mathcal{G}_\alpha \subseteq (\mathcal{H}_i, \mathcal{W})\mathcal{G}_\alpha = \{0\}.$$

(ii) The first part of the statement follows from Proposition 1.4(ii). The second part of the statement holds now by part (i). ■

Let us summarize the results obtained in the following theorem.

Theorem 2.10 *Let* $\big(\mathcal{G}, (\cdot\,,\cdot), \mathcal{H}\big)$ *be a* GRLA *with corresponding root system* R. *Then*

(i) $R = (\bigcup_{i=1}^{k} R_i) \cup R_{\rm iso}$ *where for each* i, R_i *is an* EARS. *Moreover* $R_i' = R_i \cup (\langle R_i \rangle \cap R^0)$ *is an indecomposable* GRRS.

(ii) *For* $1 \leq i \leq k$, *there exists a subspace* $\mathcal{D}_i$ *of* $\mathcal{H}$ *such that if* $\mathcal{H}_i := \mathcal{D}_i \oplus \sum_{\alpha \in R_i^\times} [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]$ *and* $\mathcal{G}^i := \mathcal{H}_i \oplus \sum_{\alpha \in R_i'} \mathcal{G}_\alpha$, *then* $\big(\mathcal{G}^i, (\cdot\,,\cdot), \mathcal{H}_i\big)$ *is an indecomposable generalized reductive subalgebra of* $\mathcal{G}$. *Moreover,*

$$\dim \mathcal{D}_i = \dim\Big(\sum_{\alpha \in R_i^\times} [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]\Big) - \operatorname{rank}(R_i).$$

(iii) $\mathcal{H}$ *has a decomposition as in* (2.9). *Moreover* $\mathcal{G} = \sum_{i=1}^{k} \mathcal{G}^i \oplus \mathcal{W} \oplus \mathcal{I}$, *where* $\mathcal{W}$ *and* $\mathcal{I}$ *are subspaces of* $\mathcal{G}$ *defined by* (2.15). *Moreover,* $[(\sum_{i=1}^{k} \mathcal{G}^i) \oplus \mathcal{W}, \mathcal{W}] = \{0\}$ *and* $[\mathcal{I}, \mathcal{G}_c] = \{0\}$.

(iv) *If* $i \neq j$, *then* $[\mathcal{G}_c^i, \mathcal{G}_c^j] = \{0\}$ *and* $\mathcal{G}_c = \sum_{i=1}^{k} \mathcal{G}_c^i$. *In particular,* $\mathcal{G}_c^i$ *is an ideal of* $\mathcal{G}$. *Moreover,* $\mathcal{Z}(\mathcal{G}_c) = \sum_{i=1}^{k} \mathcal{Z}(\mathcal{G}_c^i)$ *and*

$$\frac{\mathcal{G}_c}{\mathcal{Z}(\mathcal{G}_c)} \cong \bigoplus_{i=1}^{k} \frac{\mathcal{G}_c^i}{\mathcal{Z}(\mathcal{G}_c^i)}.$$

(v) *If* π *is the projection map* $\mathcal{V} \to \mathcal{V}^0$ *(with respect to the decomposition* (2.1)*), then for* $i \neq j$,

$$[\mathcal{G}^i, \mathcal{G}^j] \subseteq (\mathcal{H}_i^0 \cap \mathcal{H}_j^0) \oplus \sum_{\alpha \in R_i' \cup R_j', \pi(\alpha) \neq 0} \mathcal{G}_\alpha + \sum_{\alpha', \beta'} [\mathcal{G}_{\alpha'}, \mathcal{G}_{\beta'}],$$

where $(\alpha', \beta') \in \big(R_i^\times \times (R_j')^0 \setminus \{0\}\big) \cup \big(R_j^\times \times (R_i')^0 \setminus \{0\}\big)$. *In particular,* $[\mathcal{G}^i, \mathcal{G}^j] = \{0\}$ *if* $\mathcal{V}^0 = \{0\}$.

(vi) *If* $\mathcal{G}$ *is tame, then* $\mathcal{W} = \{0\}$ *and* $\mathcal{I} = \{0\}$.

Proof (i) See (2.4) and Corollary 2.2. (ii) See Corollary 2.6, (2.10), (2.13) and Proposition 2.7. (iii) See (2.16) and Lemma 2.9. (iv) See Lemma 2.8.

(v) We must check $[(\mathcal{G}^i)_\alpha, (\mathcal{G}^j)_\beta]$ for $\alpha \in R'_i, \beta \in R'_j$. First let $\alpha = 0$ and $\beta \in R'_j \setminus \{0\}$. Then $(\mathcal{G}^i)_\alpha = \mathcal{H}_i$ and $(\mathcal{G}^j)_\beta = \mathcal{G}_\beta$. Since $t_{\beta-\pi(\beta)} \in \dot{\mathcal{H}}_j$, we have from (2.8) that

$$[\mathcal{H}_i, \mathcal{G}_{\beta-\pi(\beta)}] = (t_{\beta-\pi(\beta)}, \mathcal{H}_i) \subset (\dot{\mathcal{H}}_j, \mathcal{H}_i) = \{0\}.$$

Thus

$$[\mathcal{H}_i, \sum_{\beta \in R'_j \setminus \{0\}} \mathcal{G}_\beta] \subseteq \sum_{\beta \in R'_j, \pi(\beta) \neq 0} \mathcal{G}_\beta.$$

Next let $\alpha \in R'_i \setminus \{0\}, \beta \in R'_j \setminus \{0\}$. If $-\beta = \alpha = \delta \in (R'_i)^0 \cap (R'_j)^0$, then by (1.4) and (2.6), we have $[(\mathcal{G}^i)_\alpha, (\mathcal{G}^j)_\beta] = [\mathcal{G}_\delta, \mathcal{G}_{-\delta}] = \mathbb{C}t_\delta \subseteq \mathcal{H}_i^0 \cap \mathcal{H}_j^0$. It then follows from (2.8)(i) that for $0 \neq \alpha \in R'_i$ and $0 \neq \beta \in R'_j$,

$$[\mathcal{G}_\alpha, \mathcal{G}_\beta] \subseteq \mathcal{H}_i^0 \cap \mathcal{H}_j^0 \oplus \sum_{\alpha', \beta'} [\mathcal{G}_{\alpha'}, \mathcal{G}_{\beta'}],$$

where α', β' are as in the statement.

(vi) If $\mathcal{G}$ is tame then $C_{\mathcal{G}}(\mathcal{G}_c) \subseteq \mathcal{G}_c$ and by [ABP, Lemma 3.62], $R_{\text{iso}} = \varnothing$ (and so $\mathcal{I} = \{0\}$). By part (iii), $\mathcal{W} \subseteq C_{\mathcal{G}}(\mathcal{G}_c) \cap \mathcal{H} \subseteq \mathcal{G}_c \cap \mathcal{H}$. But by Lemma 2.5, $\mathcal{G}_c \cap \mathcal{H} = \dot{\mathcal{H}} \oplus \mathcal{H}^0$. So from (2.9) we get $\mathcal{W} = \{0\}$. ■

Corollary 2.11 *Let $\big(\mathcal{G}, (\cdot, \cdot), \mathcal{H}\big)$ be a non-singular* GRLA *of nullity ν with root system R. Then $\mathcal{G} = \sum_{i=1}^k \mathcal{G}^i \oplus \mathcal{W}$, where $\mathcal{G}^i$'s are indecomposable* GRLA*s with $[\mathcal{G}_c^i, \mathcal{G}_c^j] = \{0\}$, for $i \neq j$, and $\mathcal{W}$ is contained in the centralizer of $\mathcal{G}$. Moreover,*

(i) *if $\nu \leq 2$, then each $\mathcal{G}^i$ is an* EALA*;*

(ii) *if $\nu = 0$, then $\mathcal{G}$ is a finite dimensional reductive Lie algebra.*

Proof By assumption $R_{\text{iso}} = \varnothing$ and so $\mathcal{I} = \{0\}$. If $\nu \leq 2$, we have from Remark 2.3(i) that $R'_i = R_i$. That is $(R'_i)_{\text{iso}} = \varnothing$. Thus $\mathcal{G}^i$ satisfies GR6b.

If $\nu = 0$, then from Theorem 2.10(v), we have that the sum in the statement is direct, that is $\mathcal{G} = (\bigoplus_{i=1}^k \mathcal{G}^i) \oplus \mathcal{W}$. Moreover, $R_i = \dot{R}_i$ is an irreducible finite root system and

$$\mathcal{G}^i = \bigoplus_{\alpha \in \dot{R}_i} (\mathcal{G}^i)_\alpha \text{ where } (\mathcal{G}^i)_0 = \mathcal{H}^i \text{ and } (\mathcal{G}^i)_\alpha = \mathcal{G}_\alpha \quad \text{for } \alpha \neq 0.$$

Since $\mathcal{H}_i^0 = \{0\} = \mathcal{D}_i$, we have $\dim \mathcal{H}_i = \dim \dot{\mathcal{H}}_i = \operatorname{rank} \dot{R}_i$. Now it follows from Theorem 1.3 and Serre's Theorem that each $\mathcal{G}^i$ is a finite dimensional simple Lie algebra over the field of complex numbers. That is $\mathcal{G}$ is a reductive Lie algebra. ■

It is worth mentioning that the basic structural properties of an EALA essentially come from its core modulo its center (see [AG, Proposition 1.28], [A2, Proposition 1.28] and [N2, Theorem 6]). Therefore Theorem 2.10(iv) together with Corollary 2.11(i) suggest that the structural properties of a generalized reductive Lie algebra $\mathcal{G}$ can be obtained from the indecomposable subalgebras $\mathcal{G}^i$.

3 On the Classification of GRLAs

In this section, we show that the core modulo center of an indecomposable GRLA is a centerless Lie torus. Therefore by Theorem 2.10(ii), (iv) the core modulo center of a GRLA is a direct sum of centerless Lie tori. For the classification of centerless Lie tori of types A_1 and A_2 see [Y] and [BGKN] respectively. For simply laced types of rank ≥ 3 see [BGK]. For types B_ℓ, C_ℓ, F_4 and G_2 see [AG]. For type BC_ℓ ($\ell \geq 3$) see [ABG]. Finally, for type BC_1 see [AFY]. The classification for type BC_2 is open.

Let us recall the definition of a Lie torus over $\mathbb{C}$, introduced in [N1]. Let $\mathcal{L}$ be a complex Lie algebra, $\dot{R}$ be an irreducible finite root system and Λ be a free abelian group of finite rank. Denote the set of indivisible roots of $\dot{R}$ by $\dot{R}_{\text{ind}}$, that is $\dot{R}_{\text{ind}} = \{\dot\alpha \in \dot{R} \mid \frac{1}{2}\dot\alpha \notin \dot{R}\}$. Then, the Lie algebra $\mathcal{L}$ is called a *Lie torus of type* $(\dot{R}, \Lambda)$ if it satisfies the following axioms:

LT1 $\mathcal{L}$ has a $(\langle \dot{R}\rangle \oplus \Lambda)$-grading of the form

$$\mathcal{L} = \bigoplus_{\dot\alpha \in \langle \dot{R}\rangle, \lambda \in \Lambda} \mathcal{L}^\lambda_{\dot\alpha},\ [\mathcal{L}^\lambda_{\dot\alpha}, \mathcal{L}^\mu_{\dot\beta}] \subseteq \mathcal{L}^{\lambda+\mu}_{\dot\alpha+\dot\beta},\ \text{satisfying } \mathcal{L}^\lambda_{\dot\alpha} = \{0\} \quad \text{if } \dot\alpha \notin \dot{R}.$$

LT2 For $\dot\alpha \in \dot{R}^\times := \dot{R} \setminus \{0\}$ and $\lambda \in \Lambda$, we have

- (i) $\dim \mathcal{L}^\lambda_{\dot\alpha} \leq 1$, with $\dim \mathcal{L}^0_{\dot\alpha} = 1$ if $\dot\alpha \in \dot{R}_{\text{ind}}$,
- (ii) if $\dim L^\lambda_{\dot\alpha} = 1$, then there exists $(e^\lambda_{\dot\alpha}, f^\lambda_{\dot\alpha}) \in \mathcal{L}^\lambda_{\dot\alpha} \times \mathcal{L}^{-\lambda}_{-\dot\alpha}$ such that $h^\lambda_{\dot\alpha} := [e^\lambda_{\dot\alpha}, f^\lambda_{\dot\alpha}] \in \mathcal{L}^0_0$ acts on $x \in \mathcal{L}^\mu_{\dot\beta}$ ($\dot\beta \in \dot{R}, \mu \in \Lambda$) by

$$[h^\lambda_{\dot\alpha}, x] = (\dot\beta, \dot\alpha^\vee)x,$$

where $(\dot\beta, \dot\alpha^\vee)$ is the Cartan integer of $\dot\beta$, $\dot\alpha$.

LT3 For $\lambda \in \Lambda$ we have $\mathcal{L}^\lambda_0 = \sum_{\dot\alpha \in \dot{R}^\times, \mu \in \Lambda} [\mathcal{L}^\mu_{\dot\alpha}, \mathcal{L}^{\lambda-\mu}_{-\dot\alpha}]$.

LT4 $\Lambda = \big\langle \big\{ \lambda \in \Lambda \mid \mathcal{L}^\lambda_{\dot\alpha} \neq \{0\} \text{ for some } \dot\alpha \in \dot{R} \big\} \big\rangle$.

We start with an indecomposable GRLA $\big(\mathcal{G}, (\cdot,\cdot), \mathcal{H}\big)$, that is $\mathcal{G}$ satisfies axioms GR1–GR5 and GR6a. Let R be the root system of $\mathcal{G}$. According to Section 1, we have

$$R = R_t \cup R_{\text{iso}},\ \text{where } R_t = R^\times \cup R^0_t,$$

with

$$R^0_t = \{\delta \in R^0 \mid \delta + \alpha \in R \text{ for some } \alpha \in R^\times\}.$$

Moreover, R_t is an EARS (an irreducible GRRS) and $\Lambda_t = \langle R^0_t \rangle$ is a lattice. So there exists an irreducible finite root system $\dot{R}$ with

$$R_t \subseteq \dot{R} + \Lambda_t \text{ and } \dot{R}_{\text{ind}} \subseteq R^\times.$$

(See [AABGP, Proposition II.2.11].) Thus

$$R \subseteq \dot{R} + \Lambda_t + \Lambda_0,\ \text{where } \Lambda_0 = \langle R_{\text{iso}} \rangle.$$

Set

$$\Lambda = \langle R^0 \rangle = \Lambda_t + \Lambda_0 = \langle R^0_t + R_{\text{iso}} \rangle.$$

So $\langle R \rangle = \langle \dot{R} \rangle \oplus \Lambda$.

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Theorem 3.1 *Let* $\big(\mathcal{G},(\cdot\,,\cdot),\mathcal{H}\big)$ *be an indecomposable* GRLA. *Then the Lie algebra* $\mathcal{G}_c/\mathcal{Z}(\mathcal{G}_c)$ *is a Lie torus of type* $(\dot{R},\Lambda_t)$.

Proof We start by proving that axioms LT1 and LT2 of a Lie torus hold for $\mathcal{G}$ with respect to a grading based on the abelian group $\langle\dot{R}\rangle\oplus\Lambda$. From Section 1, we know that

$$\mathcal{G}=\bigoplus_{\alpha\in R}\mathcal{G}_\alpha=\bigoplus_{\dot\alpha\in\langle\dot{R}\rangle,\lambda\in\Lambda}\mathcal{G}_{\dot\alpha+\lambda}$$

is a $(\langle\dot{R}\rangle\oplus\Lambda)$-grading for $\mathcal{G}$ with $\mathcal{G}_{\dot\alpha+\lambda}=\{0\}$ if $\dot\alpha\notin\dot{R}$. Thus by considering $\mathcal{L}^\lambda_{\dot\alpha}:=\mathcal{G}_{\dot\alpha+\lambda}$, $(\dot\alpha\in\langle\dot{R}\rangle,\lambda\in\Lambda)$, we see that axiom LT1 of a Lie torus holds for $\mathcal{G}$ with respect to this grading. Next let $\dot\alpha\in\dot{R}^\times$ and $\lambda\in\Lambda$. If $\dot\alpha+\lambda\in R$, then $\dot\alpha+\lambda\in R^\times$ and so by Theorem 1.3(d), $\dim\mathcal{G}_{\dot\alpha+\lambda}=1$. If $\dot\alpha+\lambda\notin R$, then $\dim\mathcal{G}_{\dot\alpha+\lambda}=0$. Moreover, $\dot{R}_{\mathrm{ind}}\subseteq R$ and so $\dim\mathcal{G}_{\dot\alpha}=1$ for $\dot\alpha\in\dot{R}_{\mathrm{ind}}\setminus\{0\}$. Thus part (i) of the axiom LT2 holds for $\mathcal{G}$.

Next note that if $\dot\alpha\in\dot{R}^\times$, $\lambda\in\Lambda$ and $\dim\mathcal{G}_{\dot\alpha+\lambda}=1$, then $\alpha:=\dot\alpha+\lambda\in R^\times$. So if $e_\alpha,f_\alpha,h_\alpha$ are as in (1.6), then for any $x\in\mathcal{G}_{\dot\beta+\mu}$ $(\dot\alpha\in\dot{R},\mu\in\Lambda)$

$$[h_\alpha,x]=(\dot\beta+\mu)(h_\alpha)x=\Big(t_{\dot\beta+\mu},\frac{2t_\alpha}{(\alpha,\alpha)}\Big)x=\frac{2(\dot\beta,\dot\alpha)}{(\dot\alpha,\dot\alpha)}x=(\dot\beta,\dot\alpha^\vee)x.$$

Thus part (ii) of LT2 also holds for $\mathcal{G}$.

Recall that the core $\mathcal{G}_c$ of $\mathcal{G}$ is the ideal of $\mathcal{G}$ generated by root spaces $\mathcal{G}_\alpha$, $\alpha\in R^\times=R^\times_t$. So $\mathcal{G}_c$ inherits from $\mathcal{G}$ a $(\dot{R}\oplus\Lambda)$-grading, namely

$$\mathcal{G}_c=\bigoplus_{\alpha\in\langle\dot{R}\rangle\oplus\Lambda}(\mathcal{G}_c)_\alpha,\ \text{where } (\mathcal{G}_c)_\alpha=\mathcal{G}_c\cap\mathcal{G}_\alpha.$$

Moreover, from the way $\mathcal{G}_c$ is defined, we have

$$(3.1)\qquad (\mathcal{G}_c)_\delta=\sum_{\alpha\in R^\times}[\mathcal{G}_{\alpha+\delta},\mathcal{G}_{-\alpha}]=\sum_{\dot\alpha\in\dot{R}^\times}\sum_{\lambda\in\Lambda_t}[\mathcal{G}_{\dot\alpha+\lambda+\delta},\mathcal{G}_{-\dot\alpha-\lambda}]\quad(\delta\in R^0).$$

Next let $\widetilde{\mathcal{G}}=\mathcal{G}_c/\mathcal{Z}(\mathcal{G}_c)$. Then

$$\widetilde{\mathcal{G}}=\bigoplus_{\alpha\in\langle\dot{R}\rangle\oplus\Lambda}\widetilde{\mathcal{G}}_\alpha \text{ where } \widetilde{\mathcal{G}}_\alpha=\frac{(\mathcal{G}_c)_\alpha+\mathcal{Z}(\mathcal{G}_c)}{\mathcal{Z}(\mathcal{G}_c)},$$

is a $\langle\dot{R}\rangle\oplus\Lambda$-grading for $\widetilde{\mathcal{G}}$. Note that if $\alpha\in R_{\mathrm{iso}}$ then by Proposition 1.4(ii), $\widetilde{\mathcal{G}}_\alpha\subseteq\mathcal{Z}(\mathcal{G}_c)$ and so $\widetilde{\mathcal{G}}_\alpha=\{0\}$. Therefore we may assume that $\alpha\in R_t$. Thus

$$(3.2)\qquad \widetilde{\mathcal{G}}=\bigoplus_{\alpha\in\langle\dot{R}\rangle\oplus\Lambda_t}\widetilde{\mathcal{G}}_\alpha.$$

That is $\widetilde{\mathcal{G}}$ has a $\langle\dot{R}\rangle\oplus\Lambda_t$-grading. Clearly, we have $\widetilde{\mathcal{G}}_{\dot\alpha+\lambda}=\{0\}$ if $\dot\alpha\notin\dot{R}$. Thus LT1 holds for $\widetilde{\mathcal{G}}$ with respect to the $(\langle\dot{R}\rangle\oplus\Lambda_t)$-grading (consider $\mathcal{L}^\lambda_{\dot\alpha}=\widetilde{\mathcal{G}}_{\dot\alpha+\lambda}$, $\dot\alpha\in\langle\dot{R}\rangle,\lambda\in\Lambda_t$).

For $\dot\alpha \in \dot R^\times$ and $\lambda \in \Lambda_t$, $\dim \widetilde{\mathcal{G}}_{\dot\alpha+\lambda} \leq 1$, as $\dim \mathcal{G}_{\dot\alpha+\lambda} \leq 1$. Moreover, by Theorem 1.3(d) and Proposition 1.4, $\mathcal{G}_{\dot\alpha} \cap \mathcal{Z}(\mathcal{G}_c) = \{0\}$ and so

$$\dim \widetilde{\mathcal{G}}_{\dot\alpha} = 1 \quad \text{for } \dot\alpha \in \dot R_{\mathrm{ind}} \subseteq R^\times. \tag{3.3}$$

Thus part (i) of LT2 holds for $\widetilde{\mathcal{G}}$. Now as part (ii) of LT2 holds for $\mathcal{G}$, one can see that it also holds for $\widetilde{\mathcal{G}}$ by considering

$$\tilde e_\alpha = e_\alpha + \mathcal{Z}(\mathcal{G}_c),\ \tilde f_\alpha = f_\alpha + \mathcal{Z}(\mathcal{G}_c) \text{ and } \tilde h_\alpha = h_\alpha + \mathcal{Z}(\mathcal{G}_c).$$

From (3.1) we see that LT3 holds for $\widetilde{\mathcal{G}}$.

Finally, we show that LT4 holds for $\widetilde{\mathcal{G}}$. So let $\delta \in R_t^0$. Then $\mathcal{G}_\delta \neq \{0\}$ and δ is not isolated. Thus by Proposition 1.4, $\mathcal{G}_\delta \not\subseteq \mathcal{Z}(\mathcal{G}_c)$. So we have $\widetilde{\mathcal{G}}_\delta \neq \{0\}$ and

$$\delta \in \langle \delta \in \Lambda_t \mid \widetilde{\mathcal{G}}_{\dot\alpha+\lambda} \neq \{0\} \text{ for some } \delta \in \dot R\rangle.$$

This shows that LT4 holds for $\widetilde{\mathcal{G}}$ and completes the proof. ■

Corollary 3.2 *The core modulo center of a* GRLA *is a direct sum of centerless Lie tori.*

We remark here that if $\mathcal{G}$ is an EALA, then Theorem 3.1 is a consequence of [AG, Proposition 1.28]. Also an statement similar to Theorem 3.1 is announced in [N2, Proposition 3] for a class of Lie algebras which includes the class of EALAs, however GRLAs do not necessarily satisfy GR6(b), while the Lie algebras appearing in [N2] are tame by definition and so satisfy GR6b. In [N2] a procedure is introduced for the construction of an EALA starting from a centerless Lie torus. In fact it is shown that all EALAs arise this way. It is therefore natural to ask if one can introduce a similar procedure for constructing a GRLA starting from a direct sum of centerless Lie tori.

4 Construction of New GRLAs From Old

It is known that affine Lie algebras can be realized by a process known as affinization-and-twisting [K]. It is also known that affine Lie algebras can be realized as the fixed points of some others under a finite order automorphism. This phenomenon has recently been investigated for the class of EALAs (see [ABP], [ABY] and [A2]). In this section we consider a similar method for constructing new GRLAs from old ones.

Let $\big(\mathcal{G}, (\cdot\,,\cdot), \mathcal{H}\big)$ be a GRLA with root system R. Let σ be an automorphism of $\mathcal{G}$ and set

$$\mathcal{G}^\sigma = \{x \in \mathcal{G} \mid \sigma(x) = x\} \text{ and } \mathcal{H}^\sigma = \{h \in \mathcal{H} \mid \sigma(h) = h\},$$

that is $\mathcal{G}^\sigma$ and $\mathcal{H}^\sigma$ are fixed points of $\mathcal{G}$ and $\mathcal{H}$ under σ, respectively. Assume that $\mathcal{G}$ satisfies

(4.1)
- σ is of finite order,
- $(\sigma x, \sigma y) = (x, y)$ for all $x, y \in \mathcal{G}$,
- $\sigma(\mathcal{H}) = \mathcal{H}$,
- The centralizer of $\mathcal{H}^\sigma$ in $\mathcal{G}^\sigma$ equals $\mathcal{H}^\sigma$.

Theorem 4.1 *Let* $\big(\mathcal{G}, (\cdot\,,\,\cdot), \mathcal{H}\big)$ *be a* GRLA *and* σ *be an automorphism of* $\mathcal{G}$ *satisfying* (4.1). *Then* $(\mathcal{G}^\sigma, (\cdot\,,\,\cdot), \mathcal{H}^\sigma)$ *satisfies* GR1–GR4, *where* $(\cdot\,,\,\cdot)$ *is the form on* $\mathcal{G}$ *restricted to* $\mathcal{G}^\sigma$. *In particular,* $\mathcal{G}^\sigma$ *is a* GRLA *if its root system contains some nonisotropic roots.*

Proof It is shown in [ABY] that if $\big(\mathcal{G}, (\cdot\,,\,\cdot), \mathcal{H}\big)$ is an EALA then $\mathcal{G}^\sigma$ satisfies GR1–GR4. Now checking the proof of [ABY, Theorem 2.63], one can see that the irreducibility of $\mathcal{G}$ (or its root system R) is not used at all to prove that $\mathcal{G}^\sigma$ satisfies GR1–GR4. ■

Next we consider the so called *affinization* of a Lie algebra $\mathcal{G}$ introduced in [ABP]. Let $\mathcal{G}$ be a complex Lie algebra and let c and d be two symbols. Consider the vector space

$$\mathrm{Aff}(\mathcal{G}) := (\mathcal{G} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where $\mathbb{C}[t, t^{-1}]$ is the algebra of Laurent polynomials in variable t. Then $\mathrm{Aff}(\mathcal{G})$ becomes a Lie algebra by assuming that c is central, $d = t\frac{d}{dt}$ is the degree derivation (so that $[d, x \otimes t^n] = nx \otimes t^n$), and

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n(x, y)\delta_{m+n,0}c.$$

$\mathrm{Aff}(\mathcal{G})$ is called the *affinization* of $\mathcal{G}$.

If $\mathcal{G}$ is equipped with a invariant symmetric bilinear form, then one can define an invariant symmetric bilinear form $(\cdot\,,\,\cdot)_{\mathrm{Aff}}$ on $\mathrm{Aff}(\mathcal{G})$ by

$$(\alpha x \otimes t^n + \beta c + \gamma d, \alpha' y \otimes t^m + \beta' c + \gamma' d)_{\mathrm{Aff}} = \alpha\alpha'\delta_{n,-m}(x, y) + \beta\gamma' + \beta'\gamma.$$

Moreover, this form is nondegenerate if the form on $\mathcal{G}$ is nondegenerate.

Theorem 4.2 *Let* $\mathcal{G}, (\cdot\,,\,\cdot), \mathcal{H})$ *be a* GRLA *with root system* R *and let* σ *be an automorphism of* $\mathcal{G}$ *satisfying* (4.1). *Then*

$$\big(\mathrm{Aff}(\mathcal{G}), (\cdot\,,\,\cdot)_{\mathrm{Aff}}, \mathcal{H} \oplus \mathbb{C}c \oplus \mathbb{C}d\big)$$

is a GRLA *with root system* $\tilde{R} = R + \mathbb{Z}\delta$ *where* $\delta \in \mathcal{H}t^\star$ *is defined by* $\delta(d) = 1$ *and* $\delta(\mathcal{H} + \mathbb{C}c) = 0$. *Moreover,* $\mathrm{Aff}(\mathcal{G})$ *is tame if and only if* $\mathcal{G}$ *is tame. Finally if we extend* σ *to an automorphism of* $\mathrm{Aff}(\mathcal{G})$ *by*

$$\sigma(x \otimes t^i + rc + sd) = \zeta^{-i}\sigma(x) \otimes t^i + rc + sd,$$

then σ *satisfies* (4.1) *and* $\mathrm{Aff}(\mathcal{G})^\sigma$ *satisfies* GR1–GR4. *In particular,* $\mathrm{Aff}(\mathcal{G})^\sigma$ *is a* GRLA *if its root system contains some nonisotropic roots.*

Proof It can be checked easily that $\mathrm{Aff}(\mathcal{G})$ is a GRLA with root system $\tilde{R}$ as in the statement, and that σ extended to $\mathrm{Aff}(\mathcal{G})$ satisfies (4.1). The statement regarding tameness is also easy to see. The last statement now follows from Theorem 4.1. ■

Example 4.3 Let $\mathcal{G} = \bigoplus_{i=1}^{k} \dot{\mathcal{G}}_i$ be a complex semisimple Lie algebra with $k > 1$, where each $\dot{\mathcal{G}}_i$ is a simple Lie algebra and consider $\mathrm{Aff}(\mathcal{G})$. According to Theorem 4.2, $\mathrm{Aff}(\mathcal{G})$ is a GRLA. Now if we follow the same procedure as in the proof of Theorem 2.10, we see that

$$\mathrm{Aff}(\mathcal{G}) = \sum_{i=1}^{k} \mathcal{G}_i, \text{ where } \mathcal{G}_i = \dot{\mathcal{G}}_i \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

In particular, the sum is not direct. Note that each $\mathcal{G}_i$ is an affine Lie algebra.

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