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A TRANSIENT SOLUTION TO AN $M/M/1$ QUEUE: A SIMPLE APPROACH

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Abstract

A time-dependent solution for the number in a single-server queueing system with Poisson arrivals and exponential service times is derived in a direct way.

Considerable attention has been paid to obtaining the transient solution for the system size in an $M/M/1$ queueing system. A number of methods have been put forward to solve

$$\begin{aligned} \frac{dp_n}{dt} &= \mu p_{n+1}(t) - (\lambda+\mu)p_n(t) + \lambda p_{n-1}(t), \qquad n = 1, 2, 3, \cdots, \\ \frac{dp_0}{dt} &= \mu p_1(t) - \lambda p_0(t) \end{aligned} \tag{1}$$

where $p_n(t)$ denotes the probability that the system size is n. These include the spectral method of Ledermann and Reuter (1956), the combinatorial method of Champernowne (1956) and the difference equation technique of Conolly (1958). See also Karlin and McGregor (1959) and Pegden and Rosenshine (1982).

Here we use a simple and direct approach. We assume that initially there are a customers. Define

$$\begin{aligned} q_n(t) &= \exp((\lambda+\mu)t)[\mu p_n(t) - \lambda p_{n-1}(t)], \qquad n = 1, 2, 3, \cdots, \\ &= 0, \qquad n = 0, -1, -2, \cdots \end{aligned}$$

and $H(s, t) = \sum_{n=-\infty}^{\infty} q_n(t)s^n$. From (1),

$$\begin{aligned} \frac{\partial H}{\partial t} &= \left(\lambda s + \frac{\mu}{s}\right)H - \mu q_1(t) \\ H(s, 0) &= s^a[\mu(1-\delta_{0a}) - \lambda_s] \end{aligned} \tag{2}$$

where δ is the Kronecker delta.

Solving (2),

$$H(s,t) = H(s,0)\exp\left\{\left(\lambda s + \frac{\mu}{s}\right)t\right\} - \mu\int_0^t q_1(y)\exp\left\{\left(\lambda s + \frac{\mu}{s}\right)(t-y)\right\}dy. \tag{3}$$

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We know that, if $\alpha = 2\sqrt{(\lambda\mu)}$ and $\beta = \sqrt{(\lambda/\mu)}$, then

$$\exp\left\{\left(\lambda s + \frac{\mu}{s}\right)t\right\} = \sum_{n=-\infty}^{\infty} (\beta s)^n I_n(\alpha t)$$

where $I_n(t)$ is a modified Bessel function.

Comparing the coefficients of s^n on both sides of (3) for $n = 1, 2, 3, \cdots$, we get

$$(4)\quad \beta^{-n} q_n(t) = \mu(1-\delta_{0a})\beta^{-a} I_{n-a}(\alpha t) - \lambda\beta^{-a-1} I_{n-a-1}(\alpha t) - \mu\int_0^t q_1(y) I_n(\alpha(t-y))\,dy.$$

The above holds for $n = -1, -2, -3, \cdots$, with the left-hand side replaced by zero. Using $I_{-r} = I_r$, for $n = 1, 1, 2, 3, \cdots$,

$$(5)\qquad \mu\int_0^t q_1(y) I_n(\alpha(t-y))\,dy = \mu\beta^{-a}(1-\delta_{0a}) I_{n+a}(\alpha t) - \lambda\beta^{-a-1} I_{n+a+1}(\alpha t).$$

From (4) and (5), for $n = 1, 2, 3, \cdots$,

$$(6)\quad q_n(t) = \mu\beta^{n-a}(1-\delta_{0a})[I_{n-a}(\alpha t) - I_{n+a}(\alpha t)] + \lambda\beta^{n-a-1}[I_{n+a+1}(\alpha t) - I_{n-a-1}(\alpha t)]$$

so that

$$p_n(t) = \frac{\exp\{-(\lambda+\mu)t\}}{\mu}\sum_{k=1}^{n} q_k(t)\left(\frac{\lambda}{\mu}\right)^{n-k} + \left(\frac{\lambda}{\mu}\right)^n p_0(t)$$

and

$$p_0(t) = \int_0^t q_1(y)\exp\{-(\lambda+\mu)y\}\,dy + \delta_{0a}.$$

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