INTEGRALS OF THE DIFFERENTIAL EQUATIONS $\ddot{x} + f(s)x = 0$

R. Datko

(received November 4, 1966)

Introduction. In this note we consider a relatively ancient stability problem: the behaviour of solutions of the second order differential equation $\ddot{x} + f(s)x = 0$, where $f(s)$ tends to plus infinity as s tends to plus infinity. An extensive survey of the literature concerning this problem and a resume of results may be found in [1]. More recently McShane et al. [2] have shown that the additional assumption $\dot{f}(s) \geq 0$ is not sufficient to guarantee that all solutions tend to zero as s tends to infinity. Our aim is to demonstrate a new criterion for which all solutions do have the above property. This criterion overlaps many of the cases heretofore considered.

1. Consider the system

$$\ddot{x} + f(s)x = 0 \tag{1}$$

where we assume

ASSUMPTION 1. $f(s) \to \infty$ as $s \to \infty$, and $\frac{df}{ds}$ exists and is continuous on $[s_o, \infty)$ for some $s_o \geq 0$.

We define a new variable

$$t = \int_{s_o}^{s} [f(u)]^{1/2}\, du . \tag{2}$$

Under this change of variable (1) transforms into the system

$$\ddot{x} + \frac{\dot{x}}{2}\left[\frac{df(t)}{dt}\,\frac{1}{f(t)}\right] + x = 0 , \text{ where } f(t) = f(s(t)) \tag{3}$$

(now $\cdot = \frac{d}{dt}$). For simplicity we write

Canad. Math. Bull. vol. 10, no. 2, 1967

https://doi.org/10.4153/CMB-1967-017-2 Published online by Cambridge University Press

$$(4) \qquad F(t) = \frac{df(t)}{dt}\frac{1}{f(t)} .$$

Let

$$(5) \qquad \begin{aligned} x(t) &= r(t) \cos [-\phi(t)] , \\ \dot{x}(t) &= r(t) \sin [-\phi(t)] \end{aligned}$$

i.e., we write x and $\dot{x}$ in terms of polar coordinates.

The polar form of the equations in (5) can now be written

$$(6) \qquad \begin{aligned} &(a) \qquad \frac{d[r^2]}{dt} = - F(t)\, r^2 \sin^2 (-\phi) , \\ &(b) \qquad \frac{d\phi}{dt} = 1 + \frac{F(t)}{4} \sin (-2\phi) . \end{aligned}$$

An integral representation for $[r(t)]^2$ is given by:

$$(7) \qquad [r(t)]^2 = r_o^2 \exp [- \int_{t_o}^{t} F(u) \sin^2 [-\phi(u)]\, du] .$$

Our objective is to state conditions for which the integral on the right side of (7) diverges as t tends to infintity. This leads to our second assumption.

ASSUMPTION 2. $F(t) \to 0$ as $t \to \infty$.

As can be seen from the form of $F(t)$, this is a very weak assumption since it merely implies that $\frac{df}{ds} [f(s)]^{-3/2} \to 0$ as $s \to \infty$.

Because of assumption 2 and equation 6(b) we may replace t in equation (7) by ϕ, since these imply, for t_o sufficiently large and $t \geq t_o$, that

$$\phi - \phi_o = t - t_o + \Delta_1(t) ,$$

where $\frac{\Delta_1(t)}{t} \to 0$ as $t \to \infty$. With this change of variable we can rewrite (7) as

https://doi.org/10.4153/CMB-1967-017-2 Published online by Cambridge University Press

(8) $$[r(\phi)]^2 = r_o^2 \exp\left[-\int_{\phi_o}^{\phi} F(s(\alpha)) \sin^2(-\alpha)\left[1 + \frac{F(s(\alpha))}{4} \sin(-2\alpha)\right]^{-1} d\alpha\right].$$

The term in the integrand in (8) can be rewritten as

$$[F(s(\alpha)) \sin^2(-\alpha)] \sum_{i=0}^{\infty} (-1)^i \left[\frac{F(s(\alpha))\sin(-2\alpha)}{4}\right]^i$$

(9) $$= [F(s(\alpha)) \sin^2(-\alpha)]\,[1 + F(s(\alpha))\, V(\alpha, s(\alpha))] ,$$

where V is a function which tends to zero as $s(\alpha)$ tends to infinity.

Again applying assumption 2 to equation 6(b), we see that for t_o sufficiently large $s(\alpha) = \alpha + \Delta_1(\alpha)$, where $s(\phi_o) = t_o$ and $\lim_{a \to \infty} \frac{\Delta_2(\alpha)}{\alpha} = 0$. Thus, for t_o sufficiently large, the integrand in (8) may be written

(10) $$F(\alpha + \Delta_2(\alpha)) \sin^2(-\alpha)\,[1 + F(\alpha + \Delta_2(\alpha))\, V_1(\alpha)] ,$$

where now $V_1(\alpha)$ tends to zero as α tends to infinity.

A sufficient condition for the divergence of the integral in (8) now becomes:

ASSUMPTION 3. The integral $\int^{\infty} \sin^2(-\alpha)\, F(\alpha+\Delta(\alpha))\, d\alpha$ is divergent when the perturbation $\Delta(\alpha)$ satisfies the condition $\frac{\Delta(\alpha)}{\alpha} \to 0$ as $\alpha \to \infty$.

We are now in a position to state the following results.

THEOREM 1. If the $F(t)$, induced by the $f(s)$ of equation (1) by means of equations (2) and (4), satisfies assumptions 2 and 3, and $f(s)$ satisfies assumption 1, then all solutions $x(s)$ of equation (1) tend to zero as s tends to infinity.

Example 1. Let $f(s) = \exp \alpha s$ $(\alpha > 0)$. Assumption 1 is satisfied for $s \geq 0$. The induced function is $F(t) = \frac{2\alpha}{\alpha(t+1)}$

which is easily seen to satisfy assumptions 2 and 3.

Example 2. Let $f(s) = s^n$ ($n > 0$ an integer). Again $f(s)$ satisfies assumption 1, and the function $F(t)$ induced by $f(s)$ is $F(t) = \frac{2n}{n+2} \frac{1}{t}$ which satisfies assumptions 2 and 3.

These two examples are special cases of a condition which we state as a corollary to the above theorem.

COROLLARY. If $f(s)$ satisfies assumption 1 and $\lim_{t \to \infty} t\,F(t) = B$, a positive constant, then all solutions $x(s)$ of (1) tend to zero as s tends to infinity.

Proof. The condition on F in the statement of the corollary is a special case of assumption two. Hence to prove the corollary we need only show that assumption three holds.

Suppose $\Delta(\alpha)$ is a function such that $\frac{\Delta(\alpha)}{\alpha} \to 0$ as $\alpha \to \infty$. Let ϵ and h be any two positive constants subject only to the restrictions $0 < \epsilon < B$ and $0 < h < 1$. By the hypotheses of the corollary we can find an $\alpha_o(h, \epsilon) > 0$ such that $|\Delta(\alpha)| < h\alpha$ and $F(\alpha) > \frac{B - \epsilon}{\alpha}$ for $\alpha \geq \alpha_o$. Hence

$$F(\alpha + \Delta(\alpha)) \sin^2(-\alpha) \geq \frac{B-\epsilon}{\alpha(1+h)} \sin^2(-\alpha) \text{ for } \alpha \geq \frac{\alpha_o}{1-h} .$$

Since the integral $\int^{\infty} \frac{B-\epsilon}{\alpha(1+h)} \sin^2(-\alpha) d\alpha$ is divergent assumption three holds and the corollary is proved.

Example 3. We now consider an example which is not covered by any results known to us. In this example the derivative of $f(s)$ assumes non-positive values on an infinite subset of R^+. Let f be defined such that, under the change of variables (2), $f(t)$ assumes the form

$$f(t) = t + \alpha \sin t + k$$

where $\alpha \geq 1$ and k is a constant which insures $f(t)$ is positive for all $t \geq 0$. It follows that

$$\frac{ds}{dt} = [f(t)]^{-1/2} = [t + \alpha \sin t + k]^{-1/2}$$

and

$$\frac{df}{ds} = \frac{df}{dt}\frac{dt}{ds} = (1 + \alpha \cos t)(t + \alpha \sin t + k)^{1/2} .$$

Thus $\frac{df}{ds}$ assumes non-positive values on a set of infinite measure if $\alpha > 1$ and $f(s)$ satisfies assumption 1. The induced function is:

$$F(t) = (1 + \cos t)\ (t + k + \sin t)^{-1} .$$

Hence $F(t)$ satisfies assumption 2. That it also satisfies assumption 3 is an exercise in inequalities which we bypass. Thus we see that all solutions of (1) with $f(s)$ defined as above tend to zero as s tends to infinity.

If $f(s)$ and $F(t)$ satisfy respectively assumption 1 and assumptions 2 and 3 then every integral of (1) tends to zero as s tends to infinity. On the other hand, since the Wronskian associated with any pair of linearly independent solutions of (1) is constant, this implies that all non-trivial solutions of (1) have derivatives whose lim sup is infinite.

If $(x_1, \dot{x}_1)$ and $(x_2, \dot{x}_2)$ are independent solutions of (1) then the Wronskian is constant and can be written in the form

$$w_o = f(s)^{1/2}\, x_1(s)\, f(s)^{-1/2}\, \dot{x}_2(s) - f(s)^{1/2}\, x_2(s)\, f(s)^{-1/2}\, \dot{x}_1(s) .$$

By what we have shown already $f(s)^{-1/2}\,\dot{x}_i(s) = \frac{dx_i(t)}{dt}$ $(i = 1, 2)$ which tends to zero as s tends to infinity. Since each $x_i(s)$ vanishes an infinite number of times on R^+ we conclude that $\limsup |x_i(s)| f(s)^{1/2} = \infty$. This result can also be found in [3].

REFERENCES

1. L. Cesari, Asymptotic Behaviour and Stability Problems in Ordinary Differential Equations. Ergebnisse der Mathematik and Ihrer Grenzgebiete, NF 16 (1963) New York.

2. A.S. Galbraith, E.J. McShane and G.B. Parrish, On the solutions of linear second order equations. Proc. Nat.

https://doi.org/10.4153/CMB-1967-017-2 Published online by Cambridge University Press

Acad. Scie. U.S.A. 53 (1965), pages 247-249.

3. M. Biernacki, Sur l'equation $x + A(t)\ddot{x} = 0$. Prace Mat. - F17 40 (1932), pages 163-171.

McGill University