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THE Γ-OPIAL PROPERTY

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In this short paper we show that if $(X, \|\cdot\|)$ is a Banach space, Γ a norming set for X and C is a nonempty, bounded and Γ sequentially compact subset of X, then in C the Γ-Opial condition for nets is equivalent to the Γ-Opial condition.

1. INTRODUCTION

In 1967 Opial [**11**] introduced a property which is very useful in metric fixed point theory (for the characterisation of this property and its generalisations, see [**1, 2, 4, 5, 9, 12, 13**],). The Opial property deals with weakly convergent sequences but in many applications we often need this property for nets especially when we study either convergence (in the Hausdorff linear topology $\mathcal{T}$) of almost orbits of various types of semigroups of self-mappings of a $\mathcal{T}$-compact subset C of X or nonexpansive retractions on fixed point sets (see, for example [**8, 6, 14**]). Therefore in such cases the basic question is whether the Opial property for nets is equivalent to the Opial property for sequences. For the weak topology this equivalence was proved by W. Kaczor and S. Prus in [**7**]. In our paper we show that if $(X, \|\cdot\|)$ is a Banach space, Γ a norming set for X and C is a nonempty, bounded and sequentially compact, in the Γ-topology, subset of X, then in C the Γ-Opial condition for nets is equivalent to the Γ-Opial condition. However, in our proof we use a different method than that of [**7**].

2. PRELIMINARIES

Throughout this paper all Banach spaces are over the reals.

Let $(X, \|\cdot\|)$ be a Banach space and let Γ be a nonempty subspace of its dual X^*. If

$$\sup\{f(x) : f \in \Gamma, \|f\| = 1\} = \|x\|$$

for each $x \in X$, then we say that Γ is a norming set for X.

It is obvious that a norming set generates a Hausdorff linear topology $\sigma(X, \Gamma)$ on X which is weaker than the weak topology $\sigma(X, X^*)$.

Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{T}$ the Hausdorff vector topology in X. We say that a nonempty set $C \subset X$ satisfies the $\mathcal{T}$-Opial condition (or has the $\mathcal{T}$-Opial

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property), if whenever the bounded sequence $\{x_n\}$ of elements of C converges in the topology $\mathcal{T}$ to $x \in C$ we have

$$\limsup_n \|x_n - x\| < \limsup_n \|x_n - y\|$$

for $y \in C \backslash \{x\}$, or equivalently

$$\liminf_n \|x_n - x\| < \liminf_n \|x_n - y\|.$$

We say that C satisfies the $\mathcal{T}$-Opial condition for nets, (or has the $\mathcal{T}$-Opial property) if whenever the bounded net $\{x_\alpha\}_{\alpha\in I}$ of elements of C converges in the topology $\mathcal{T}$ to $x \in C$ we have

$$\limsup_{\alpha\in I} \|x_\alpha - x\| < \limsup_{\alpha\in I} \|x_\alpha - y\|$$

for $y \in C \backslash \{x\}$, or equivalently

$$\liminf_{\alpha\in I} \|x_\alpha - x\| < \liminf_{\alpha\in I} \|x_\alpha - y\|.$$

In case when Γ is a norming subset of X^* and C a nonempty subset of X we say that C satisfies the Γ-Opial condition (the Γ-Opial condition for nets), if C satisfies the $\sigma(X, \Gamma)$-Opial condition (the $\sigma(X, \Gamma)$-Opial condition for nets).

3. Equivalence of the Γ-Opial property for nets and the Γ-Opial property

The basic question which we meet with in many applications of the Opial property is whether we can use the Opial property for nets when we know that the Opial property is valid. Now, we show that under suitable assumptions on the set C and the Γ-topology the Γ-Opial condition for nets is equivalent to the Γ-Opial condition.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a Banach space, Γ be a norming set for X and C a nonempty, bounded and Γ sequentially compact subset of X. Then in C the Γ-Opial condition for nets is equivalent to the Γ-Opial condition.*

Proof: We only need to prove that the Γ-Opial condition implies the Γ-Opial condition for nets. Let us assume that the net $\{x_\alpha\}_{\alpha\in I}$ of elements of C converges in $\sigma(X, \Gamma)$ to $x \in C$ and let $y \in C$ be such that

$$\limsup_{\alpha\in I} \|x_\alpha - y\| \leqslant \limsup_{\alpha\in I} \|x_\alpha - x\| = \lambda.$$

Then there exist sequences $\{\alpha_n\}$ in I and $\{f_n\}$ in Γ such that

$$\alpha_n \leqslant \alpha_{n+1},$$
$$\lambda - \frac{1}{n} \leqslant \|x_{\alpha_n} - x\| \leqslant \lambda + \frac{1}{n},$$
$$\|x_{\alpha_n} - y\| \leqslant \lambda + \frac{1}{n},$$
$$\|f_n\| = 1,$$
$$f_n(x_{\alpha_n} - x) \geqslant \|x_{\alpha_n} - x\| - \frac{1}{n}$$

for $n = 1, 2, \dots$ and $|f_l(x_{\alpha_n} - x)| \leqslant 1/n$ for $l = 1, 2, \dots, n-1$ and $n = 2, 3, \dots$ Since C is Γ sequentially compact we can choose a subsequence $\{x_{\alpha_{n_k}}\}$ of $\{x_{\alpha_n}\}$ which is Γ-convergent, say to $w \in C$. Then for $l \in \mathcal{N}$ and for each $n_k > l$ we have

$$|f_l(x_{\alpha_{n_k}} - x)| \leqslant \frac{1}{n_k},$$

which implies $|f_l(w - x)| = 0$ for $l = 1, 2, \dots$ Next, we have

$$\begin{aligned} \|x_{\alpha_n} - w\| &= \|f_n\|\,\|x_{\alpha_n} - w\| \geqslant f_n(x_{\alpha_n} - w) = f_n(x_{\alpha_n} - x) + f_n(x - w) \\ &= f_n(x_{\alpha_n} - x) \geqslant \|x_{\alpha_n} - x\| - \frac{1}{n} \geqslant \lambda - \frac{1}{n} - \frac{1}{n} = \lambda - \frac{2}{n} \end{aligned}$$

for $n = 1, 2, \dots$ and therefore $\liminf_n \|x_{\alpha_n} - w\| \geqslant \lambda$. Hence

$$\liminf_k \|x_{\alpha_{n_k}} - x\| = \lim_k \|x_{\alpha_{n_k}} - x\| = \lambda \leqslant \liminf_n \|x_{\alpha_n} - w\| \leqslant \liminf_k \|x_{\alpha_{n_k}} - w\|.$$

By the Γ-Opial property of C we see that $x = w$ and finally we obtain

$$\limsup_k \|x_{\alpha_{n_k}} - y\| \leqslant \lambda = \lim_k \|x_{\alpha_{n_k}} - x\|,$$

which gives $y = x$. This completes the proof. ∎

EXAMPLE 3.1. In [**15**] Van Dulst proved, that every separable Banach space X can be equivalently renormed in such a way that the space X with this new norm has the Opial property with respect to the weak topology. We can apply the Van Dulst result to the Banach space $C([0,1],\mathbb{R})$ and we obtain the space which is not dual and has the Opial property with respect to the weak topology. Next, it is known that the Banach space l^1 treated as a dual to c_0 has the Opial property with respect to the weak* topology [**3, 10**]. Now taking the Cartesian product of these two spaces furnished with the l^2-norm we get the Banach space with the Opial property with respect to the Γ-topology, where $\Gamma = c_0 \times C([0,1],\mathbb{R})^*$. Γ is also a norming set. It is easy to observe that in $l^1 \times C([0,1],\mathbb{R})$ there exist nontrivial convex and bounded (in norm) sets which are sequentially compact and compact in the Γ-topology. The Γ-topology is obviously neither the weak topology nor the weak* topology.

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