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ON C^*-ALGEBRAS WHICH DETECT NUCLEARITY

FLORIN POP

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Abstract

A C^*-algebra A is said to detect nuclearity if, whenever a C^*-algebra B satisfies $A \otimes_{\min} B = A \otimes_{\max} B$, it follows that B is nuclear. In this note, we survey the main results associated with this topic and present the background and tools necessary for proving the main results. In particular, we show that the C^*-algebra $A = C^*(\mathbb{F}_\infty) \otimes_{\min} B(\ell^2)/K(\ell^2)$ detects nuclearity. This result is known to experts, but has never appeared in the literature.

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1. Introduction

Generally speaking, a C^*-algebra A is said to detect a certain property $\mathcal{P}$ (or is a $\mathcal{P}$ detector) if, for any C^*-algebra B, we have $A \otimes_{\min} B = A \otimes_{\max} B$ if and only if B has property $\mathcal{P}$. Examples illustrating this notion go back to Kirchberg's seminal paper [6], where he proved that $C^*(\mathbb{F}_\infty)$ detects Lance's weak expectation property (WEP) and that $B(H)$ detects the local lifting property (LLP). Recently, more examples of WEP detectors were found in [2].

The existence of a C^*-algebra which detects nuclearity remained an open problem for a long time. When an answer emerged, it was circulated as folklore among several experts, but it has never appeared in the literature. It stated that the C^*-algebra $A = C^*(\mathbb{F}_\infty) \otimes_{\min} B(\ell^2)/K(\ell^2)$ is a nuclearity detector. More recently, Kavruk (Corollary 2.3 in [4]) constructed an example of a unital, separable C^*-algebra which detects nuclearity. Kavruk's example comes from operator systems via a universal construction. Namely, the universal C^*-algebra $C^*_u(\mathcal{W}_{32})$ of the operator system

$$\mathcal{W}_{32} = \left\{ \begin{bmatrix} a & b & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & c & 0 & 0 \\ 0 & 0 & c & a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & d \\ 0 & 0 & 0 & 0 & d & a \end{bmatrix} : a, b, c, d \in \mathbb{C} \right\} \subset M_6(\mathbb{C})$$

is shown to detect nuclearity. As explained in [8], universal algebras of this kind are very difficult to describe. While several additional nonself-adjoint detectors were presented in [3], the fact remains that, so far, $C^*(\mathbb{F}_\infty) \otimes_{\min} B(\ell^2)/K(\ell^2)$ remains the only concrete example of a nuclearity detecting C^*-algebra. The goal of this note is to present a detailed account of this fact. To make it easier for the reader, we include several background and auxiliary results required for the proof.

While Kavruk's universal construction represents a separable nuclearity detector, finding a concrete example of such an algebra remains, for the moment, an open problem. Another open problem is whether or not the Calkin algebra alone detects nuclearity. Equivalently (see Section 4), must an exact C^*-algebra with the LLP be necessarily nuclear?

2. Background and basics

We begin by recalling the definitions of the minimal and maximal C^*-cross-norms and refer the reader to [10, 11] for more details. Throughout the paper, we will denote an arbitrary Hilbert space by H and by ℓ^2 a separable Hilbert space. The notion of complete positivity is shortened to c.p., while u.c.p. stands for a unital, completely positive map. Let A_1 and A_2 be unital C^*-algebras. A C^*-cross-norm on the algebraic tensor product $A_1 \otimes A_2$ is a C^*-norm γ satisfying the additional condition $\|a \otimes b\|_\gamma = \|a\| \cdot \|b\|$. If $\pi_1 : A_1 \to B(H_1)$ and $\pi_2 : A_2 \to B(H_2)$ are unital $*$-homomorphisms, we get a unital, $*$-preserving homomorphism $\pi_1 \otimes \pi_2 : A_1 \otimes A_2 \to B(H_1 \otimes H_2)$ by letting $\pi_1 \otimes \pi_2(a \otimes b) = \pi_1(a) \otimes \pi_2(b)$. Thus, if for $x \in A_1 \otimes A_2$, we set $\|x\|_{\min} = \sup\{\pi_1 \otimes \pi_2(x) : \pi_i : A_i \to B(H_i) \text{ unital } *\text{-homomorphism}, i = 1, 2\}$, then we obtain a C^*-cross-norm on $A_1 \otimes A_2$. The completion of $A_1 \otimes A_2$ in this norm is denoted by $A_1 \otimes_{\min} A_2$ and is called the minimal (or the spatial) tensor norm. It is the smallest possible C^*-cross-norm on the algebraic tensor product $A_1 \otimes A_2$. It is not difficult to see that the minimal tensor norm is given by $\| \sum_{i=1}^n a_i \otimes b_i\|_{\min} = \| \sum_{i=1}^n \pi_1(a_i) \otimes \pi_2(b_i)\|$ in $B(H) \otimes B(K) \subset B(H \otimes K)$, where $\pi_1 : A \to B(H)$ and $\pi_2 : B \to B(K)$ are arbitrary faithful $*$-representations.

Let now $\pi_1 : A \to B(H)$ and $\pi_2 : B \to B(H)$ be unital $*$-homomorphisms such that $\pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$ for all $a \in A_1$ and $b \in A_2$. Define a unital $*$-homomorphism $\pi : A \otimes B \to B(H)$ by $\pi(x) = \sum_{i=1}^n \pi_1(a_i)\pi_2(b_i)$, where $x = \sum_{i=1}^n a_i \otimes b_i$. Conversely, if we have a unital $*$-homomorphism $\pi : A \otimes B \to B(H)$ and we define $\pi_1(a) = \pi(a \otimes I)$, $\pi_2(b) = \pi(I \otimes b)$, we obtain a pair of unital $*$-homomorphisms of A and B, respectively, with commuting ranges such that $\pi(a \otimes b) = \pi_1(a)\pi_2(b)$. We define $\|x\|_{\max} = \sup\{\|\pi(x)\| : \pi : A \otimes B \to B(H) \text{ unital } *\text{-homomorphisms}\}$. The completion of $A \otimes B$ in this norm, denoted by $A \otimes_{\max} B$, is called the maximal tensor norm, and is the largest possible C^*-cross-norm on the algebraic tensor product $A \otimes B$.

In the nonunital case, both min and max tensor products are taken to be the respective subalgebras of $A^+ \otimes_{\min} B^+$ and $A^+ \otimes_{\max} B^+$, where A^+ and B^+ are the unitisations of A and B. A C^*-algebra is called nuclear if $A \otimes_{\min} B = A \otimes_{\max} B$ for

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every C^*-algebra B. It is well known that abelian C^*-algebras, $M_n(\mathbb{C})$, and the algebra of compact operators $K(H)$ are nuclear. We recall that a C^*-algebra A is exact if the sequence

$$0 \to A \otimes_{\min} B/J \to A \otimes_{\min} J \to A \otimes_{\min} B \to 0$$

is exact for every C^*-algebra B and every closed, two-sided ideal $J \subset B$.

Two important and well-known results that we will use repeatedly are collected in the next proposition. We refer the reader to [11], where the first is a particular case of Corollary 4.18 and the second is Proposition 7.15.

PROPOSITION 2.1.

(a) *If $\varphi : A \to B$ is a u.c.p. map, then φ extends to a u.c.p. map $\varphi \otimes \mathrm{id} : A \otimes_{\max} C \to B \otimes_{\max} C$.*

(b) *The short exact sequence $0 \to A/J \to J \to A \to 0$ extends to the short exact sequence $0 \to A/J \otimes_{\max} B \to J \otimes_{\max} B \to A \otimes_{\max} B \to 0$.*

It can be proved without difficulty that if $A_i \subset B_i$, $i = 1, 2$, then $A_1 \otimes_{\min} A_2 \subset B_1 \otimes_{\min} B_2$, but this is no longer the case for the max norm. Specifically, if $A \subset B$, then it is possible that the inclusion $A \otimes C \subset B \otimes_{\max} C$ does not induce the max norm on $A \otimes C$. If it does, we write $A \otimes_{\max} C \subset B \otimes_{\max} C$.

In [9], Lance observed that $A \otimes_{\max} C \subset A^{**} \otimes_{\max} C$ for all C^*-algebras A and C, and introduced the following property: given a unital inclusion $A \subset B$, we say that A is weakly c.p. complemented in B if there exists a u.c.p. map $\psi : B \to A^{**}$ such that $\psi(a) = a$ for all $a \in A$. If the above map ψ takes values in A, then A is said to be c.p. complemented in B. It is easy to see that if $A \subset B \subset C$ and A is (weakly) c.p. complemented in C, then A will be (weakly) c.p. complemented in B as well.

Lance proved in [9] that if A is weakly c.p. complemented in B, then $A \otimes_{\max} C \subset B \otimes_{\max} C$ for any C. The particular case when A is c.p. complemented in B will be used subsequently, so we present it as Lemma 3.1 for the benefit of the reader. A C^*-algebra A has the WEP if, for some faithful representation on a Hilbert space H, A is weakly c.p. complemented in $B(H)$. It can be seen [9] that this definition does not depend on the particular representation of A.

A u.c.p. map $\varphi : A \to B/J$ is said to be u.c.p. liftable if there exists a u.c.p. map $\psi : E \to B$ such that $\varphi|_E = q \circ \psi$, where q is the quotient map. The map $\varphi : A \to B/J$ is locally u.c.p. liftable if, for every finite dimensional operator system $E \subset A$, there exists a u.c.p. map $\psi : E \to B$ such that $\varphi|_E = q \circ \psi$. A C^*-algebra has the lifting property (LP) (respectively the LLP) if every u.c.p. map from A to B/J is u.c.p. liftable (respectively locally u.c.p. liftable) to B.

In [1], Choi and Effros proved that separable nuclear C^*-algebras have the LP, while Kirchberg [7] proved the same for the nonnuclear C^*-algebra $C^*(\mathbb{F}_\infty)$, the full C^*-algebra of the free group on countably many generators.

The next proposition contains several fundamental results based on Kirchberg's work [6].

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PROPOSITION 2.2.

(i) *A C^*-algebra A has the WEP if and only if $A \otimes_{\max} C^*(F_\infty) = A \otimes_{\min} C^*(F_\infty)$.*
(ii) *A C^*-algebra A has the LLP if and only if $A \otimes_{\max} B(H) = A \otimes_{\min} B(H)$.*
(iii) *$A \otimes_{\max} B = A \otimes_{\min} B$ if A has the LLP and B has the WEP.*
(iv) *If $A \otimes_{\max} B = A \otimes_{\min} B$ for every separable C^*-algebra B with the WEP, then A has the LLP.*

PROOF. Statements (i), (ii) and (iii) constitute Proposition 1.1 in [6]. To prove statement (iv), suppose that $A \otimes_{\max} B(H) \neq A \otimes_{\min} B(H)$ and choose operators $a_i \in A$, $b_i \in B(H)$, $i = 1, \ldots, n$, such that

$$\Big\| \sum_{i=1}^n a_i \otimes b_i \Big\|_{A\otimes_{\min} B(H)} < \Big\| \sum_{i=1}^n a_i \otimes b_i \Big\|_{A\otimes_{\max} B(H)}.$$

By Lemma 2.4 in [6], there exists a separable C^*-subalgebra $B \subset B(H)$ with the WEP such that $b_i \in B$, $i = 1, \ldots, n$. Then B is weakly c.p. complemented in $B(H)$ and

$$\Big\| \sum_{i=1}^n a_i \otimes b_i \Big\|_{A\otimes_{\max} B} = \Big\| \sum_{i=1}^n a_i \otimes b_i \Big\|_{A\otimes_{\max} B(H)} > \Big\| \sum_{i=1}^n a_i \otimes b_i \Big\|_{A\otimes_{\min} B(H)} = \Big\| \sum_{i=1}^n a_i \otimes b_i \Big\|_{A\otimes_{\min} B},$$

a contradiction. □

3. Preliminary results

In this section, we collect several facts that will be used subsequently in proving the main results.

LEMMA 3.1. *Let $A \subset B$ be an inclusion of C^*-algebras and suppose that there exists a u.c.p. map $\theta : B \to A$ such that $\theta(a) = a$ for all $a \in A$. Then $A \otimes_{\max} C \subset B \otimes_{\max} C$ for every C^*-algebra C.*

PROOF. By Proposition 2.1(b), the inclusion $\iota : A \to B$ extends to a u.c.p. map $\iota \otimes \mathrm{id} : A \otimes_{\max} C \to B \otimes_{\max} C$. Therefore, for all $a_1, \ldots, a_n \in A$ and $c_1, \ldots, c_n \in C$,

$$\begin{aligned}\Big\| \sum_{i=1}^n a_i \otimes c_i \Big\|_{B\otimes_{\max} C} &= \Big\| \sum_{i=1}^n \iota(a_i) \otimes c_i \Big\|_{B\otimes_{\max} C} = \Big\| (\iota \otimes \mathrm{id})\Big(\sum_{i=1}^n a_i \otimes c_i \Big) \Big\|_{B\otimes_{\max} C} \\ &\leq \Big\| \sum_{i=1}^n a_i \otimes c_i \Big\|_{A\otimes_{\max} C}.\end{aligned}$$

Similarly, θ extends to a u.c.p. map $\theta \otimes \mathrm{id} : B \otimes_{\max} C \to A \otimes_{\max} C$, so

$$\begin{aligned}\Big\| \sum_{i=1}^n a_i \otimes c_i \Big\|_{A\otimes_{\max} C} &= \Big\| \sum_{i=1}^n \theta(a_i) \otimes c_i \Big\|_{A\otimes_{\max} C} = \Big\| (\theta \otimes \mathrm{id})\Big(\sum_{i=1}^n a_i \otimes c_i \Big) \Big\|_{A\otimes_{\max} C} \\ &\leq \Big\| \sum_{i=1}^n a_i \otimes c_i \Big\|_{B\otimes_{\max} C}.\end{aligned}$$

□

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COROLLARY 3.2.

(a) *For any unital C^*-algebras A, B and C, we have*

$$A \otimes_{\max} C \subset (A \otimes_{\min} B) \otimes_{\max} C \quad \text{and} \quad A \otimes_{\max} C \subset (B \otimes_{\min} A) \otimes_{\max} C.$$

(b) *For any unital C^*-algebras $A_1, A_2, \dots, A_n$ and B and for $1 \le i \le n$, we have $A_i \otimes_{\max} B \subset (A_1 \otimes_{\min} A_2 \otimes_{\min} \cdots \otimes_{\min} A_n) \otimes_{\max} B$.*

PROOF. (a) We apply Lemma 3.1 for B replaced by $A \otimes_{\min} B$ (respectively $B \otimes_{\min} A$) and identify A with the subalgebra $A \otimes I$ (respectively $I \otimes A$) of $A \otimes_{\min} B$. As far as θ is concerned, choose an arbitrary state φ on B and define $\theta : A \otimes_{\min} B \to A$ by

$$\theta\Big(\sum_{i=1}^{n} a_i \otimes b_i\Big) = \sum_{i=1}^{n} \varphi(b_i) a_i.$$

Respectively, define $\theta : B \otimes_{\min} A \to A$ by

$$\theta\Big(\sum_{i=1}^{n} b_i \otimes a_i\Big) = \sum_{i=1}^{n} \varphi(b_i) a_i.$$

(b) We use induction. The case $n = 2$ is part (a). Suppose now that the property is true for n and consider $n + 1$ algebras $A_1, \dots, A_{n+1}$. For $i = 1$,

$$A_1 \otimes_{\max} B \subset [A_1 \otimes_{\min} (A_2 \otimes_{\min} \cdots \otimes_{\min} A_{n+1})] \otimes_{\max} B$$

by part (a), while for $2 \le i \le n + 1$,

$$A_i \otimes_{\max} B \subset (A_2 \otimes_{\min} \cdots \otimes_{\min} A_{n+1}) \otimes_{\max} B$$

by the inductive hypothesis and

$$(A_2 \otimes_{\min} \cdots \otimes_{\min} A_{n+1}) \otimes_{\max} B \subset [A_1 \otimes_{\min} (A_2 \otimes_{\min} \cdots \otimes_{\min} A_{n+1})] \otimes_{\max} B$$

by part (a). The conclusion follows. □

4. Tensor products with the Calkin algebra

In this section, we prove that a C^*-algebra whose max and min tensor products with the Calkin algebra are equal must be exact and have the LLP. This result, which represents the core technical ingredient needed in the proof of the main result, has also been folklore for a while. It first appeared in print recently as Corollary 10.10 in [11]. The proof we present here is different and shorter than the one in [11].

PROPOSITION 4.1. *A C^*-algebra A satisfies $A \otimes_{\min} B(\ell^2)/K(\ell^2) = A \otimes_{\max} B(\ell^2)/K(\ell^2)$ if and only if A has the LLP and is exact.*

PROOF. $\Leftarrow$ The sequence

$$0 \to A \otimes_{\max} B(\ell^2)/K(\ell^2) \to A \otimes_{\max} K(\ell^2) \to A \otimes_{\max} B(\ell^2) \to 0$$

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is exact and, since A is exact, so is the sequence

$$0 \to A \otimes_{\min} B(\ell^2)/K(\ell^2) \to A \otimes_{\min} K(\ell^2) \to A \otimes_{\min} B(\ell^2) \to 0.$$

Since A has the LLP, $A \otimes_{\min} B(\ell^2) = A \otimes_{\max} B(\ell^2)$. This, together with the fact that $A \otimes_{\min} K(\ell^2) = A \otimes_{\max} K(\ell^2)$ (since $K(\ell^2)$ is nuclear), leads to the conclusion.

$\Rightarrow$ First we prove that A has the LLP. To get a contradiction, suppose, by using Proposition 2.2(iv), that there exists a separable WEP C^*-algebra B with the property $A \otimes_{\min} B \neq A \otimes_{\max} B$. As a consequence, there exist operators $a_i \in A, b_i \in B$, $i = 1, \dots, n$, and $\varepsilon > 0$ such that

$$\Big\| \sum_{i=1}^{n} a_i \otimes b_i \Big\|_{A \otimes_{\max} B} = 1 \quad \text{and} \quad \Big\| \sum_{i=1}^{n} a_i \otimes b_i \Big\|_{A \otimes_{\min} B} < 1 - \varepsilon.$$

We may assume without loss of generality that B is faithfully represented on a separable Hilbert space, viewed as ℓ^2, in such a way that $B \cap K(\ell^2) = \{0\}$. Then the quotient map $\pi : B(\ell^2) \to B(\ell^2)/K(\ell^2)$ induces a $*$-isomorphism between B and $\pi(B) \subset B(\ell^2)/K(\ell^2)$. If we view the Calkin algebra represented faithfully on some (nonseparable) Hilbert space H, then

$$\pi(B) \subset B(\ell^2)/K(\ell^2) \subset B(H).$$

Since $\pi(B)$ has the WEP, it is weakly c.p. complemented in $B(H)$; therefore, also weakly c.p. complemented in $B(\ell^2)/K(\ell^2)$, as remarked in Section 2. In particular, $A \otimes_{\max} \pi(B) \subset A \otimes_{\max} B(\ell^2)/K(\ell^2)$. It follows that

$$\Big\| \sum_{i=1}^{n} a_i \otimes b_i \Big\|_{A \otimes_{\max} B} = \Big\| \sum_{i=1}^{n} a_i \otimes \pi(b_i) \Big\|_{A \otimes_{\max} \pi(B)} = \Big\| \sum_{i=1}^{n} a_i \otimes \pi(b_i) \Big\|_{A \otimes_{\max} B(\ell^2)/K(\ell^2)}.$$

However, by using the hypothesis, we obtain

$$\begin{aligned} \Big\| \sum_{i=1}^{n} a_i \otimes \pi(b_i) \Big\|_{A \otimes_{\max} B(\ell^2)/K(\ell^2)} &= \Big\| \sum_{i=1}^{n} a_i \otimes \pi(b_i) \Big\|_{A \otimes_{\min} B(\ell^2)/K(\ell^2)} \\ = \Big\| \sum_{i=1}^{n} a_i \otimes \pi(b_i) \Big\|_{A \otimes_{\min} \pi(B)} &= \Big\| \sum_{i=1}^{n} a_i \otimes b_i \Big\|_{A \otimes_{\min} B} < 1 - \varepsilon. \end{aligned}$$

It follows that $\| \sum_{i=1}^{n} a_i \otimes b_i \|_{A \otimes_{\max} B} < 1 - \varepsilon$, which contradicts the assumption that $\| \sum_{i=1}^{n} a_i \otimes b_i \|_{A \otimes_{\max} B} = 1$. We conclude that A has the LLP.

To prove exactness, we will show that in the short exact sequence

$$0 \to A \otimes_{\max} B(\ell^2)/K(\ell^2) \to A \otimes_{\max} K(\ell^2) \to A \otimes_{\max} B(\ell^2) \to 0,$$

we can replace 'max' by 'min'. Since A has the LLP, it follows that $A \otimes_{\max} B(\ell^2) = A \otimes_{\min} B(\ell^2)$. By hypothesis, we also have $A \otimes_{\max} B(\ell^2)/K(\ell^2) = A \otimes_{\min} B(\ell^2)/K(\ell^2)$ which, together with $A \otimes_{\max} K(\ell^2) = A \otimes_{\min} K(\ell^2)$, leads to

$$0 \to A \otimes_{\min} B(\ell^2)/K(\ell^2) \to A \otimes_{\min} K(\ell^2) \to A \otimes_{\min} B(\ell^2) \to 0.$$

This implies that A is exact by Theorem 1.1 in [5]. □

5. Detecting nuclearity

This section is devoted to the main results.

DEFINITION 5.1.

(i) A unital C^*-algebra A is said to detect nuclearity if, whenever a C^*-algebra B satisfies $A \otimes_{\min} B = A \otimes_{\max} B$, it follows that B is nuclear.

(ii) A set $\mathbb{S} = \{A_1, A_2, \ldots, A_n\}$ of unital C^*-algebras is said to detect nuclearity if, whenever a C^*-algebra B satisfies $A_i \otimes_{\min} B = A_i \otimes_{\max} B$ for all $i = 1, \ldots, n$, it follows that B is nuclear.

REMARK 5.2. Without loss of generality, we can assume that B is also unital. Otherwise, if B^+ denotes the unitisation of B, $A \otimes_{\min} B = A \otimes_{\max} B$ implies $A \otimes_{\min} B^+ = A \otimes_{\max} B^+$ by Lemma 5.4 in [3]. If A detects nuclearity among unital C^*-algebras, then B^+ is nuclear; therefore, B is nuclear as an ideal of B^+.

The connection between the two notions in the above definition is given in the next proposition.

PROPOSITION 5.3. *If the set $\mathbb{S} = \{A_1, A_2, \ldots, A_n\}$ of unital C^*-algebras detects nuclearity, then so does $A_1 \otimes_{\min} A_2 \otimes_{\min} \cdots \otimes_{\min} A_n$.*

PROOF. Suppose that B satisfies

$$(A_1 \otimes_{\min} A_2 \otimes_{\min} \cdots \otimes_{\min} A_n) \otimes_{\min} B = (A_1 \otimes_{\min} A_2 \otimes_{\min} \cdots \otimes_{\min} A_n) \otimes_{\max} B.$$

By Corollary 3.2(b),

$$\begin{aligned} A_i \otimes_{\max} B &\subset (A_1 \otimes_{\min} A_2 \otimes_{\min} \cdots \otimes_{\min} A_n) \otimes_{\max} B \\ &= (A_1 \otimes_{\min} A_2 \otimes_{\min} \cdots \otimes_{\min} A_n) \otimes_{\min} B; \end{aligned}$$

therefore, $A_i \otimes_{\min} B = A_i \otimes_{\max} B$ for $1 \le i \le n$. Since $\mathbb{S}$ detects nuclearity, the conclusion follows. □

We arrive at the main result of this section.

PROPOSITION 5.4. *The C^*-algebra $C^*(\mathbb{F}_\infty) \otimes_{\min} B(\ell^2)/K(\ell^2)$ detects nuclearity.*

PROOF. By Proposition 5.3, it suffices to show that the set $\mathbb{S} = \{C^*(\mathbb{F}_\infty), B(\ell^2)/K(\ell^2)\}$ detects nuclearity. Let B be a C^*-algebra satisfying $A \otimes_{\min} B = A \otimes_{\max} B$ for all $A \in \mathbb{S}$. From $B(\ell^2)/K(\ell^2) \otimes_{\min} B = B(\ell^2)/K(\ell^2) \otimes_{\max} B$, we see that B is exact and has the LLP by Proposition 4.1. However, $C^*(\mathbb{F}_\infty) \otimes_{\min} B = C^*(\mathbb{F}_\infty) \otimes_{\max} B$ implies that B has the WEP (Proposition 2.2(i)). Since B is both WEP and exact, it must be nuclear (see for example [12, 4.2] or [11, 10.9]). □

REMARK 5.5. In the category of QWEP C^*-algebras (quotients of C^*-algebras with the WEP), the Calkin algebra detects nuclearity. This is a consequence of the fact that exactness and LLP plus QWEP imply nuclearity. Indeed, a C^*-algebra with QWEP and LLP must have the WEP (Corollary 2.6(ii) in [6]), and WEP plus exactness imply nuclearity as seen in the previous paragraph.

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6. Other tensor product characterisations of nuclearity

In the same circle of ideas as nuclearity detection, we mention the issue of characterising nuclearity by the equality of the min and max norms on certain tensor products.

To illustrate this idea, we recall that a C^*-algebra is nuclear if and only if any Hilbert space representation generates an injective von Neumann algebra. Equivalently, A is nuclear if and only if A^{**} is injective [11, 8.16]. Since, for von Neumann algebras, injectivity and the WEP are equivalent [11, 9.26], we obtain the following characterisation of nuclearity [11, 9.27].

PROPOSITION 6.1. *A C^*-algebra A is nuclear if and only if*

$$C^*(\mathbb{F}_\infty) \otimes_{\min} A^{**} = C^*(\mathbb{F}_\infty) \otimes_{\max} A^{**}.$$

Recall that, given a C^*-algebra A, the opposite C^*-algebra, denoted by A^{op}, is the same algebra with the same involution but with reversed product, so that the product of a and b in A^{op} is defined as ba. We conclude with one more characterisation of nuclearity.

PROPOSITION 6.2. *A C^*-algebra A is nuclear if and only if*

$$A \otimes_{\min} (A^{\mathrm{op}} \otimes_{\min} B(\ell^2)/K(\ell^2)) = A \otimes_{\max} (A^{\mathrm{op}} \otimes_{\min} B(\ell^2)/K(\ell^2)).$$

PROOF. We use Corollary 3.2(a) to get $A \otimes_{\min} (B(\ell^2)/K(\ell^2)) = A \otimes_{\max} (B(\ell^2)/K(\ell^2))$ and $A \otimes_{\min} A^{\mathrm{op}} = A \otimes_{\max} A^{\mathrm{op}}$. The former implies that A is exact, by Proposition 4.1, while the latter shows that A has the WEP, as a consequence of Theorem 23.43 in [11]. As previously seen, exactness and WEP imply nuclearity. □

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FLORIN POP, Department of Mathematics,
Wagner College, Staten Island, NY 10301, USA
e-mail: fpop@wagner.edu