ON APPROXIMATION IN WEIGHTED SPACES OF CONTINUOUS VECTOR-VALUED FUNCTIONS

by LIAQAT ALI KHAN

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1. Introduction. The fundamental work on approximation in weighted spaces of continuous functions on a completely regular space has been done mainly by Nachbin ([**5**], [**6**]). Further investigations have been made by Summers [**10**], Prolla ([**7**], [**8**]), and other authors (see the monograph [**8**] for more references). These authors considered functions with range contained in the scalar field or a locally convex topological vector space. In the present paper we prove some approximation results without local convexity of the range space.

2. Definitions and terminology. Throughout this paper we shall assume, unless stated otherwise, that X is a completely regular Hausdorff space and E a Hausdorff topological vector space. Let $C(X, E)$ be the vector space of all continuous E-valued functions on X, and let $C_b(X, E)(C_{00}(X, E))$ be the subspace of $C(X, E)$ consisting of those functions which are bounded (have compact support). When E is the real or complex field, these spaces are denoted by $C(X)$, $C_b(X)$, and $C_{00}(X)$. We shall denote by $C(X) \otimes E$ the vector space spanned by the set of all functions of the form $\varphi \otimes a$, where $\varphi \in C(X)$, $a \in E$, and $(\varphi \otimes a)(x) = \varphi(x)a(x \in X)$.

A *Nachbin family* V on X is a set of non-negative upper semi-continuous functions on X, called *weights*, such that, given $u, v \in V$ and $t \geqslant 0$, there exists a $w \in V$ such that $tu, tv \leqslant w$ (pointwise). Let $CV_b(X, E)(CV_0(X, E))$ denote the subspace consisting of those $f \in C(X, E)$ such that vf is bounded (vanishes at infinity) for all $v \in V$. The *weighted topology* ω_V on $CV_b(X, E)$ is defined as the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form

$$N(v, W) = \{f \in CV_b(X, E) : (vf)(X) \subseteq W\},$$

where $v \in V$ and W is a neighbourhood of 0 in E. $(CV_b(X, E), \omega_V)$ is called a *weighted space.*

The following are some instances of weighted spaces.

(1) If $V = K^+(X)$, the set of all non-negative constant functions on X, then $CV_b(X, E) = C_b(X, E)$ and ω_V is the *uniform topology* σ.

(2) If $V = S_0^+(X)$, the set of all non-negative upper semi-continuous functions on X which vanish at infinity, then $CV_b(X, E) = CV_0(X, E) = C_b(X, E)$ and ω_V is the *strict topology* β_0 ([**2**], Theorem 3.7)

(3) If $V = \{t\chi_K : t \geqslant 0 \text{ and } K \subseteq X, K \text{ compact}\}$, then $CV_b(X, E) = CV_0(X, E) = C(X, E)$ and ω_V is the *compact-open topology* κ.

A neighbourhood W of 0 in E is called *shrinkable* [**3**] if $r\bar{W} \subseteq \operatorname{int} W$ for $0 \leq r < 1$. By ([**3**], Theorems 4 and 5), every Hausdorff topological vector space has a base of shrinkable neighbourhoods of 0 and also the Minkowski functionals of such neighbourhoods are continuous. E is said to be *admissible* ([**3**, p. 284]) if the identity map on E can be approximated uniformly on compact sets by continuous maps with range contained in finite dimensional subspaces of E. By ([**4**], [**9**]), locally convex spaces, topological vector spaces having the approximation property, and ultrabarrelled topological vector spaces with a Schauder basis (in particular, F-spaces with a basis) are admissible.

3. Approximation results. Throughout this section V denotes a Nachbin family on X. The following result extends ([**6**], Propositions 1 and 2, p. 64).

THEOREM 3.1. (1) *$CV_0(X, E)$ is ω_V-closed in $CV_b(X, E)$.* (2) *If X is locally compact, then $C_{00}(X, E)$ is ω_V-dense in $CV_0(X, E)$.*

Proof. (1) Let f belong to the ω_V-closure of $CV_0(X, E)$ in $CV_b(X, E)$, and let $v \in V$ and W a neighbourhood of 0 in E. Choose a balanced neighbourhood G of 0 in E with $G + G \subseteq W$. There exists a function $g \in CV_0(X, E)$ such that $g - f \in N(v, G)$. It is easily seen that $\{x \in X : v(x)g(x) \in G\} \subseteq \{x \in X : v(x)f(x) \in W\}$. Hence the set $\{x \in X : v(x)f(x) \notin W\}$ is compact and so $f \in CV_0(X, E)$.

(2) Let $f \in CV_0(X, E)$, and let $v \in V$ and W a balanced neighbourhood of 0 in E. Then $K = \{x \in X : v(x)f(x) \notin W\}$ is compact. Since X is locally compact, there exists a $\varphi \in C_{00}(X)$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on K. Then $\varphi f \in C_{00}(X, E)$ and $\varphi f - f \in N(v, W)$. Thus $C_{00}(X, E)$ is ω_V-dense in $CV_0(X, E)$.

We now obtain a generalization of ([**6**], Proposition 5, p. 66).

THEOREM 3.2. *Suppose E is a locally bounded topological vector space. Then $C_b(X, E) \cap CV_0(X, E)$ is ω_V-dense in $CV_0(X, E)$.*

Proof. Let $f \in CV_0(C, E)$, and let $v \in V$ and W a balanced neighbourhood of 0 in E. Let H be a bounded neighbourhood of 0 in E. There exists a closed shrinkable neighbourhood S of 0 in E with $S \subseteq H$. Choose $r \geqslant 1$ such that $H \subseteq rW$ and $H \subseteq rS$. The set $K = \{x \in X : v(x)f(x) \notin (1/r)S\}$ is compact and so we choose $t \geqslant 1$ with $f(K) \subseteq (t/r)H$. The Minkowski functional ρ of S is continuous and positively homogeneous and, consequently, the function $h_t : E \to E$ defined by

$$h_t(a) = \begin{cases} a & \text{if} \quad a \in tS, \\ (t/\rho(a))a & \text{if} \quad a \notin tS, \end{cases}$$

is continuous with $h_t(E) \subseteq tS$. Let $g = h_t \circ f$. Clearly, $g \in C_b(X, E)$ and it is not difficult to see that g also belongs to $CV_0(X, E)$. Now, let $x \in X$. Then, since $f(K) \subseteq tS$, we have

$$v(x)(g(x) - f(x)) = \begin{cases} 0 \in W & \text{if} \quad f(x) \in tS, \\ (t/\rho(f(x)) - 1)v(x)f(x) \in (1/r)S \subseteq W & \text{if} \quad f(x) \notin tS. \end{cases}$$

Thus $g - f \in N(v, W)$, as required.

The following result is proved in [**1**, Theorem 3.3] in the case of $V = S_0^+(X)$. We present here a more direct proof than the one given in [**1**].

THEOREM 3.3. *Suppose E is an admissible topological vector space and $V \subseteq S_0^+(X)$. Then $C_b(X) \otimes E$ is ω_V-dense in $C_b(X, E)$.*

Proof. Let $f \in C_b(X, E)$, and let $v \in V$ and W a neighbourhood of 0 in E. Choose an open balanced neighbourhood G of 0 in E such that $G+G+G \subseteq W$. Choose $r > \|v\|$ with $f(X) \subseteq rG$, and let $K = \{x \in X : v(x) \geqslant 1/r\}$. Then $f(K)$ is a compact subset of E and so, by hypothesis, there exists a continuous map $\varphi : f(K) \to E$ with range contained in a finite dimensional subspace of E such that $\varphi(f(x)) - f(x) \in (1/r)G$ for all $x \in K$. We can write $\varphi \circ f = \sum_{i=1}^{n} (\varphi_i \circ f) \otimes a_i$, where $\varphi_i \circ f \in C(K)$ and $a_i \in E$. By the Tietze extension theorem, there exist $\psi_i (1 \leq i \leq n)$ in $C_b(X)$ such that $\psi_i = \varphi_i \circ f$ on K. Let $h = \sum_{i=1}^{n} \psi_i \otimes a_i$. Then $K \subseteq h^{-1}(rG + rG) = F$ (say), which is open in X, and so there exists a $\psi \in C_b(X)$ with $0 \leq \psi \leq 1$, $\psi = 1$ on K and $\psi = 0$ on $X \backslash F$. Let $g = \psi h$. Then $g \in C_b(X) \otimes E$ and $g = h = \varphi \circ f$ on K. Further, $g(X) \subseteq rG + rG$. It is now easily verified that $g - f \in N(v, W)$. This completes the proof.

COROLLARY 3.4. *If E is locally bounded and admissible and $V \subseteq S_0^+(X)$, then $C_b(X) \otimes E$ is ω_V-dense in $CV_0(X, E)$.*

We do not know whether, for any Nachbin family V and E admissible, $CV_0(X) \otimes E$ is ω_V-dense in $CV_0(X, E)$. However, under some restrictions on X, this is true for E any topological vector space.

THEOREM 3.5. *Let X be a locally compact space of finite covering dimension. Then $C_{00}(X) \otimes E$ is ω_V-dense in $CV_0(X, E)$.*

Proof. In view of Theorem 3.1(2), it suffices to show that $C_{00}(X) \otimes E$ is ω_V-dense in $C_{00}(X, E)$. Let $f \in C_{00}(X, E)$, let $v \in V$ and W a balanced neighbourhood of 0 in E. There exists a compact set $K \subseteq X$ such that $f(x) = 0$ for $x \notin K$. Choose $r \geqslant 1$ with $v(x) \leq r$ for all $x \in K$. Since X is of finite covering dimension, it follows from ([**9**, Theorem 1]) that there exists a function $g \in C_{00}(X) \otimes E$ with $g = 0$ outside K and such that $g(x) - f(x) \in (1/r)W$ for all $x \in X$. Then $g - f \in N(v, W)$, as required.

REMARK. If E is assumed to be locally convex, then Theorem 3.5 holds without restricting X to have a finite covering dimension (see [**8**, p. 96]).

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DEPARTMENT OF MATHEMATICS,
FACULTY OF SCIENCE,
GARYOUNIS UNIVERSITY,
P.O. BOX 9480,
BENGHAZI, LIBYA.

Present address:
DEPARTMENT OF MATHEMATICS,
FEDERAL GOVERNMENT COLLEGE,
H-8, ISLAMABAD, PAKISTAN.

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