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PROOF OF SOME CONJECTURAL CONGRUENCES INVOLVING APÉRY AND APÉRY-LIKE NUMBERS

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Abstract In this paper, we mainly prove the following conjectures of Sun [16]: Let $p>3$ be a prime. Then

$$A_{2p}\equiv A_2-\frac{1648}{3}p^3B_{p-3}\ (\mathrm{mod}\ p^4),$$
$$A_{2p-1}\equiv A_1+\frac{16p^3}{3}B_{p-3}\ (\mathrm{mod}\ p^4),$$
$$A_{3p}\equiv A_3-36738p^3B_{p-3}\ (\mathrm{mod}\ p^4),$$

where $A_n=\sum_{k=0}^n\binom{n}{k}^2\binom{n+k}{k}^2$ is the nth Apéry number, and B_n is the nth Bernoulli number.

Keywords: congruences; Apéry numbers; Apéry-like numbers; harmonic numbers; Bernoulli numbers

Mathematics subject classification: Primary 11A07; Secondary 05A10; 11B65; 11B68

1. Introduction

It is well known that the Riemann zeta function was defined by $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$, where s is a complex number with real part larger than 1. In 1979, Apéry [1] introduced the Apéry numbers A_n and A_n' to prove that $\zeta(2)$ and $\zeta(3)$ are irrational, and these numbers are defined by:

$$A_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2\quad\text{and}\quad A_n'=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}.$$

It is well known (see [2]) that:

$$(n+1)^3A_{n+1}=(2n+1)(17n(n+1)+5)A_n-n^3A_{n-1}\quad(n\geq 1),$$
$$(n+1)^2A_{n+1}'=(11n(n+1)+3)A_n'+n^2A_{n-1}'\quad(n\geq 1).$$

The Apéry-like numbers $\{u_n\}$ of the first kind satisfy:

$$u_0 = 1, \quad u_1 = b, \quad (n+1)^3 u_{n+1} = (2n+1)(an(n+1)+b)u_n - cn^3 u_{n-1},$$

where a, b, c are integers and $c \neq 0$. The well-known Domb numbers $D_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k}$ are Apéry-like numbers of this kind, and the following numbers are also Apéry-like numbers of the first kind,

$$T_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}^2.$$

In 2009, Zagier [20] studied the Apéry-like numbers $\{u_n\}$ of the second kind given by:

$$u_0 = 1, \quad u_1 = b, \text{ and } (n+1)^2 u_{n+1} = (an(n+1)+b)u_n - cn^2 u_{n-1} \quad (n \geq 1),$$

where a, b, c are integers and $c \neq 0$. And the famous Franel numbers $f_n = \sum_{k=0}^{n} \binom{n}{k}^3$ and $a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$ are Apéry-like sequences of the second kind. For more congruences involving Apéry-like numbers, we refer the readers to [6–8, 11, 12].
In [16], Sun proposed many congruence conjectures involving these numbers, for example:

Conjecture 1.1. ([16, Conjectures 5.1 and 5.3]) *Let p be a prime with $p > 3$. Then*

$$\begin{aligned}
&A_p \equiv A_1 - \frac{14}{3} p^3 B_{p-3} \pmod{p^4}, \; A_p' \equiv A_1' - \frac{5}{3} p^3 B_{p-3} \pmod{p^4}, \\
&T_p \equiv T_1 - p^3 B_{p-3} \pmod{p^4}, \; D_p \equiv D_1 + \frac{16}{3} p^3 B_{p-3} \pmod{p^4}, \\
&f_p \equiv f_1 + \frac{1}{2} p^3 B_{p-3} \pmod{p^4}, \; a_p \equiv a_1 + \frac{p^2}{2} \left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}.
\end{aligned}$$

Remark 1.1. Actually,

$$a_p \equiv a_1 + \frac{1}{2} p^2 \left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3},$$

has been proved by the first author [9] in 2017, which is earlier than the above conjecture. The congruences of A_p and D_p were proved by Zhang [21].

The above $\{B_n\}$ and $\{B_n(x)\}$ are Bernoulli numbers and Bernoulli polynomials given by:

$$\begin{aligned}
B_0 = 1, \quad &\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad (n \geq 2), \\
B_n(x) = &\sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad (n = 0, 1, 2, \ldots).
\end{aligned}$$

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For $n, m \in \{1, 2, 3, \ldots\}$, define:

$$H_n^{(m)} = \sum_{1 \le k \le n} \frac{1}{k^m},$$

these numbers with $m = 1$ are often called the classic harmonic numbers.

Let $p > 3$ be a prime. Wolstenholme [19] proved that:

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}. \tag{1.1}$$

In 1990, Glaisher [3, 4] showed further that:

$$\binom{2p-1}{p-1} \equiv 1 - \frac{2}{3}p^3 B_{p-3} \pmod{p^4}. \tag{1.2}$$

In this paper, our first goal is to prove the rest unsolved congruences in Conjecture 1.1.

Theorem 1.1. *Let p be a prime with $p > 3$. Then*

$$A_p' \equiv A_1' - \frac{5}{3}p^3 B_{p-3} \pmod{p^4}, \quad T_p \equiv T_1 - p^3 B_{p-3} \pmod{p^4},$$
$$f_p \equiv f_1 + \frac{1}{2}p^3 B_{p-3} \pmod{p^4}.$$

And, we also confirm some conjectures of Sun [16, Conjectures 5.1 and 5.3] involving $()_{2p}$:

Theorem 1.2. *For any prime $p > 3$, we have:*

$$A_{2p} \equiv A_2 - \frac{1648}{3}p^3 B_{p-3} \pmod{p^4}, \; A_{2p}' \equiv A_2' - \frac{280}{3}p^3 B_{p-3} \pmod{p^4},$$
$$T_{2p} \equiv T_2 - 136p^3 B_{p-3} \pmod{p^4}, \; D_{2p} \equiv D_2 + \frac{448}{3}p^3 B_{p-3} \pmod{p^4},$$
$$f_{2p} \equiv f_2 - 8p^3 B_{p-3} \pmod{p^4},$$
$$a_{2p} \equiv a_2 + 6p^2 \left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}.$$

We also proved some conjecture of Sun [16, Conjecture 5.2] involving $()_{2p-1}$:

Theorem 1.3. *Let $p > 3$ be a prime. Then,*

$$A_{2p-1} \equiv A_1 + \frac{16}{3}p^3 B_{p-3} \pmod{p^4},$$
$$T_{2p-1} \equiv 16^{2(p-1)} T_1 - 6p^3 B_{p-3} \pmod{p^4}.$$

At last, we prove some conjectures of Sun [16, Conjecture 5.1] involving $()_{3p}$:

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Theorem 1.4. *Let $p>3$ be a prime. Then,*

$$A_{3p}\equiv A_3-36738p^3B_{p-3}\pmod{p^4},\ A'_{3p}\equiv A'_3-2475p^3B_{p-3}\pmod{p^4},$$
$$T_{3p}\equiv T_3-6696p^3B_{p-3}\pmod{p^4},\ \ D_{3p}\equiv D_3+3168p^3B_{p-3}\pmod{p^4}$$
$$f_{3p}\equiv f_3-189p^3B_{p-3}\pmod{p^4},$$
$$a_{3p}\equiv a_3+\frac{135p^2}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)\pmod{p^3}.$$

We are going to prove Theorem 1.1 in the next section. Sections 3–5 are devoted to proving Theorems 1.2–1.4.

2. Proof of Theorem 1.1

Lemma 2.1. (Lemma [15]). *Let $p>5$ be a prime. Then,*

$$H_{(p-1)/2}\equiv -2q_p(2)+pq_p^2(2)\pmod{p^2},\ \ H^{(2)}_{\frac{p-1}{2}}\equiv\frac{7}{3}pB_{p-3}\pmod{p^2},$$
$$H^{(3)}_{\frac{p-1}{2}}\equiv -2B_{p-3}\pmod{p},\ \ H_{p-1}\equiv-\frac{1}{3}p^2B_{p-3}\pmod{p^3},$$
$$H^{(2)}_{p-1}\equiv\frac{2}{3}pB_{p-3}\pmod{p^2},\ \ H^{(3)}_{p-1}\equiv 0\pmod{p}.$$

Proof of Theorem 1.1. $p=5$ can be checked directly. We will assume $p>5$ from now on. It is easy to check that:

$$\binom{p+k}{k}^2=\frac{(p+k)^2\cdots(p+1)^2}{k!^2}\equiv 1+2pH_k\pmod{p^2},\tag{2.1}$$

and

$$\binom{p-1}{k-1}^2=\frac{(p-1)^2\cdots(p-k+1)^2}{(k-1)!^2}\equiv 1-2pH_{k-1}\pmod{p^2}.\tag{2.2}$$

These yield that:

$$\begin{aligned}A'_p&=\sum_{k=0}^{p}\binom{p}{k}^2\binom{p+k}{k}=1+\binom{2p}{p}+\sum_{k=1}^{p-1}\binom{p}{k}^2\binom{p+k}{k}\\&\equiv 1+\binom{2p}{p}+p^2\sum_{k=1}^{p-1}\frac{1}{k^2}\left(1-pH_k+\frac{2p}{k}\right)\\&\equiv 1+\binom{2p}{p}+p^2H^{(2)}_{p-1}+2p^3H^{(3)}_{p-1}-p^3\sum_{k=1}^{p-1}\frac{H_k}{k^2}\pmod{p^4}.\end{aligned}$$

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In view of [18, (3.17)], we have

$$\sum_{k=1}^{p-1}\frac{H_k}{k^2}\equiv B_{p-3}\pmod p. \tag{2.3}$$

This, with (1.2), Lemma 2.1 yields that:

$$A_p^{'}\equiv 3-\frac{5}{3}p^3B_{p-3}=A_1^{'}-\frac{5}{3}p^3B_{p-3}\pmod{p^4}.$$

Now, we are ready to evaluate T_p modulo p^4. In the same way of proving (2.1), we have the following congruence for each $1\le k\le (p-1)/2$,

$$\binom{2p-2k}{p-2k}^2\equiv 1+2pH_{p-2k}\equiv 1+2pH_{2k-1}\pmod{p^2}.$$

This with (2.2) yields that:

$$\begin{aligned}T_p&=\sum_{k=0}^{p}\binom{p}{k}^2\binom{2k}{p}^2=\binom{2p}{p}^2+\sum_{k=\frac{p+1}{2}}^{p-1}\binom{p}{k}^2\binom{2k}{p}^2\\&=\binom{2p}{p}^2+\sum_{k=1}^{\frac{p-1}{2}}\binom{p}{k}^2\binom{2p-2k}{p-2k}^2\\&\equiv\binom{2p}{p}^2+p^2\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k^2}(1+2pH_{2k-1}-2pH_{k-1})\\&\equiv\binom{2p}{p}^2+p^2H_{\frac{p-1}{2}}^{(2)}+2p^3\sum_{k=1}^{\frac{p-1}{2}}\frac{H_{2k}-H_k}{k^2}+p^3H_{\frac{p-1}{2}}^{(3)}\pmod{p^3}.\end{aligned}$$

In view of [9, 13], we have

$$\sum_{k=1}^{\frac{p-1}{2}}\frac{H_{2k}}{k^2}\equiv\frac{3}{2}B_{p-3}\pmod p\quad\text{and}\quad\sum_{k=1}^{\frac{p-1}{2}}\frac{H_k}{k^2}\equiv-\frac{1}{2}B_{p-3}\pmod p. \tag{2.4}$$

So with (1.2) and Lemma 2.1, we immediately obtain the desired result:

$$T_p\equiv 4-p^3B_{p-3}=T_1-p^3B_{p-3}\pmod{p^4}.$$

At last, we evaluate f_p modulo p^4. This is much easier. By (2.2),

$$f_p=\sum_{k=0}^{p}\binom{p}{k}^3=2+\sum_{k=1}^{p-1}\binom{p}{k}^3\equiv 2-p^3\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}$$

$$= 2 - p^3 \sum_{k=1}^{p-1} \frac{1+(-1)^k}{k^3} + p^3 H_{p-1}^{(3)}$$
$$= 2 - \frac{1}{4} p^3 H_{\frac{p-1}{2}}^{(3)} + p^3 H_{p-1}^{(3)} \pmod{p^4}.$$

So we immediately get the desired result:

$$f_p \equiv 2 + \frac{1}{2} p^3 B_{p-3} = f_1 + \frac{1}{2} p^3 B_{p-3} \pmod{p^4},$$

with the help of Lemma 2.1.
Now the proof of Theorem 1.1 is complete. □

3. Proof of Theorem 1.2

Lemma 3.1. *Let $p > 3$ be a prime. If $1 \le k \le (p-1)/2$, then*

$$\binom{2k}{k}\binom{4p-2k}{2p-k} \equiv -\frac{12p}{k}(1 + 4pH_{2k-1} - 4pH_{k-1}) \pmod{p^3}. \tag{3.1}$$

If $(p+1)/2 \le k \le p-1$, then

$$\binom{4p-2k}{2p-k} \equiv 2\binom{2p-2k}{p-k}(1 + 2pH_{2p-2k} - 2pH_{k-1}) \pmod{p^2}. \tag{3.2}$$

Proof. If $1 \le k \le (p-1)/2$. Since $H_{p-1} \equiv 0 \pmod{p^2}$ and $H_{p-1-k} \equiv H_k \pmod{p}$ for each $0 \le k \le p-1$, we have

$$\begin{aligned}
&\binom{4p-2k}{2p-k} \\
&= \frac{6p(4p-2k)\cdots(3p+1)(3p-1)\cdots(2p+1)(2p-1)\cdots(2p-k+1)}{(2p-k)\cdots(p+1)(p-1)!} \\
&\equiv \frac{6p(p-2k)!(1+3pH_{p-2k})(-1)^{k-1}(k-1)!(1-2pH_{k-1})}{(p-k)!(1+pH_{p-k})} \\
&\equiv \frac{6p}{k}\frac{(-1)^{k-1}(1+3pH_{2k-1}-2pH_{k-1})}{\binom{p-k}{k}(1+pH_{k-1})} \pmod{p^3},
\end{aligned}$$

and

$$\begin{aligned}
\binom{p-k}{k} &= \frac{(p-k)\cdots(p-2k+1)}{k!} \\
&\equiv \frac{(-1)^k k\cdots(2k-1)(1-pH_{2k-1}+pH_{k-1})}{k!} \\
&\equiv \frac{(-1)^k}{2}\binom{2k}{k}(1-pH_{2k-1}+pH_{k-1}) \pmod{p^2}.
\end{aligned} \tag{3.3}$$

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Hence

$$\binom{2k}{k}\binom{4p-2k}{2p-k} \equiv \frac{-12p}{k}\frac{1+3pH_{2k-1}-3pH_{k-1}}{1-pH_{2k-1}+pH_{k-1}}$$
$$\equiv \frac{-12p}{k}(1+4pH_{2k-1}-4pH_{k-1}) \pmod{p^3}.$$

If $(p+1)/2 \le k \le p-1$. It is easy to see that:

$$\binom{4p-2k}{2p-k}$$
$$= \frac{2(4p-2k)\cdots(2p+1)(2p-1)\cdots(2p-k+1)}{(2p-k)\cdots(p+1)(p-1)!}$$
$$\equiv \frac{2(2p-2k)!(1+2pH_{2p-2k})(-1)^{k-1}(k-1)!(1-2pH_{k-1})}{(p-k)!(1+pH_{p-k})}$$
$$\equiv 2\binom{2p-2k}{p-k}\frac{(-1)^{k-1}(1+2pH_{2p-2k}-2pH_{k-1})}{\binom{p-1}{k-1}(1+pH_{k-1})}$$
$$\equiv 2\binom{2p-2k}{p-k}(1+2pH_{2p-2k}-2pH_{k-1}) \pmod{p^2}.$$

Now the proof of Lemma 3.1 is complete. □

Proof of Theorem 1.2. We can check case $p=5$ directly. So we will assume that $p>5$ in the following process. As the same way of proving (2.1) and (2.2), we have

$$\binom{2p-1}{k-1}^2 \equiv 1-4pH_{k-1} \pmod{p^2}, \tag{3.4}$$

$$\binom{2p+k}{k}^2 \equiv 1+4pH_k \pmod{p^2}, \tag{3.5}$$

$$\binom{4p-k}{2p-k}^2 \equiv 9(1+4pH_{k-1}) \pmod{p^2}. \tag{3.6}$$

So we have

$$A_{2p}-1-\binom{4p}{2p}^2-\binom{3p}{p}^2\binom{2p}{p}^2$$
$$=\sum_{k=1}^{p-1}\binom{2p}{k}^2\binom{2p+k}{k}^2+\sum_{k=p+1}^{2p-1}\binom{2p}{k}^2\binom{2p+k}{k}^2$$
$$=\sum_{k=1}^{p-1}\binom{2p}{k}^2\binom{2p+k}{k}^2+\sum_{k=1}^{p-1}\binom{2p}{k}^2\binom{4p-k}{2p-k}^2.$$

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Thus, in view of (3.4), (3.5) and (3.6), we have

$$A_{2p}-1-\binom{4p}{2p}^2-\binom{2p}{p}^2\binom{3p}{p}^2\equiv 4p^2\sum_{k=1}^{p-1}\frac{1}{k^2}\left(1+\frac{4p}{k}\right)+4p^2\sum_{k=1}^{p-1}\frac{9}{k^2}$$
$$=4p^2H_{p-1}^{(2)}+16p^3H_{p-1}^{(3)}+36p^2H_{p-1}^{(2)}=40p^2H_{p-1}^{(2)}+16p^3H_{p-1}^{(3)} \pmod{p^4}.$$

Mao [10, Lemma 4.1] proved that:

$$\binom{4p}{2p}\equiv 6-32p^3B_{p-3} \pmod{p^4}. \tag{3.7}$$

Similarly, with (1.2), Lemma 2.1 and (2.3) we can get that:

$$\binom{3p}{p}=\sum_{k=0}^{p}\binom{2p}{k}\binom{p}{k}=1+\binom{2p}{p}+\sum_{k=1}^{p-1}\binom{2p}{k}\binom{p}{k}$$
$$=1+\binom{2p}{p}+\sum_{k=1}^{p-1}\frac{2p^2}{k^2}\binom{2p-1}{k-1}\binom{p-1}{k-1}$$
$$\equiv 1+\binom{2p}{p}+\sum_{k=1}^{p-1}\frac{2p^2}{k^2}(1-3pH_{k-1})\equiv 3-6p^3B_{p-3} \pmod{p^4}. \tag{3.8}$$

This, with (1.2), (3.7) and Lemma 2.1 yields that:

$$A_{2p}\equiv 73-\frac{1648}{3}p^3B_{p-3}=A_2-\frac{1648}{3}p^3B_{p-3} \pmod{p^4}.$$

Now, we consider A'_{2p} modulo p^4. Similarly, we have

$$A'_{2p}-1-\binom{4p}{2p}-\binom{2p}{p}^2\binom{3p}{p}$$
$$=\sum_{k=1}^{p-1}\binom{2p}{k}^2\binom{2p+k}{k}+\sum_{k=1}^{p-1}\binom{2p}{k}^2\binom{4p-k}{2p-k}.$$

In view of (3.4), (3.5) and (3.6), we have

$$A'_{2p}-1-\binom{4p}{2p}-\binom{2p}{p}^2\binom{3p}{p}$$
$$\equiv 4p^2\sum_{k=1}^{p-1}\frac{1}{k^2}(1+2pH_k-4pH_{k-1})+12p^2\sum_{k=1}^{p-1}\frac{1}{k^2}(1-2pH_{k-1})$$
$$=16p^2H_{p-1}^{(2)}-32p^3\sum_{k=1}^{p-1}\frac{H_k}{k^2}+40p^3H_{p-1}^{(3)} \pmod{p^4}.$$

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Therefore, with (1.2), (3.7), (3.8), Lemma 2.1 and (2.3), we can deduce that:

$$A_{2p}^{'}\equiv 19-\frac{280}{3}p^3B_{p-3}=A_2^{'}-\frac{280}{3}p^3B_{p-3} \pmod{p^4}.$$

Now we evaluate T_{2p} modulo p^4. In the same way of proving Lemma 3.1, we have, if $1\le k\le (p-1)/2$,

$$\binom{4p-2k}{2p-2k}^2\equiv 9(1+4pH_{2k-1}) \pmod{p^2},$$

and if $(p+1)/2\le k\le p-1$,

$$\binom{4p-2k}{2p-2k}^2\equiv 9(1+4pH_{2p-2k}) \pmod{p^2}.$$

So with (3.4) we can deduce that:

$$\begin{aligned}T_{2p}-\binom{2p}{p}^2-\binom{4p}{2p}^2&=\sum_{k=p+1}^{2p-1}\binom{2p}{k}^2\binom{2k}{2p}^2=\sum_{k=1}^{p-1}\binom{2p}{k}^2\binom{4p-2k}{2p-2k}^2\\ \equiv&36p^2\sum_{k=1}^{\frac{p-1}{2}}\frac{1+4pH_{2k-1}-4pH_{k-1}}{k^2}+4p^2\sum_{k=\frac{p+1}{2}}^{p-1}\frac{1+4pH_{2p-2k}-4pH_{k-1}}{k^2}\\ \equiv&36p^2\sum_{k=1}^{\frac{p-1}{2}}\frac{1+4pH_{2k-1}-4pH_{k-1}}{k^2}+\sum_{k=\frac{p+1}{2}}^{p-1}\frac{4p^2}{k^2}+16p^3\sum_{k=1}^{\frac{p-1}{2}}\frac{H_{2k}-H_{p-k-1}}{k^2}\\ \equiv&32p^2H_{\frac{p-1}{2}}^{(2)}+4p^2H_{p-1}^{(2)}+72p^3H_{\frac{p-1}{2}}^{(3)}+160p^3\sum_{k=1}^{\frac{p-1}{2}}\frac{H_{2k}-H_k}{k^2} \pmod{p^4}.\end{aligned}$$

This, with (1.2), (3.7), Lemma 2.1 and (2.4) yields that:

$$T_{2p}\equiv 40-136p^3B_{p-3}=T_2-136p^3B_{p-3} \pmod{p^4}.$$

Next, we consider D_{2p} modulo p^4. It is easy to see that:

$$\begin{aligned}&D_{2p}-2\binom{4p}{2p}-\binom{2p}{p}^4\\ &=\sum_{k=1}^{p-1}\binom{2p}{k}^2\binom{2k}{k}\binom{4p-2k}{2p-k}+\sum_{k=p+1}^{2p-1}\binom{2p}{k}^2\binom{2k}{k}\binom{4p-2k}{2p-k}\\ &=2\sum_{k=1}^{p-1}\binom{2p}{k}^2\binom{2k}{k}\binom{4p-2k}{2p-k}.\end{aligned}$$

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So by Lemma 3.1 and (3.4), we obtain that:

$$D_{2p}-2\binom{4p}{2p}-\binom{2p}{p}^4\equiv -96p^3H^{(3)}_{\frac{p-1}{2}}+16p^2\sum_{k=\frac{p+1}{2}}^{p-1}\frac{1}{k^2}\binom{2k}{k}\binom{2p-2k}{p-k}$$
$$\equiv -96p^3H^{(3)}_{\frac{p-1}{2}}+16p^2\sum_{k=\frac{p+1}{2}}^{p-1}\frac{2p}{k^3}=-128p^3H^{(3)}_{\frac{p-1}{2}}+32p^3H^{(3)}_{p-1} \pmod{p^4}.$$

Then, we can obtain the desired result:

$$D_{2p}\equiv 28+\frac{448}{3}p^3B_{p-3}=D_2+\frac{448}{3}p^3B_{p-3} \pmod{p^4},$$

with the help of (1.2), (3.7) and Lemma 2.1.

Similarly, f_{2p} modulo p^4 is also easier. It is easy to check that by (3.4),

$$f_{2p}-2-\binom{2p}{p}^3=\sum_{k=1}^{p-1}\binom{2p}{k}^3+\sum_{k=p+1}^{2p-1}\binom{2p}{k}^3=2\sum_{k=1}^{p-1}\binom{2p}{k}^3$$
$$\equiv -16p^3\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}=-16p^3\sum_{k=1}^{p-1}\frac{1+(-1)^k}{k^3}+16p^3H^{(3)}_{p-1}$$
$$=-4p^3H^{(3)}_{\frac{p-1}{2}}+16p^3H^{(3)}_{p-1} \pmod{p^4}.$$

In view of (1.2) and Lemma 2.1, we immediately get the desired result:

$$f_{2p}\equiv 10-8p^3B_{p-3}=f_2-8p^3B_{p-3} \pmod{p^4}.$$

At last, we evaluate a_{2p} modulo p^3. By (1.1), (3.4) and (3.7), we have:

$$a_{2p}=1+\binom{4p}{2p}+\binom{2p}{p}^3+\sum_{k=1}^{p-1}\binom{2p}{k}^2\binom{2k}{k}+\sum_{k=p+1}^{2p-1}\binom{2p}{k}^2\binom{2k}{k}$$
$$\equiv 15+\sum_{k=1}^{p-1}\binom{2p}{k}^2\binom{2k}{k}+\sum_{k=1}^{p-1}\binom{2p}{k}^2\binom{4p-2k}{2p-k}$$
$$\equiv 15+4p^2\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}+4p^2\sum_{k=1}^{p-1}\frac{1}{k^2}\binom{4p-2k}{2p-k} \pmod{p^3}.$$

And then in view of Lemma 3.1, we have

$$a_{2p} \equiv 15 + 4p^2 \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2} + 8p^2 \sum_{k=\frac{p+1}{2}}^{p-1} \frac{1}{k^2}\binom{2p-2k}{p-k}$$
$$\equiv 15 + 4p^2 \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2} + 8p^2 \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k^2}\binom{2k}{k}$$
$$\equiv 15 + 12p^2 \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2} \pmod{p^3}.$$

In view of [14], we have

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2} \equiv \frac{1}{2}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p}. \tag{3.9}$$

Therefore, we immediately get the desired result:

$$a_{2p} \equiv 15 + 6p^2 \left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) = a_2 + 6p^2 \left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}.$$

Now the proof of Theorem 1.2 is complete. □

4. Proof of Theorem 1.3

In the same way of proving Lemma 3.1, we have, for $1 \le k \le p-1$

$$\binom{2p-1}{k}^2 \binom{2p-1+k}{k}^2 \equiv \frac{4p^2}{k^2}\left(1 - \frac{4p}{k}\right) \pmod{p^4}, \tag{4.1}$$

and for $0 \le k \le p-2$:

$$\binom{2p-1}{k}^2 \binom{4p-2-k}{2p-1-k}^2 \equiv \frac{36p^2}{(p-1-k)^2}\left(1 + \frac{6p}{k+1}\right) \pmod{p^4}. \tag{4.2}$$

So we have

$$A_{2p-1} - 1 - \binom{2p-1}{p-1}^2 \binom{3p-1}{2p-1}^2$$
$$= \sum_{k=1}^{p-1} \binom{2p-1}{k}^2 \binom{2p-1+k}{k}^2 + \sum_{k=p+1}^{2p-1} \binom{2p-1}{k}^2 \binom{2p-1+k}{k}^2$$
$$= \sum_{k=1}^{p-1} \binom{2p-1}{k}^2 \binom{2p-1+k}{k}^2 + \sum_{k=0}^{p-2} \binom{2p-1}{k}^2 \binom{4p-2-k}{2p-1-k}^2$$

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$$\equiv 4p^2\sum_{k=1}^{p-1}\frac{1-\frac{4p}{k}}{k^2}+\sum_{k=0}^{p-2}\frac{36p^2}{(p-1-k)^2}\left(1+\frac{6p}{k+1}\right)$$
$$\equiv 4p^2\sum_{k=1}^{p-1}\frac{1-\frac{4p}{k}}{k^2}+36p^2\sum_{k=1}^{p-1}\frac{k+2p}{k^3}\left(1+\frac{6p}{k}\right)$$
$$\equiv 40p^2H_{p-1}^{(2)}+272p^3H_{p-1}^{(2)} \pmod{p^4}.$$

In view of (1.2), (3.8) and Lemma 2.1, we immediately get the desired result:

$$A_{2p-1}\equiv 5+\frac{16}{3}p^3B_{p-3}=A_1+\frac{16}{3}p^3B_{p-3} \pmod{p^4}.$$

Now we consider T_{2p-1} modulo p^4. It is easy to see that:

$$T_{2p-1}=\sum_{k=p}^{2p-1}\binom{2p-1}{k}^2\binom{2k}{2p-1}^2=\sum_{k=0}^{p-1}\binom{2p-1}{k}^2\binom{4p-2-2k}{2p-1-2k}^2.$$

So

$$T_{2p-1}-\binom{2p-1}{\frac{p-1}{2}}^2\binom{3p-1}{2p-1}^2$$
$$=\sum_{k=0}^{\frac{p-3}{2}}\binom{2p-1}{k}^2\binom{4p-2-2k}{2p-1-k}^2+\sum_{k=\frac{p+1}{2}}^{p-1}\binom{2p-1}{k}^2\binom{4p-2-2k}{2p-1-2k}^2.$$

In the same way of proving Lemma 3.1, we have, for $0\le k\le (p-3)/2$:

$$\binom{2p-1}{k}^2\binom{4p-2-2k}{2p-1-2k}^2$$
$$\equiv\frac{36p^2}{(p-1-2k)^2}\left(1-4pH_k+4pH_{2k}+\frac{6p}{2k+1}\right) \pmod{p^4},$$

and for $(p+1)/2\le k\le p-1$,

$$\binom{2p-1}{k}^2\binom{4p-2-2k}{2p-1-2k}^2$$
$$\equiv\frac{4p^2}{(2p-1-2k)^2}(1+4pH_{2p-2-2k}-4pH_k) \pmod{p^4}.$$

So we have

$$T_{2p-1}-\binom{2p-1}{\frac{p-1}{2}}^2\binom{3p-1}{2p-1}^2$$

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$$\equiv 36p^2\sum_{k=0}^{\frac{p-3}{2}}\frac{2k+1+8p}{(2k+1)^3}+4p^2\sum_{k=0}^{\frac{p-3}{2}}\frac{1-4pH_k+4pH_{2k}}{(2k+1)^2}$$
$$\equiv 40p^2\sum_{k=0}^{\frac{p-3}{2}}\frac{1}{(2k+1)^2}+160p^3\sum_{k=0}^{\frac{p-3}{2}}\frac{H_{2k}-H_k}{(2k+1)^2}+288p^3\sum_{k=0}^{\frac{p-3}{2}}\frac{1}{(2k+1)^3}$$
$$\equiv 10p^2H^{(2)}_{\frac{p-1}{2}}-26p^3H^{(3)}_{\frac{p-1}{2}}+160p^3\sum_{k=0}^{\frac{p-3}{2}}\frac{H_{2k}-H_k}{(2k+1)^2} \pmod{p^4}.$$

In view of [10, (5.1)], we have

$$\binom{2p-1}{\frac{p-1}{2}}^2=\frac{(2p-1)^2\cdots(2p-\frac{p-1}{2})^2}{(\frac{p-1}{2})!^2}\equiv(16^{p-1}+\frac{11}{6}p^3B_{p-3})^2$$
$$\equiv 16^{2(p-1)}+\frac{11}{3}p^3B_{p-3} \pmod{p^4}.$$

This, with (3.8), [10, Lemma 2.3] and Lemma 2.1 yields that:

$$T_{2p-1}\equiv 4\cdot 16^{2(p-1)}-6p^3B_{p-3}=16^{2(p-1)}V_1-6p^3B_{p-3} \pmod{p^4}.$$

Now the proof of Theorem 1.3 is complete. □

5. Proof of Theorem 1.4

For any $n\geq m$, we define the alternating multiple harmonic sum as:

$$H(a_1,a_2,\ldots,a_m;n)=\sum_{1\leq k_1<k_2<\ldots<k_m\leq n}\prod_{i=1}^{m}\frac{\operatorname{sign}(a_i)^{k_i}}{k_i^{|a_i|}}.$$

The integers m and $\sum_{i=1}^{m}|a_i|$ are respectively the depth and the weight of the harmonic sum. As a matter of convenience, we remember $H(1;n)$ as H_n. In view of [5], we have

$$H(\{a\}^r;p-1)\equiv\begin{cases}(-1)^r\frac{a(ar+1)}{2(ar+2)}p^2B_{p-ar-2} & \pmod{p^3} \quad \text{if } ar \text{ is odd,}\\ (-1)^{r-1}\frac{a}{ar+1}pB_{p-ar-1} & \pmod{p^2} \quad \text{if } ar \text{ is even.}\end{cases} \tag{5.1}$$

Lemma 5.1. *For any prime $p>3$, we have*

$$\binom{4p}{p}\equiv 4-16p^3B_{p-3} \pmod{p^4},\quad \binom{5p}{2p}\equiv 10-100p^3B_{p-3} \pmod{p^4},$$
$$\binom{6p}{3p}\equiv 20-360p^3B_{p-3} \pmod{p^4}.$$

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Proof. It is easy to see that

$$\binom{4p}{p} = 4\binom{4p-1}{p-1} = \frac{(4p-1)\cdots(3p+1)}{(p-1)!}$$
$$\equiv 1 + 3pH_{p-1} + \frac{9p^2}{2}(H_{p-1}^2 - H_{p-1}^{(2)}) + 27p^3H(1,1,1,p-1) \pmod{p^4}.$$

This, with (5.1) and Lemma 2.1 yields that:

$$\binom{4p}{p} \equiv 4 - 16p^3B_{p-3} \pmod{p^4}.$$

Then with this, Lemma 2.1 and (5.1) we have

$$\binom{5p}{2p} = \frac{5}{2}\binom{5p-1}{2p-1} = \frac{5}{2}\frac{(5p-1)\cdots(4p+1)}{(2p-1)\cdots(p+1)}\binom{4p}{p}$$
$$\equiv \frac{5}{2}\binom{4p}{p}\frac{1+4pH_{p-1}+8p^2(H_{p-1}^2-H_{p-1}^{(2)})+64p^3H(1,1,1,p-1)}{1+pH_{p-1}+\frac{p^2}{2}(H_{p-1}^2-H_{p-1}^{(2)})+p^3H(1,1,1,p-1)}$$
$$\equiv 10 - 100p^3B_{p-3} \pmod{p^4}.$$

Similarly, with this and (5.1) and Lemma 2.1, we have

$$\binom{6p}{3p} \equiv 2\binom{5p}{2p}\frac{(6p-1)\cdots(5p+1)}{(3p-1)\cdots(2p+1)}$$
$$\equiv 2\binom{5p}{2p}\frac{1+5pH_{p-1}+\frac{25}{2}p^2(H_{p-1}^2-H_{p-1}^{(2)})+125p^3H(1,1,1,p-1)}{1+2pH_{p-1}+2p^2(H_{p-1}^2-H_{p-1}^{(2)})+8p^3H(1,1,1,p-1)}$$
$$\equiv 20 - 360p^3B_{p-3} \pmod{p^4}.$$

Now the proof of Lemma 5.1 is complete. □

In the same way of proving (2.1) and (2.2), we have

$$\binom{3p-1}{k-1}^2 \equiv 1 - 6pH_{k-1} \pmod{p^2}, \tag{5.2}$$

$$\binom{3p-1}{p+k-1}^2 \equiv 4(1 - 6pH_{k-1}) \pmod{p^2}, \tag{5.3}$$

$$\binom{3p+k}{k}^2 \equiv 1 + 6pH_k \pmod{p^2}, \tag{5.4}$$

$$\binom{4p+k}{p+k}^2 \equiv 16(1 + 6pH_k) \pmod{p^2}, \tag{5.5}$$

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$$\binom{6p-k}{3p-k}^2 \equiv 100(1+6pH_{k-1}) \pmod{p^2}. \tag{5.6}$$

So, we have

$$\begin{aligned}
&A_{3p} - 1 - \binom{3p}{p}^2\binom{4p}{2p}^2 - \binom{3p}{2p}^2\binom{5p}{2p}^2 - \binom{6p}{3p}^2 \\
=&\sum_{k=1}^{p-1}\binom{3p}{k}^2\binom{3p+k}{k}^2 + \sum_{k=1}^{p-1}\binom{3p}{p+k}^2\binom{4p+k}{p+k}^2 \\
&\qquad + \sum_{k=1}^{p-1}\binom{3p}{k}^2\binom{6p-k}{3p-k}^2 \\
\equiv&9p^2\sum_{k=1}^{p-1}\frac{1+\frac{6p}{k}}{k^2} + 576p^2\sum_{k=1}^{p-1}\left(\frac{1}{k^2}+\frac{4p}{k^3}\right) + 900p^2H_{p-1}^{(2)} \\
\equiv&1485p^2H_{p-1}^{(2)} + (54+36\cdot 64)p^3H_{p-1}^{(3)} \pmod{p^4}.
\end{aligned}$$

Then with Lemma 2.1, (3.7), (3.8) and Lemma 5.1, we immediately obtain the desired result:

$$A_{3p} \equiv 1445 - 36738p^3B_{p-3} = A_3 - 36738p^3B_{p-3} \pmod{p^4}.$$

Next we consider $A_{3p}^{'}$ modulo p^4. Similarly,

$$\begin{aligned}
&A_{3p}^{'} - 1 - \binom{3p}{p}^2\binom{4p}{2p} - \binom{3p}{2p}^2\binom{5p}{2p} - \binom{6p}{3p} \\
=&\sum_{k=1}^{p-1}\binom{3p}{k}^2\binom{3p+k}{k} + \sum_{k=1}^{p-1}\binom{3p}{p+k}^2\binom{4p+k}{p+k} \\
&+ \sum_{k=1}^{p-1}\binom{3p}{k}^2\binom{6p-k}{3p-k} \equiv 9p^2\sum_{k=1}^{p-1}\frac{1-3pH_k+\frac{6p}{k}}{k^2} \\
&+ 144p^2\sum_{k=1}^{p-1}\left(\frac{1}{k^2}+\frac{4p}{k^3}-\frac{3pH_k}{k^2}\right) + 90p^2\sum_{k=1}^{p-1}\frac{1-3pH_k+\frac{3p}{k}}{k^2} \\
\equiv&243p^2H_{p-1}^{(2)} + 900p^3H_{p-1}^{(3)} - 729p^3\sum_{k=1}^{p-1}\frac{H_k}{k^2} \pmod{p^4}.
\end{aligned}$$

This, with (2.3), (3.7), (3.8), Lemma 2.1 and Lemma 5.1 yields that:

$$A_{3p}^{'} \equiv 147 - 2475p^3B_{p-3} = A_3^{'} - 2475p^3B_{p-3} \pmod{p^4}.$$

Now we evaluate T_{3p} modulo p^4. It is easy to see that modulo p^4,

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$$\begin{aligned}
&T_{3p}-\binom{6p}{3p}^2-\binom{3p}{p}^2\binom{4p}{p}^2\\
=&\sum_{k=1}^{p-1}\binom{3p}{k}^2\binom{6p-2k}{3p-2k}^2+\sum_{k=1}^{\frac{p-1}{2}}\binom{3p}{p+k}^2\binom{4p-2k}{p-2k}^2\\
\equiv&\sum_{k=1}^{p-1}\frac{9p^2}{k^2}\binom{3p-1}{k-1}^2\binom{6p-2k}{3p-2k}^2+\sum_{k=1}^{\frac{p-1}{2}}\frac{9p^2}{(p+k)^2}\binom{3p}{p+k}^2\binom{4p-2k}{p-2k}^2.
\end{aligned}$$

Similar to prove (2.1) and (2.2), we have, for any $1\le k\le (p-1)/2$,

$$\binom{4p-2k}{p-2k}^2\equiv 1+6pH_{2k-1}\pmod{p^2},$$

$$\binom{6p-2k}{3p-2k}^2\equiv 100(1+6pH_{2k-1})\pmod{p^2},$$

and for each $(p+1)/2\le k\le p-1$,

$$\binom{6p-2k}{3p-2k}^2\equiv 16(1+6pH_{2p-2k})\pmod{p^2}.$$

So we have

$$\begin{aligned}
&T_{3p}-\binom{6p}{3p}^2-\binom{3p}{p}^2\binom{4p}{p}^2\\
\equiv&900p^2\sum_{k=1}^{\frac{p-1}{2}}\frac{1+6pH_{2k-1}-6pH_{k-1}}{k^2}\\
&\qquad+144p^2\sum_{k=\frac{p+1}{2}}^{p-1}\frac{1+6pH_{2p-2k}-6pH_{k-1}}{k^2}\\
&\qquad\qquad+36p^2\sum_{k=1}^{\frac{p-1}{2}}\left(\frac{1}{k^2}+\frac{p}{k^3}+\frac{6pH_{2k}-6pH_k}{k^2}\right)\\
\equiv&1080p^2H^{(2)}_{\frac{p-1}{2}}+3024p^3H^{(3)}_{\frac{p-1}{2}}+6480p^3\sum_{k=1}^{\frac{p-1}{2}}\frac{H_{2k}-H_k}{k^2}\pmod{p^4}.
\end{aligned}$$

Therefore, we immediately obtain the desired result:

$$T_{3p}\equiv 544-6696p^3B_{p-3}=T_3-6696p^3B_{p-3}\pmod{p^4},$$

with the help of (3.8), Lemma 5.1, (2.4) and Lemma 2.1.

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Then, we consider f_{3p} modulo p^4. This is easier; it is easy to check that:

$$\binom{3p-1}{p+k-1} = \binom{2p+p-1}{p+k-1} \equiv \binom{2p}{p}\binom{p-1}{k-1} \equiv 2(-1)^{k-1} \pmod p.$$

So

$$\begin{aligned} f_{3p} - 2 - 2\binom{3p}{p}^2 &= 2\sum_{k=1}^{p-1}\binom{3p}{k}^3 + \sum_{k=1}^{p-1}\binom{3p}{p+k}^3 \\ &\equiv 54p^3\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k^3} + 216p^3\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k^3} \\ &= -270p^3\sum_{k=1}^{p-1}\frac{1+(-1)^k}{k^3} + 270p^3H_{p-1}^{(3)} \\ &= -\frac{135}{2}p^3H_{\frac{p-1}{2}}^{(3)} + 270p^3H_{p-1}^{(3)} \pmod{p^4}. \end{aligned}$$

This, with (3.8) and Lemma 2.1 yields that:

$$f_{3p} \equiv 56 - 189p^3B_{p-3} = f_3 - 189p^3B_{p-3} \pmod{p^4}.$$

Now we consider D_{3p} modulo p^4. In the same way of proving Lemma 3.1, modulo p^2 we have, for $1 \le k \le (p-1)/2$,

$$\binom{2k}{k}\binom{6p-2k}{3p-k} \equiv \frac{-60p}{k}, \quad \binom{4p-2k}{2p-k}\binom{2p+2k}{p+k} \equiv \frac{-24p}{k},$$

and for $(p+1)/2 \le k \le p-1$,

$$\binom{2k}{k}\binom{6p-2k}{3p-k} \equiv \frac{12p}{k}, \quad \binom{4p-2k}{2p-k}\binom{2p+2k}{p+k} \equiv \frac{24p}{k}.$$

In view of [17, Lemma 2.1], we have

$$j\binom{2j}{j}\binom{2(p-j)}{p-j} \equiv 2p(-1)^{\lfloor 2k/p\rfloor - 1} \pmod{p^2}. \tag{5.7}$$

These, with (5.2), (5.3) yield that:

$$\begin{aligned} &D_{3p} - 2\binom{6p}{3p} - 2\binom{3p}{p}^2\binom{2p}{p}\binom{4p}{2p} \\ =&2\sum_{k=1}^{p-1}\binom{3p}{k}^2\binom{2k}{k}\binom{6p-2k}{3p-k} + \sum_{k=1}^{p-1}\binom{3p}{p+k}^2\binom{2p+2k}{p+k}\binom{4p-2k}{2p-k} \end{aligned}$$

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$$
\begin{aligned}
&\equiv 18p^2\sum_{k=1}^{p-1}\binom{3p-1}{k-1}^2\binom{2k}{k}\binom{6p-2k}{3p-k}\\
&\qquad\qquad + 9p^2\sum_{k=1}^{p-1}\binom{3p-1}{p+k-1}^2\binom{2p+2k}{p+k}\binom{4p-2k}{2p-k}\\
&\equiv 18p^2\sum_{k=1}^{\frac{p-1}{2}}\frac{-60p}{k^3}+18p^2\sum_{k=\frac{p+1}{2}}^{p-1}\frac{6}{k^2}\binom{2k}{k}\binom{2p-2k}{p-2}\\
&\qquad\qquad + 9p^2\sum_{k=1}^{\frac{p-1}{2}}\frac{-96p}{k^3}+9p^2\sum_{k=\frac{p+1}{2}}^{p-1}\frac{96p}{k^3}\\
&\equiv -3024p^3H^{(3)}_{\frac{p-1}{2}}+864p^3B_{p-3} \pmod{p^4}.
\end{aligned}
$$

Finally, with the help of Lemma 5.1, Lemma 2.1, (1.2), (3.7) and (3.8), we immediately get the desired result:

$$
D_{3p}\equiv 256+3168p^3B_{p-3}=D_3+3168p^3B_{p-3} \pmod{p^4}.
$$

At last, we evaluate a_{3p} modulo p^3. It is easy to verify that, for each $1\le k\le (p-1)/2$,

$$
\binom{2p+2p}{p+k}\equiv 2\binom{2k}{k} \pmod p,\quad \binom{6p-2k}{3p-k}\equiv 0 \pmod p,
$$

and for $(p+1)/2\le k\le p-1$,

$$
\binom{2p+2p}{p+k}\equiv 0 \pmod p,\quad \binom{6p-2k}{3p-k}\equiv 6\binom{2p-2k}{p-k} \pmod p.
$$

These, with (5.2), (5.3) yield that:

$$
\begin{aligned}
&a_{3p}-1-\binom{3p}{p}^2\binom{2p}{p}-\binom{3p}{p}^2\binom{4p}{2p}-\binom{6p}{3p}\\
&=\sum_{k=1}^{p-1}\binom{3p}{k}^2\binom{2k}{k}+\sum_{k=1}^{p-1}\binom{3p}{p+k}^2\binom{2p+2k}{p+k}+\sum_{k=1}^{p-1}\binom{3p}{k}^2\binom{6p-2k}{3p-k}\\
&\equiv 9p^2\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}+36p^2\sum_{k=1}^{p-1}\frac{\binom{2p+2k}{p+k}}{k^2}+9p^2\sum_{k=1}^{p-1}\frac{\binom{6p-2k}{3p-k}}{k^2}\\
&\equiv 9p^2\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}+36p^2\sum_{k=1}^{\frac{p-1}{2}}\frac{2\binom{2k}{k}}{k^2}+9p^2\sum_{k=\frac{p+1}{2}}^{p-1}\frac{6\binom{2p-2k}{p-k}}{k^2}\\
&\equiv 135p^2\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2} \pmod{p^3}.
\end{aligned}
$$

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Hence with (1.1), (3.7), (3.8), (3.9) and Lemma 5.1, we immediately obtain the desired result:

$$\begin{aligned} a_{3p} &\equiv 93 + \frac{135}{2}p^2 \left(\frac{p}{3}\right) p^2 B_{p-2}\left(\frac{1}{3}\right) \\ &= a_3 + \frac{135}{2}p^2 \left(\frac{p}{3}\right) p^2 B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}. \end{aligned}$$

Therefore the proof of Theorem 1.4 is complete. □

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