LOWER CENTRAL DEPTH IN FINITELY GENERATED SOLUBLE-BY-FINITE GROUPS

by JOHN C. LENNOX

(Received 28 March, 1977)

We say that a group G has finite lower central depth (or simply, finite depth) if the lower central series of G stabilises after a finite number of steps.

In [**1**], we proved that if G is a finitely generated soluble group in which each two generator subgroup has finite depth then G is a finite-by-nilpotent group. Here, in answer to a question of R. Baer, we prove the following stronger version of this result.

THEOREM. *Suppose that G is a finitely generated soluble-by-finite group in which every subgroup of the form $\langle x, x^y \rangle$, x and y in G, has finite lower central depth. Then G is finite-by-nilpotent.*

It is easy to deduce, by methods analogous to those in [**1**], that the result also holds for the classes of finitely generated linear groups and finitely generated hyper-(abelian-by-finite) groups.

Proof. Arguing as in the first part of the proof of Theorem 3 of [**1**], we may assume that G has a residually nilpotent soluble subgroup N of finite index.

Let x, y be elements of N. Then $X = \langle x, x^y \rangle$ is residually nilpotent and has finite lower central depth by hypothesis. Hence X is nilpotent. Let $\bar{N}$ be any finite homomorphic image of N. Then the image $\bar{X}$ of X in $\bar{N}$ is nilpotent and hence, if α and β are the images of x and y in $\bar{N}$, we have that there exists a positive integer n such that the repeated commutator

$$[\alpha^\beta, {}_n \alpha] = 1.$$

Since $\bar{N}$ is finite and y is arbitrary it follows from a result of Peng [**2**] (see also [**3**, 7.22]) that the normal closure of α in $\bar{N}$ is nilpotent. Since x is arbitrary and $\bar{N}$ is finite, it follows that $\bar{N}$ is nilpotent.

Thus N has all of its finite homomorphic images nilpotent and is finitely generated since it is of finite index in a finitely generated group. It now follows from a theorem of Robinson [**3**, 10.51] that N is nilpotent.

The fact that G is finite-by-nilpotent follows as in the last three paragraphs of the proof of Theorem 3 of [**1**].

REFERENCES

1. J. C. Lennox, Finitely generated soluble groups in which all subgroups have finite lower central depth, *Bull. London Math. Soc.* **7** (1975), 273–278.

https://doi.org/10.1017/S0017089500003554 Published online by Cambridge University Press

2. T. A. Peng, Engel elements of groups with maximal condition on abelian subgroups, *Nanta Math.* **1** (1966), 23–28.

3. D. J. S. Robinson, *Finiteness conditions and generalised soluble groups*, Vol II (Springer, 1972).

Department of Pure Mathematics
University College
P.O. Box 78
Cardiff CF1 1XL

https://doi.org/10.1017/S0017089500003554 Published online by Cambridge University Press