GOLDIE CRITERIA FOR SOME SEMIPRIME RINGS

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We principally consider rings R of the form $R = S[G]$, generated as a ring by the subring S of R and the subgroup G of the group of units of R normalizing S. (All our rings have identities except the nilrings.) We wish to deduce that certain semiprime images of R are Goldie rings from ring theoretic information about S and group theoretic information about G. Usually the latter is given in the form that G/N has some solubility or finiteness property, where N is some specified normal subgroup of G contained in S. Note we do *not* assume that $N = G \cap S$; in particular $N = \langle 1 \rangle$ is always an option.

We call a ring R *thin* (resp. *right thin*, resp. *left thin*) if every prime image of R with the maximal and the minimal condition on right annihilators (in brief max-ra and min-ra) is Goldie (right Goldie, left Goldie). A ring is thin, note, if and only if it is both right and left thin. Clearly every commutative ring and every Noetherian ring is thin. It follows from Lemma 1.3 below that if R is a thin ring then every semiprime image of R with max-ra and min-ra is Goldie (similarly for right thin rings and left thin rings).

In the notation of P. Hall, $\langle P, L\rangle(\mathfrak{A}\mathfrak{F})$ denotes the smallest class of groups containing all abelian and all finite groups and closed under the poly and local operators P and L; see the opening pages of [**4**]. This class may look esoteric, but it contains a large number of interesting groups, including the soluble groups, the finite groups, the locally soluble groups, the locally finite groups and even some torsion-free infinite simple groups.

THEOREM. *Let $R = S[G]$ be a ring generated by its subring S and subgroup G of its group of units normalizing S. Suppose N is a normal subgroup of G contained in S. If S is thin and $G/N \in \langle P, L\rangle(\mathfrak{A}\mathfrak{F})$ then R is thin.*

There has to be some restriction on G; for example, if $R = SG$ is the group algebra over the field S of a free group G of rank 2 then S is thin while R is not. The theorem has a number of immediate corollaries.

COROLLARY 1. *If J is a commutative ring and if G is a $\langle P, L\rangle(\mathfrak{A}\mathfrak{F})$-group then the group ring JG is thin.*

For any group G, denote the unique maximal locally finite normal subgroup of G by $\tau(G)$ and define $\alpha(G)$ to be the inverse image in G of the centre of the Hirsch-Plotkin radical of $G/\tau(G)$. If G is soluble-by-finite Zal G denotes a canonical characteristic FC-subgroup of G defined in paragraph 3 of [**11**]. (If G is a soluble it coincides with the Zalesskiĭ subgroup of G as defined in [**3**] and [**5**].) The following is an immediate corollary of Corollary 1 and [**11**, Corollary 1]. (See [**3**] for the definition of control.)

COROLLARY 2. *Let F be a field, G a $\langle P, L\rangle(\mathfrak{A}\mathfrak{F})$-group* and $\mathfrak{p}$ *an ideal of the group algebra FG. Suppose that $G \cap (1 + \mathfrak{p}) = \langle 1 \rangle$, $FG/\mathfrak{p}$ satisfies max-ra and min-ra and $\mathfrak{p} \cap FN$ is a prime ideal of FN for every characteristic subgroup N of G. Then Corollary* 1 *of* [**11**] *applies; in particular $\mathfrak{p}$ is controlled by $\alpha(G)$. If $\tau(G)$ is soluble-by-finite, for example if* char $F = 0$, *then $\mathfrak{p}$ is controlled by the characteristic FC-subgroup* Zal $\alpha(G)$ *of G.*

COROLLARY 3. *Let F be a field, G a* $\langle P, L\rangle(\mathfrak{A}\mathfrak{F})$*-group and* $\mathfrak{p}$ *a completely prime ideal of the group algebra FG such that* $G \cap (1+\mathfrak{p}) = \langle 1\rangle$. *Then* $\mathfrak{p}$ *is controlled by a normal subgroup B of G with B/T abelian and T trivial, quaternion of order* 8, *binary tetrahedral of order* 24 *or binary icosahedral of order* 120.

Corollary 3 is an immediate consequence of Corollary 1 and paragraph 13 of **[11]**. If, in Corollaries 2 and 3, in fact $G \in \langle P, L\rangle\mathfrak{A}$ then **[10**, 1.3] and **[9**, Corollary 1] are applicable.

We do not know whether there is a one-sided version of the theorem. (But see the addendum.) Specifically let $R = S[G]$ be a ring with S a subring and G a subgroup normalizing S and let N be a normal subgroup of G contained in S.

QUESTION 1. *If S is right thin and* $G/N \in \langle P, L\rangle(\mathfrak{A}\mathfrak{F})$ *is R right thin?*

Some partial answers are contained in Sections 6 and 7 below. In particular we have the following extension of the theorem of **[8]**. (This can be proved by adapting the proof given in **[8]**, but we give a proof based on the proof of our theorem above.)

PROPOSITION 1. *Let A be a ring with a nilpotent ideal* τ *such that* A/τ *is Artinian. Suppose S is a right thin subring of A and G is a subgroup of the group of units of A normalizing S such that for some normal subgroup N of G contained in S we have* $G/N \in \langle P, L\rangle(\mathfrak{A}\mathfrak{F})$. *Then the subring* $R = S[G]$ *of A generated by S and G has a nilpotent semiprime ideal* $\mathfrak{n}$ *such that* $R/\mathfrak{n}$ *is right Goldie.*

The following corollary of the theorem is actually an alternative statement of it. The proof we give in Section 6.

COROLLARY 4. *Let* $R = S[G]$ *be a prime ring satisfying max-ra and min-ra, where S is a Goldie subring of R and G is a subgroup of the group of units of R normalizing S. Suppose that G has a normal subgroup N contained in S such that* $G/N \in \langle P, L\rangle(\mathfrak{A}\mathfrak{F})$. *Then R is a Goldie ring.*

Note that it is not sufficient in Corollary 4 to assume that R is semiprime, even if R is a subring of an Artinian ring; see the main counter-example in the introduction of **[8]**, the obstruction being that a minimal prime of R need not intersect S in an intersection of minimal primes of S. The following corollary stands in the same relation to Proposition 1 as Corollary 4 does to the theorem, though we will in fact prove it first. It generalises 24 of **[8]**.

COROLLARY 5. *Let A be a ring with a nilpotent ideal* τ *such that* A/τ *is Artinian. Suppose S is a right Goldie subring of A and G is a subgroup of the group of units of A normalizing S such that for some normal subgroup N of G contained in S we have* $G/N \in \langle P, L\rangle(\mathfrak{A}\mathfrak{F})$. *Assume the subring* $R = S[G]$ *of A is prime. Then R too is right Goldie.*

The next proposition is an important step in the proof of the theorem. By analogy with group theoretical usage we will say a ring R is "locally" in some class of rings if every finite subset of R lies in a subring of R in that class. Now a locally Goldie ring need not be Goldie; if $\bar{R}$ is the cartesian product of infinitely many copies of some field and if R is the subring of $\bar{R}$ of elements whose entries are virtually constant, the R is locally (semiprime Goldie), indeed locally (semisimple Artinian), but clearly R is not Goldie.

Proposition 2. *Let R be a ring satisfying max-ra and min-ra and suppose R is locally (semiprime-Goldie). Then R is semiprime and Goldie.*

The huge class of locally Goldie rings seems to have received relatively little attention and much of its behaviour under the imposition of finiteness conditions remains to be resolved. As a sample of the obvious open questions we ask the following.

Question 2. *Let R be a ring satisfying max-ra and min-ra and suppose R is locally right Goldie. Is R always a right Goldie ring?*

Clearly Proposition 2 settles one special case of Question 2; Proposition 3 below settles another.

Proposition 3. *Let R be a semiprime ring satisfying max-ra and min-ra and suppose R is locally right Artinian. Then R is semisimple Artinian.*

Suppose R is a semiprime ring and $\mathscr{L}$ is a set of subrings of R such that each finite subset X of R lies in some member of $\mathscr{L}$. There is no need for X to lie in a semiprime member of $\mathscr{L}$. Indeed possibly no member of $\mathscr{L}$ is semiprime. For example let F be a field of positive characteristic p, G an infinite locally finite p-group with no non-trivial finite normal subgroups, R the group algebra FG and L the set of subrings FH of R as H runs over the finite non-trivial subgroups of G. Here R is even prime [**3**, 4.2.10]. However, in certain circumstances, each X will indeed lie in a semiprime member of $\mathscr{L}$: see Section 3 below for details; this forms the second main ingredient of the proof of the theorem. We have not investigated the extent to which semiprime can be replaced by prime in Section 3, although 3.3 below gives a partial result.

If a is an ideal of a ring R then $\mathscr{C}_R(a)$ denotes the set of elements of R that are regular modulo a. If X is a subset of R then $r_R(X)$ and $l_R(X)$ denote the right and left annihilators of X in R. We now embark on the proofs.

1. Preliminary lemmas.

1.1. *Let $\mathfrak{b}$ be an ideal of the ring R and set $\mathfrak{a} = r_R(\mathfrak{b})$. If X is a subset of R then*

$$r_{R/\mathfrak{a}}(X + \mathfrak{a}/\mathfrak{a}) = r_R(\mathfrak{b}X)/\mathfrak{a}.$$

In particular if R satisfies max-ra (resp. min-ra), then so does $R/\mathfrak{a}$.

Proof. Here $\mathfrak{b}X$ denotes $\{bx : b \in \mathfrak{b} \text{ and } x \in X\}$. Set $Y = \{r \in R : Xr \subseteq \mathfrak{a}\}$. The claim is that $Y = r_R(\mathfrak{b}X)$. Now $XY \subseteq \mathfrak{a}$ by the definition of Y, so $\mathfrak{b}X \,.\, Y \subseteq \mathfrak{b}\mathfrak{a} = \{0\}$. Thus $Y \subseteq r_R(\mathfrak{b}X)$. Conversely $\mathfrak{b}X \,.\, r_R(\mathfrak{b}X) = \{0\}$, so $X \,.\, r_R(\mathfrak{b}X) \subseteq r_R(\mathfrak{b}) = \mathfrak{a}$ and $r_R(\mathfrak{b}X) \subseteq Y$. All parts of 1.1 follow easily.

1.2. *Let R be a semiprime ring with max-ra (resp. min-ra). Then R has only a finite number of minimal prime ideals and, modulo each, R also has max-ra (resp. min-ra).*

Proof. By [**1**, 1.16] the ring R has only a finite number of minimal prime ideals and each is an annihilator ideal. The result now follows from 1.1.

1.3. *Let R be a ring such that every prime image of R with max-ra (resp. max-ra and min-ra) is right Goldie. Then every semiprime image of R with max-ra (resp. max-ra and min-ra) is right Goldie.*

Proof. Suppose R is also semiprime with max-ra (resp. and also min-ra). Then R has only a finite number of minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ by 1.2 and each $R/\mathfrak{p}_i$ is right Goldie. Then R has finite right uniform dimension (e.g. [**7**, 2.1]). Since R at least has max-ra, so R is right Goldie.

1.4. *Let R be a ring and $\mathscr{C}$ a (left and right) divisor subset of regular elements of R. If X is a subset of $R\mathscr{C}^{-1} = \mathscr{C}^{-1}R$ then*

$$r_{R\mathscr{C}^{-1}}(X) = r_R(R \cap \mathscr{C}X) \,.\, \mathscr{C}^{-1}.$$

In particular if R satisfies max-ra (resp. min-ra), then so does the ring $R\mathscr{C}^{-1}$ of quotients.

Proof. Clearly we may assume X is not empty. If $x \in \mathscr{C}^{-1}R$ then $r_R(x) = r_R(R \cap \mathscr{C}x)$. Thus $r_R(X) = r_R(R \cap \mathscr{C}X)$. Also $r_{R\mathscr{C}^{-1}}(X)$ is a right ideal of $R\mathscr{C}^{-1}$, so

$$r_{R\mathscr{C}^{-1}}(X) = (R \cap r_{R\mathscr{C}^{-1}}(X))\mathscr{C}^{-1} = r_R(X) \,.\, \mathscr{C}^{-1} = r_R(R \cap \mathscr{C}X) \,.\, \mathscr{C}^{-1}.$$

If $Y \subseteq X$ then $r_R(R \cap \mathscr{C}Y) \supseteq r_R(R \cap \mathscr{C}X)$. The lemma follows easily.

The *annihilator dimension* a-dim R of a ring R is the dimension (length of a chain of maximal length) of the poset of right annihilators of subsets of R; so a-dim R is a non-negative integer or ∞. Replacing right annihilators by left annihilators does not change this invariant of R.

1.5. *Let R be a semisimple Artinian ring. Then every right ideal of R is a right annihilator. In particular* a-dim R *is the composition length of R_R and every maximal chain of right annihilators of R has length* a-dim R.

Proof. If A is a right ideal of R then $A = eR$ for some idempotent e, and so $A = r_R(1-e)$. The claim follows.

1.6. *Let R be a semiprime (left and right) Goldie ring with quotient ring $Q = R\mathscr{C}^{-1} = \mathscr{C}^{-1}R$, where $\mathscr{C} = \mathscr{C}_R(0)$. Then*

$$\text{a-dim } R = \text{a-dim } Q = \text{u-dim } Q_Q = \text{u-dim } R_R = \text{u-dim } {}_RR$$

and every maximal chain of right annihilators has length a-dim R.

Here u-dim M_R denotes the *uniform dimension* of the right R-module M. Similarly we define u-dim ${}_RM$.

Proof. If $X \subseteq R$ then $r_R(X) = R \cap r_Q(X)$. Thus a-dim $R \leq$ a-dim Q, and the converse follows from 1.4. It is well known that u-dim $R_R =$ u-dim Q_Q (e.g. [**2**, 2.2.12]). In view of 1.5 the chain of equalities is now clear.

Suppose X and Z are subsets of R with $r_R(X) < r_R(Z)$ and let Y be a subset of Q with $r_Q(X) < r_Q(Y) < r_Q(Z)$. Then by 1.4 we have

$$r_R(X) = r_R(R \cap \mathscr{C}X) < r_R(R \cap \mathscr{C}Y) < r_R(R \cap \mathscr{C}Z) = r_R(Z).$$

Thus a maximal chain of right annihilators of R is the intersection with R of a maximal chain of right annihilators of Q. Hence the final part of 1.6 follows from 1.5.

We need a very slight generalisation of annihilators in rings to annihilators in subsets. If X and Y are *subsets* of a ring R then the right annihilator of Y in X is

$r_X(Y) = \{x \in X : Yx = \{0\}\} = X \cap r_R(Y)$. Similarly we define $l_X(Y)$. Denote by $\mathcal{R}(X)$ the poset $\{r_X(Y) : Y \subseteq X\}$.

1.7. *Let $S \subseteq T$ be subsets of a ring R and let $X \in \mathcal{R}(S)$. Then*

$$X = S \cap r_T l_S(X) \quad \text{and} \quad l_S(X) = S \cap l_T r_T l_S(X).$$

If also $X \subseteq X' \in \mathcal{R}(S)$ then $l_S(X') \subseteq l_S(X)$ and $r_T l_S(X') \supseteq r_T l_S(X)$.

Proof. If X is *any* subset of S, clearly $l_S(X) . X = \{0\} = l_S(X) . r_T l_S(X)$, and so $X \subseteq S \cap r_T l_S(X)$ and $l_S(X) \subseteq S \cap l_T r_T l_S(X)$. Hence

$$l_S r_T l_S(X) . X \subseteq l_S r_T l_S(X) . r_T l_S(X) = \{0\},$$

so $S \cap l_T r_T l_S(X) = l_S r_T l_S(X) \subseteq l_S(X)$, which proves the second equality.

If $X \in \mathcal{R}(S)$, say $X = r_S(Y)$, then replacing X by Y and left by right in the second equality yields $r_S(Y) = S \cap r_T l_T r_S(Y)$, i.e. $X = S \cap r_T l_T(X)$, and setting $S = T$ gives $X = r_S l_S(X)$, which thus equals $S \cap r_T l_S(X)$ in general. The claims concerning X' are obvious.

2. Local criteria for Goldie rings.

2.1. *The proof of Proposition* 2. Suppose a-dim R is infinite. Then there exist finite subsets A_i of R such that $i <$ a-dim A_i (=dim $\mathcal{R}(A_i)$) for $i = 0, 1, \ldots$. Inductively define finite subsets $S_1 \subseteq S_2 \subseteq \ldots$ of R and, for each i, a chain $\mathcal{C}_i$ in $\mathcal{R}(S_i)$ as follows.

Let $S_1 = A_0$ and let $\mathcal{C}_1$ be any chain in $\mathcal{R}(S_1)$ of length 1. Suppose S_i is chosen and $\mathcal{C}_i$ is the chain $X_0 \subset X_1 \subset \ldots \subset X_r$. There exists a semiprime subring R_i of R containing $S_i \cup A_i$. Refine the chain

$$r_{R_i} l_{S_i}(X_0) \subset r_{R_i} l_{S_i}(X_1) \subset \ldots \subset r_{R_i} l_{S_i}(X_r)$$

to a maximal chain $\mathcal{C}_i'$ in $\mathcal{R}(R_i)$. By 1.6 this maximal chain $\mathcal{C}_i'$ has length a-dim R_i, and since $A_i \subseteq R_i$ note that a-dim $R_i > i$. By max-ra every right annihilator in R_i is the right annihilator of a finite subset of R_i. For each term in $\mathcal{C}_i'$ choose such a finite subset of R_i. Let S_{i+1} be any finite subset of R_i containing S_i and all these finite subsets such that the $S_{i+1} \cap Y$ for $Y \in \mathcal{C}_i'$ are all distinct. Then $\mathcal{C}_{i+1} = \{S_{i+1} \cap Y : Y \in \mathcal{C}_i'\}$ is a chain in $\mathcal{R}(S_{i+1})$ of length at least $i+1$ and containing

$$r_{S_{i+1}} l_{S_i}(X_0) \subset \ldots \subset r_{S_{i+1}} l_{S_i}(X_r).$$

Set $S = \bigcup_i S_i$. We define a set $\mathcal{C}$ of subsets of S as follows. Let $X_m \in \mathcal{C}_m$. Set

$$X = \bigcup_{i>m} r_{S_i} l_{S_{i-1}} r_{S_{i-1}} \ldots l_{S_m}(X_m)$$

and let $\mathcal{C}$ be the set of all such X as X_m runs over $\mathcal{C}_m$ and m runs over $1, 2, \ldots$. We claim that $\mathcal{C}$ is an infinite chain in $\mathcal{R}(S)$, not necessarily well ordered.

Given $X_m \in \mathcal{C}_m$ define X_i and Y_i inductively by $Y_i = l_{S_i}(X_i)$ and $X_{i+1} = r_{S_{i+1}}(Y_i)$. Then $X = \bigcup X_i$. Set $Y = \bigcup Y_i$. We prove that $X = r_S(Y)$. (A similar argument shows that $Y = l_S(X)$.) By 1.7 and a simple induction, $S_i \cap X = X_i$ and $S_i \cap Y = Y_i$ for $i \geq m$. Let $x \in X$ and $y \in Y$. There exists $i \geq m$ with $x, y \in S_i$. Then $x \in X_i$ and $y \in Y_i = l_{S_i}(X_i)$. Hence $yx = 0$ and consequently $YX = \{0\}$ and $X \subseteq r_S(Y)$. Suppose $x \in r_S(Y)$. Pick $i \geq m$ with $x \in S_i$. Then $Y_i x \subseteq Yx = \{0\}$ and $x \in r_{S_i}(Y_i) = X_i \subseteq X$. Hence $r_S(Y) = X$.

Now suppose $X_m \in \mathscr{C}_m$ and $X'_n \in \mathscr{C}_n$, where $m \leqslant n$. Then

$$X_n = r_{S_n} l_{S_{n-1}} \ldots l_{S_m}(X_m) \in \mathscr{C}_n$$

and either $X_n \subseteq X'_n$ or $X'_n \subseteq X_n$. If X and X' denote the members of $\mathscr{C}$ generated respectively by X_m and X'_n, a simple induction using 1.7 shows that either $X \subseteq X'$ or $X' \subseteq X$. Therefore $\mathscr{C}$ is a chain. Again, by 1.7, we have $\mathscr{C}_i \subseteq \{S_i \cap X : X \in \mathscr{C}\}$. Thus $\mathscr{C}$ contains at least as many terms as $\mathscr{C}_i$. Hence $\mathscr{C}$ is infinite. We have now proved the claims concerning $\mathscr{C}$.

This is a contradiction of the max-ra and min-ra conditions on R and hence on S. Thus a-dim R is finite. Suppose $U_1 \oplus \ldots \oplus U_n$ is a direct sum of non-zero right ideals of R. There is a semiprime Goldie subring T of R such that each $T \cap U_i \neq \{0\}$. Then

$$n \leqslant \text{u-dim } T_T = \text{a-dim } T \leqslant \text{a-dim } R < \infty,$$

by 1.6 again. Thus u-dim R_R is finite, and R satisfies max-ra by hypothesis. Therefore R is right Goldie. Similarly R is left Goldie. Clearly any locally semiprime ring is semiprime.

2.2. *The proof of Proposition* 3.

(a) *A non-zero right ideal A of R contains a non-zero idempotent.* If A is nil then A is nilpotent [**1**, 1.34] and consequently so is the ideal RA of the semiprime ring R. But $A \neq \{0\}$ and so A is not nil. Let $a \in A$ be a non-nil element of R. There exists a right Artinian subring S of R containing a. Then the right ideal $A \cap S$ of S is not nil and it therefore contains a non-zero idempotent.

(b) *Let A be a right ideal of R with* u-dim $A_R \geqslant 2$. *Then* $A = B \oplus C$ *for some non-zero right ideals B and C of R.* By hypothesis there exist non-zero right ideals X and Y of R with $X \oplus Y \leqslant A$. There is by (a) a non-zero idempotent e of R in X. Then $R = eR \oplus (1-e)R$ and $eR \leqslant A$, so $A = eR \oplus (A \cap (1-e)R)$. Now $e \neq 0$, so $eR \neq \{0\}$, and $eR \leqslant X < A$, so $eR \neq A$. This proves (b).

(c) u-dim R_R *is finite.* Suppose otherwise. Assume we have constructed non-zero right ideals $A_1, \ldots, A_n$ and B_n of R with $R = A_1 \oplus \ldots \oplus A_n \oplus B_n$ and u-dim B_n infinite; this is certainly possible if $n = 0$. By (b), we have $B_n = A_{n+1} \oplus B_{n+1}$ for some right ideals A_{n+1} and B_{n+1} of R with $A_{n+1} \neq \{0\}$ and u-dim B_{n+1} infinite. By induction, we define A_n and B_n for all $n \geqslant 1$.

Let $i \geqslant 1$, pick $j \geqslant i$ and let $e_i \in R \simeq \text{End } R_R$ be the projection of R onto A_i along

$$A_1 \oplus \ldots \oplus A_{i-1} \oplus B_i = A_1 \oplus \ldots \oplus A_{i-1} \oplus A_{i+1} \ldots \oplus A_j \oplus B_j.$$

Then $e_1, e_2, \ldots$ are orthogonal idempotents. Set $R_n = r_R(e_1, e_2, \ldots, e_n)$. Then $R_n \supseteq R_{n+1}$ and $e_{n+1} \in R_n \backslash R_{n+1}$. This contradicts min-ra and (c) is proved.

(d) R is *semisimple Artinian.* By hypothesis and (c) the ring R is semiprime and right Goldie, so R has a semisimple Artinian ring Q of right quotients. Let c be a regular element of R. There exists a right Artinian subring S of R containing c. Then c is regular in S and hence is a unit of S and hence of R. It follows that $R = Q$, which is semisimple Artinian. The proof of Proposition 3 is complete.

3. Semi-prime subring criteria.

3.1. *Let R be a prime ring with max-ra such that every nil subring of R is nilpotent. Suppose R is the union of its subrings R_α for $\alpha<\lambda$, α and λ ordinals, where*

$$R_0 \leq R_1 \leq \dots \leq R_\alpha \leq \dots \leq R.$$

Then for some $\alpha<\lambda$ the ring R_β is semiprime for $\alpha \leq \beta < \lambda$.

Proof. Trivially $\lambda>0$ and if $\lambda-1$ exists we can set $\alpha=\lambda-1$. Thus assume λ is a limit ordinal. Let $\mathfrak{n}_\alpha$ denote the nil radical of R_α and set $N=\sum_{\alpha<\lambda}\mathfrak{n}_\alpha$. If $\alpha\leq\beta<\lambda$ then $\mathfrak{n}_\beta$ is a nilpotent ideal of R_β with $\mathfrak{n}_\alpha\mathfrak{n}_\beta+\mathfrak{n}_\beta\mathfrak{n}_\alpha\leq\mathfrak{n}_\beta$. Therefore N is a nil subring of R and as such is nilpotent. Pick the integer n minimal with $N^n=\{0\}$. If $n=1$ then each R_α is semiprime, so assume otherwise. In particular $N^{n-1}\neq\{0\}$.

For each $\alpha<\lambda$ set $A_\alpha=r_R(\sum_{\alpha\leq\beta<\lambda}\mathfrak{n}_\beta)$. If $\alpha\leq\beta<\lambda$ then $A_\alpha\leq A_\beta$. By max-ra, there exists $\alpha<\lambda$ with $A_\alpha=A_\beta$ for $\beta\geq\alpha$. Set $\mathfrak{a}=A_\alpha$. If $\alpha\leq\beta<\lambda$ then

$$R_\beta\cap\mathfrak{a}=R_\beta\cap A_\beta=\bigcap_{\beta\leq\gamma<\lambda}R_\beta\cap r_{R_\gamma}(\mathfrak{n}_\gamma).$$

Thus $R_\beta\cap\mathfrak{a}$ is an ideal of R_β whenever $\alpha\leq\beta<\lambda$ and consequently $\mathfrak{a}$ is an ideal of $\bigcup_{\beta\geq\alpha}R_\beta=R$. Also $N^{n-1}\subseteq\mathfrak{a}$, so $\mathfrak{a}\neq\{0\}$. But R is a prime ring. Therefore

$$\{0\}=l_R(\mathfrak{a})=l_R(A_\alpha)\supseteq\sum_{\alpha\leq\beta<\lambda}\mathfrak{n}_\beta.$$

Hence $\mathfrak{n}_\beta=\{0\}$ and R_β is semiprime whenever $\alpha\leq\beta<\lambda$.

3.2. *Let R be a semiprime ring with max-ra and min-ra. Suppose R is the union of its subrings R_α for $\alpha<\lambda$, α and λ ordinals, where*

$$R_0 \leq R_1 \leq \dots \leq R_\alpha \leq \dots.$$

Then for some $\alpha<\lambda$ the ring R_β is semiprime for $\alpha\leq\beta<\lambda$.

Proof. If R is actually prime the conclusion follows immediately from 3.1 and [**1**, 1.34]. In general R has only a finite number of minimal prime ideals $\mathfrak{p}_1,\dots,\mathfrak{p}_m$, each $R/\mathfrak{p}_i$ satisfies max-ra and min-ra and $\bigcap\mathfrak{p}_i=\{0\}$, see 1.2. Then, for $i=1,2,\dots,m$, there exists $\alpha_1,\dots,\alpha_m<\lambda$ such that, for each i, if $\alpha_i\leq\beta<\lambda$ then $R_\beta/R_\beta\cap\mathfrak{p}_i\cong R_\beta+\mathfrak{p}_i/\mathfrak{p}_i$ is semiprime. Let $\alpha=\max\{\alpha_1,\dots,\alpha_m\}$. Then $\alpha<\lambda$ and if $\alpha\leq\beta<\lambda$ then each $R_\beta/R_\beta\cap\mathfrak{p}_i$ is semiprime, $\bigcap_i R_\beta\cap\mathfrak{p}_i=\{0\}$ and R_β is semiprime.

If R is actually prime in 3.2, are the R_β also prime for large enough β? The following gives a positive partial answer.

3.3. *Assume the notation of* 3.2. *Suppose R is a prime subring of A, where A has a semiprime nilpotent ideal $\mathfrak{r}$ such that $A/\mathfrak{r}$ is a subring of an Artinian ring (e.g. $A/\mathfrak{r}$ might be left or right Goldie). Then for some $\alpha<\lambda$, whenever $\alpha\leq\beta<\lambda$ the ring R_β is prime.*

Proof. Clearly $R\cap\mathfrak{r}=\{0\}$, so we may pass to $A/\mathfrak{r}$ and assume $\mathfrak{r}=\{0\}$. Hence we may choose A to be Artinian and, reducing again to $\mathfrak{r}=\{0\}$, semiprime. Then since R is prime, R embeds into a simple component of A and we may assume that A is the matrix ring $D^{n\times n}$ for some division ring D.

Let $V = D^{(n)}$ denote row n-space over D, regarded as a D–A-bimodule in the obvious way. Let $V = V_0 > V_1 > \ldots > V_m = \{0\}$ be a composition series of V as D–R-bimodule and denote the annihilator of V_{i-1}/V_i in R by $\mathfrak{a}_i$. Then $V\mathfrak{a}_1\mathfrak{a}_2 \ldots \mathfrak{a}_m = \{0\}$, $\mathfrak{a}_1\mathfrak{a}_2 \ldots \mathfrak{a}_m = \{0\}$ and the primeness of R ensures that some $\mathfrak{a}_i = \{0\}$. Hence we may replace A by $\mathrm{End}_D(V_{i-1}/V_i)$ and assume that V is D–R-irreducible.

There exists $\alpha_1 < \lambda$ such that among the $\alpha < \lambda$ the composition length of V as D–R_{α_1}-bimodule is minimal. Suppose $\alpha_1 \leq \alpha \leq \beta < \lambda$. Then a composition series of V as D–R_β-bimodule is also one as D–R_α-bimodule. In particular every irreducible D–R_β-submodule of V is D–R_α-irreducible. Let S_α denote the socle of V as D–R_α-bimodule. We have now proved that $S_\alpha \geq S_\beta > \{0\}$ whenever $\alpha_1 \leq \alpha \leq \beta < \lambda$.

Choose α_2 with $\alpha_1 \leq \alpha_2 < \lambda$ and $\dim_D S_{\alpha_2}$ minimal. Then $S_\beta = S_{\alpha_2}$ whenever $\alpha_2 \leq \beta < \lambda$ and S_{α_2} is a non-zero submodule of the irreducible D–R-bimodule V. Consequently $S_\beta = V$ and V is completely D–R_β-reducible for $\alpha_2 \leq \beta < \lambda$. If $\alpha_2 \leq \beta \leq \gamma < \lambda$ the homogeneous components of V as D–R_γ-bimodule are also D–R_β-homogeneous. Thus the homogeneous D–R_β-components are direct sums of homogeneous D–R_γ-components, and so are D–R_γ-submodules. Hence they are D–R-submodules, and yet V is D–R-irreducible. Therefore V is D–R_β-homogeneous for $\alpha_2 \leq \beta < \lambda$. Pick β with $\alpha_2 \leq \beta < \lambda$ and let W be an irreducible D–R_β-submodule of V. Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of R_β with $\mathfrak{a}\mathfrak{b} = \{0\}$. If $W = W\mathfrak{a}$ then $W\mathfrak{b} = W\mathfrak{a}\mathfrak{b} = \{0\}$, $V\mathfrak{b} = \{0\}$ by homogeneity and so $\mathfrak{b} = \{0\}$. If $W \neq W\mathfrak{a}$ then $W\mathfrak{a} = \{0\}$ by irreducibility, $V\mathfrak{a} = \{0\}$ by homogeneity and so $\mathfrak{a} = \{0\}$. Consequently R_β is prime and the proof is complete.

4. Classes of groups.

4.1. *Let $\mathfrak{X}$ be a subgroup-closed class of groups. Then $\mathfrak{X}$ is locally closed if (and trivially only if) $\mathfrak{X}$ is closed under unions of ascending chains of $\mathfrak{X}$-groups.*

Proof. Thus whenever $G_1 \leq G_2 \leq \ldots \leq G_\alpha \ldots$ is an ascending chain of $\mathfrak{X}$-subgroups of some group G, we assume that $\bigcup G_\alpha \in \mathfrak{X}$. Let $G \in L\mathfrak{X}$. We prove by induction on $|G|$ that $G \in \mathfrak{X}$. If G is finite this is trivial so assume G is infinite. Let λ be the least ordinal of cardinal $|G|$ and enumerate G; say $G = \{g_\alpha : 0 \leq \alpha < \lambda\}$. Set $G_\beta = \langle g_\alpha : 0 \leq \alpha < \beta \rangle$ for $1 \leq \beta < \lambda$.

Suppose α is finite. Then G_α is finitely generated. But $G \in L\mathfrak{X}$, so $G_\alpha \leq H$ for some $\mathfrak{X}$-subgroup H of G. Hence $G_\alpha \in S\mathfrak{X} = \mathfrak{X}$. Now suppose G is uncountable and consider any $\alpha < \lambda$. Then $|G_\alpha| < |G|$. Also $G_\alpha \in SL\mathfrak{X} \subseteq LS\mathfrak{X} = L\mathfrak{X}$. By induction, we assume that $G_\alpha \in \mathfrak{X}$. Thus now in all cases G is the union of an ascending chain of $\mathfrak{X}$-subgroups. By hypothesis, $G \in \mathfrak{X}$.

4.2. *Let $\mathfrak{X}$ be any class of groups closed under unions of ascending chains of subgroups. Then the class $\mathfrak{X}^S$ is locally closed.*

Here, $\mathfrak{X}^S$ denotes the class of all groups G such that every subgroup of G lies in $\mathfrak{X}$.

Proof. Suppose $G_1 \leq G_2 \leq \ldots \leq G_\alpha \leq \ldots$ is an ascending chain of $\mathfrak{X}^S$-subgroups of a group G with union G. If H is a subgroup of G then $G_1 \cap H \leq G_2 \cap H \leq \ldots \leq G_\alpha \cap H \leq \ldots \cup G_\alpha \cap H = H$. Each $G_\alpha \cap H \in S\mathfrak{X}^S \subseteq \mathfrak{X}$. By hypothesis, $H \in \mathfrak{X}$. This is for all such H, so $G \in \mathfrak{X}^S$. Now $\mathfrak{X}^S$ is locally closed by 4.1.

5. Extension lemmas. Throughout this section $R = S[G]$ is a *prime* ring satisfying max-ra and min-ra generated by its subring S and its subgroup G of its group of units normalizing S. Further N denotes some normal subgroup of G lying in S.

5.1. *The subring S is semiprime with max-ra and min-ra.*

Proof. If $\mathfrak{n}$ is the (say upper) nilradical of S then $\mathfrak{n}$ is nilpotent [**1**, 1.34] and normalized by G, so $\mathfrak{n}G$ is a nilpotent ideal of the prime ring R. Then $\mathfrak{n} = \{0\}$ and S is semiprime. The remaining claims are trivial.

5.2. *Suppose S is right Goldie and G/N is polycyclic-by-finite. Then R is right Goldie.*

Proof. Let $\mathscr{C} = \mathscr{C}_S(0)$. Then $\mathscr{C}$ is a right divisor subset of S by Goldie's theorem, and it is normalized by G. Hence $\mathscr{C}$ is a right divisor subset of R by [**5**, 5.6.3a]. Since R is prime with min-ra, R is right-$\mathscr{C}$-torsion-free [**6**, 13]; that is, the elements of $\mathscr{C}$ are left regular in R. Thus $\mathscr{C} \subseteq \mathscr{C}_R(0)$ by [**1**, 1.30a].

We now form the partial quotient ring $R\mathscr{C}^{-1}$ of R. Since G normalizes $\mathscr{C}$, we have $R\mathscr{C}^{-1} = S\mathscr{C}^{-1}[G]$ and $S\mathscr{C}^{-1}$ is semiprime Artinian. By a theorem of P. Hall (cf. [**3**, 10.2.6]), the ring $R\mathscr{C}^{-1}$ is Noetherian. Thus $R\mathscr{C}^{-1}$ has finite right uniform dimension, and consequently so does R [**2**, 2.2.12]. Therefore R is right Goldie as claimed.

5.3. *Suppose every finite subset of G lies in a subgroup H of G such that the subring S[H] of R is semiprime and Goldie. Then R is Goldie.*

Proof. This is an immediate consequence of Proposition 2.

5.4. *Suppose* $G = \bigcup_{\alpha<\gamma} H_\alpha$ *is the union of an ascending chain of subgroups* H_α *such that whenever the subring* $S[H_\alpha]$ *of R is semiprime it is Goldie. Then R is Goldie.*

Proof. By 3.2, there is an ordinal $\alpha < \gamma$ such that $S[H_\beta]$ is semiprime whenever $\alpha \leq \beta < \gamma$. By hypothesis, each such $S[H_\beta]$ is Goldie. Hence R too is Goldie by 5.3.

6. Thin rings and thin groups. In this section we use the following notation: $R = S[G]$ is a ring generated by its subring S and its subgroup G of its group of units normalizing S, and N is a normal subgroup of G contained in S.

Call a group K *thin* (resp. *right thin*, *left thin*) if whenever we have $R = S[G]$ as above with G/N isomorphic to an image of K then R is thin (resp. right thin, left thin) whenever S is. The class of all such groups K we denote by $\mathfrak{T}$ (resp. $\mathfrak{r}\mathfrak{T}$, $\mathfrak{l}\mathfrak{T}$). Note that trivially $\mathfrak{r}\mathfrak{T} \cap \mathfrak{l}\mathfrak{T} \subseteq \mathfrak{T}$, but equality here seems unlikely. By definition these three classes $\mathfrak{T}$, $\mathfrak{r}\mathfrak{T}$ and $\mathfrak{l}\mathfrak{T}$ are Q-closed and it is almost immediate that they are P-closed. By 5.2 we have the following. We let $\mathfrak{P}\mathfrak{F}$ denote the class of polycyclic-by-finite groups.

6.1. $\mathfrak{P}\mathfrak{F} \subseteq \mathfrak{r}\mathfrak{T} = \langle P, Q\rangle\mathfrak{r}\mathfrak{T}$ *and similarly with* $\mathfrak{l}\mathfrak{T}$.

6.2. $\mathfrak{T}$ *is closed under unions of ascending chains.*

Proof. Consider $K = \bigcup_{\alpha<\gamma} K_\alpha$, where each $K_\alpha \in \mathfrak{T}$ and $K_1 \leq K_2 \leq \ldots \leq K_\alpha \leq \ldots$. Suppose we have $R = S[G]$ as above with S thin, and a homomorphism of K onto G/N. Let H_α/N be the image of K_α in G/N. We need to consider certain prime images of R. It suffices to assume also that R is prime with max-ra and min-ra and to prove that R is Goldie. Since $K_\alpha \in \mathfrak{T}$, the ring $S[H_\alpha]$ is thin, so if it is also semiprime then it is Goldie by the definition of thinness and 1.3. Consequently R is Goldie by 5.4 and the proof is complete.

6.3. THEOREM. $\langle P, L\rangle(\mathfrak{A}\mathfrak{F}) \subseteq \mathfrak{T}^S = \langle P, L, Q, S\rangle\mathfrak{T}^S \subseteq \mathfrak{T} = \langle P, Q\rangle\mathfrak{T}$.

Proof. By 6.1 we have $\mathfrak{P}\mathfrak{F} \subseteq \mathfrak{r}\mathfrak{T} \cap \mathfrak{l}\mathfrak{T} \subseteq \mathfrak{T}$, so by the remarks preceding 6.1 we have

$$\mathfrak{P}\mathfrak{F} \subseteq \mathfrak{T}^S = \langle P, Q, S\rangle\mathfrak{T}^S \subseteq \mathcal{T} = \langle P, Q\rangle\mathfrak{T}.$$

Also $L\mathfrak{T}^S = \mathfrak{T}^S$ by 4.2 and 6.2 and clearly $\langle P, L\rangle(\mathfrak{P}\mathfrak{F}) = \langle P, L\rangle(\mathfrak{A}\mathfrak{F})$. The proof is complete.

QUESTION 3. Are any of $\mathfrak{T}$, $\mathfrak{r}\mathfrak{T}$ and $\mathfrak{l}\mathfrak{T}$ locally closed?

The theorem of the introduction is an immediate consequence of 6.3 and its first three corollaries also follow with no further argument. Its Corollary 4 is a consequence of (a) implies (c) of the next result.

6.4. *For a group K the following are equivalent.*

(a) *K is thin.*

(b) *Whenever we have $R = S[G]$ and N as above, with S Artinian, G/N an image of K and R prime with max-ra and min-ra, then R is Goldie.*

(c) *Whenever we have $R = S[G]$ and N as above, with S Goldie, G/N an image of K and R prime with max-ra and min-ra, then R is Goldie.*

Proof. (a) *implies* (b). Every Artinian ring is thin, so here S is thin, R is thin and therefore R is Goldie.

(b) *implies* (c). S is also semiprime by 5.1, so $\mathscr{C} = \mathscr{C}_S(0)$ is a divisor subset of S normalized by G. Hence $\mathscr{C}$ is a divisor subset of R, and further $\mathscr{C} \subseteq \mathscr{C}_R(0)$ by [**6**, 13] and [**1**, 1.30]. Thus we can form the ring $\mathscr{C}^{-1}R = R\mathscr{C}^{-1} = S\mathscr{C}^{-1}[G]$ of quotients. Now $S\mathscr{C}^{-1}$ is Artinian and $R\mathscr{C}^{-1}$ is prime and, by 1.4, satisfies max-ra and min-ra. By the definition of thinness the ring $R\mathscr{C}^{-1}$ is Goldie. It follows that R is too.

(c) *implies* (a). Suppose we have $R = S[G]$ and N with S thin and G/N an image of K. If $\mathfrak{p}$ is a prime ideal of R such that $R/\mathfrak{p}$ satisfies max-ra and min-ra then $(S+\mathfrak{p})/\mathfrak{p}$ is semiprime with max-ra and min-ra (5.1) and S is thin, so $(S+\mathfrak{p})/\mathfrak{p}$ is Goldie. Consequently $R/\mathfrak{p} = (S+\mathfrak{p}/\mathfrak{p})[G+\mathfrak{p}/\mathfrak{p}]$ is Goldie by (c). Therefore K is thin.

7. Subrings of Artinian rings. It remains only to prove Proposition 1 and 2. We need a few lemmas.

7.1. *Let A be a ring with nilpotent ideal $\mathfrak{r}$ such that $A/\mathfrak{r}$ is Artinian and suppose R is a subring of A. Then R has a semiprime nilpotent ideal $\mathfrak{n}$ such that $R/\mathfrak{n}$ is embeddable in a semisimple Artinian ring.*

Proof. Both A and R have nilpotent nilradicals. Let $\mathfrak{n}$ be the nilradical of R; we may assume $\mathfrak{r}$ is the nilradical of A. Clearly $\mathfrak{r} \cap R \subseteq \mathfrak{n}$, so we may pass to $A/\mathfrak{r}$ and assume that A is semisimple Artinian. Now $\mathfrak{n}$ is nilpotent, say $\mathfrak{n}^n = \{0\}$. Set $B = \bigoplus_{i=1}^{n} (A\mathfrak{n}^{i-1}/A\mathfrak{n}^i)$.

Clearly B is an A–R-bimodule. Let $\mathfrak{k}$ be the kernel of the action of R on B. Then $\mathfrak{n} \leq \mathfrak{k}$. Also $A\mathfrak{n}^{i-1}\mathfrak{k} \leq A\mathfrak{n}^i$ for each i, so $A\mathfrak{k}^n = \{0\}$, $\mathfrak{k}$ is nilpotent and $\mathfrak{k} = \mathfrak{n}$. Consequently $R/\mathfrak{n}$ embeds into $\text{End}_A B$. But A is semisimple Artinian, so ${}_AB$ is completely reducible and $\text{End}_A B$ is also semisimple Artinian.

7.2. *Assume the notation of* 7.1. *If* $\mathfrak{p}$ *is a minimal prime ideal of* R *then* $R/\mathfrak{p}$ *is embeddable in a simple Artinian ring.*

Proof. By 7.1, we may assume R is semiprime and A is semiprime Artinian. Let

$$A = A_0 > A_1 > \ldots > A_n = \{0\}$$

be a composition series of A as A–R-bimodule and set $\mathfrak{p}_i = \text{Ann}_R(A_{i-1}/A_i)$. Since A_{i-1}/A_i is irreducible, $\mathfrak{p}_i$ is prime. Also $A\mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_n = \{0\}$, so $\prod \mathfrak{p}_i = \{0\} \subseteq \mathfrak{p}$ and the minimality of $\mathfrak{p}$ yields that $\mathfrak{p} = \mathfrak{p}_i$ for some i. Then $R/\mathfrak{p}$ embeds into the semisimple Artinian ring $E = \text{End}_A(A_{i-1}/A_i)$. Since $\mathfrak{p}$ is prime, in fact $R/\mathfrak{p}$ will embed into a simple (Artinian) component of E.

7.3. *Let R be a semiprime subring of the matrix ring $D^{n\times n}$ over the division ring D and suppose $\mathscr{C}$ is a right divisor subset of regular elements of R. Then, for some $m \leq n$, the ring $R\mathscr{C}^{-1}$ of right quotients is isomorphic to a subring of $D^{m\times m}$.*

Proof. Repeat the proof of [**5**, 5.7.7]. (This is the special case of 7.3 where $\mathscr{C} = \mathscr{C}_R(0)$, but the proof given makes no use of this condition.)

7.4. *The proof of Corollary* 5. By 7.1 we may assume that A is semisimple Artinian. But R is prime, so we may in fact assume that $A = D^{n\times n}$ for some division ring D and positive integer n. Also S is semiprime and right Goldie. Set $\mathscr{C} = \mathscr{C}_S(0)$. Then $\mathscr{C}$ is a right divisor set of regular elements of R([**5**, 5.6.3], [**6**, 13] and [**1**, 1.30a]). Thus we can form the over-ring $R\mathscr{C}^{-1} = S\mathscr{C}^{-1}[G]$ of R and, by 7.3, embed it into $D^{m\times m}$ for some m. Now $S\mathscr{C}^{-1}$, being Artinian, is certainly thin. By the Theorem, $R\mathscr{C}^{-1}$ is thin, as well as prime with max-ra and min-ra. Therefore $R\mathscr{C}^{-1}$ is Goldie, from which it follows that R is right Goldie.

7.5. *The Proof of Proposition* 1. (If S is thin then R is thin by the theorem and a simple application of 7.1 and 1.2 yields the desired conclusion. In general we need to use 7.2 and Corollary 5 and hence also 7.3.)

By 7.1, we may assume that R is semiprime and that A is Artinian. Then, by 1.2, there is only a finite number of minimal prime ideals $\mathfrak{p}$ of R and, by 7.2, each $R/\mathfrak{p}$ embeds into an Artinian ring. If each $R/\mathfrak{p}$ right Goldie, so is R. Hence we may assume that $\mathfrak{p} = \{0\}$ and R is prime. Then $S \leq A$ is semiprime with max-ra and min-ra, and also S is right thin. Consequently S is right Goldie and therefore R is too by Corollary 5.

Addendum

In response to our paper, Hajarnavis has produced his note [**12**], which is devoted to an ingenious proof of the following extension of our 1.6.

1.6′. (Hajarnavis). *Let R be a semiprime right Goldie ring. Then every maximal chain of right annihilators in R has length u-*dim R_R.

Using this in place of 1.6, our proof of Proposition 2 immediately yields the following.

Proposition 2′. *Let R be a ring satisfying max-ra and min-ra and suppose R is locally (semiprime and right Goldie). Then R too is semiprime and right Goldie.*

This proposition in turn produces one-sided versions of 5.3, 5.4, 6.2 and 6.3 and hence answers Question 1 as follows.

Theorem′. *Let R, S, G and N be as in the Theorem. If S is right thin and $G/N \in \langle P, L\rangle(\mathfrak{A}\mathfrak{F})$ then R is right thin.*

Remark (added May 1990). In a revised version of **[12]**, Hajarnavis has now pushed his methods further, to answer Question 2 positively in the special case where R is semiprime.

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