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ON METANILPOTENT VARIETIES OF GROUPS

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1. Introduction. Let $\mathfrak{N}_c\mathfrak{N}_d$ $(c, d \geqq 1)$ denote the variety of all groups which are extensions of a nilpotent-of-class-c group by a nilpotent-of-class-d group, and let $\mathfrak{M}$ denote the variety of all metabelian groups. The main result of this paper is the following theorem.

THEOREM. *Let $\mathfrak{B}$ be a subvariety of $\mathfrak{N}_c\mathfrak{N}_d$ which does not contain $\mathfrak{M}$. Then every $\mathfrak{B}$-group is an extension of a group of finite exponent by a nilpotent group by a group of finite exponent. In particular, a finitely generated torsion-free $\mathfrak{B}$-group is a nilpotent-by-finite group.*

This generalizes the main theorem of Šmel′kin [**4**], where the same result is proved for subvarieties of $\mathfrak{N}_c\mathfrak{A}$, where $\mathfrak{A}$ is the variety of abelian groups. See also Lewin and Lewin [**2**] for a related discussion.

2. Notation. For unexplained notation, the reader is referred to Neumann [**3**]. The most frequently used notation is the following:

$[x, y] = x^{-1}y^{-1}xy$;

$[x, y, z] = [[x, y], z]$;

$[H, K] = gp\{[x, y]; x \in H, y \in K\}$ where H, K are subgroups;

$[H, 1K] = [H, K]$ and $[H, tK] = [H, (t-1)K, K]$ for $t \geqq 2$;

G^m: the subgroup of G generated by mth powers of elements of G;

$\gamma_m(G)$: the mth term of the lower central series of G;

$\delta_m(G)$: the mth term of the derived series of G.

3. Preliminary lemmas. For a variety $\mathfrak{U}$, let $\mathfrak{U}^{(2)}$ denote the variety all of whose 2-generator groups are in $\mathfrak{U}$.

LEMMA 1. *If $\mathfrak{M} \subseteq \mathfrak{U}^{(2)}$, then $\mathfrak{M} \subseteq \mathfrak{U}$.*

Proof. This follows from [**3**, 25.34].

LEMMA 2. *Let $\mathfrak{B}$ be a variety which contains $\mathfrak{U}$ but does not contain $\mathfrak{M}$ and let $G \in \mathfrak{B}$. Then for every normal subgroup N of G contained in G',*

$$[N, tG^m]^n \leqq [N, G'],$$

where t, m, and n are fixed positive integers.

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Proof. Let F_2 be the free group on x, y and let Φ be the fully invariant subgroup of F_2 corresponding to $\mathfrak{V}$. Since $\mathfrak{M} \not\subseteq \mathfrak{V}$, it follows by Lemma 1 that $\mathfrak{M} \not\subseteq \mathfrak{V}^{(2)}$. Thus for some $\varphi \in \Phi$, $\varphi = [x, y]^{p(x,y)}$, where $p(x, y)$ is a non-zero polynomial of the integral group ring $\mathbf{Z}(F_2/F_2')$. In φ, replacing y by $x^l u$ ($u \in F_2'$) for a suitable large integer l shows that Φ contains an element $\varphi' = u^{q(x)}$, where $q(x)$ is a non-zero polynomial.

For the rest of the proof, we may assume that $[N, G'] = E$. Since $G \in \mathfrak{V}$, it follows that $u^{q(x)} = 1$ for all $u \in N$ and all $x \in G$. By [**1**, Lemma 1], $[u, x_1, x_2^2, \ldots, x_t^t]^n = 1$ for all $u \in N$, $x_1, \ldots, x_t \in G$, where t is the degree of $q(x)$ and n is its leading coefficient. Putting $m = t!$ and replacing x_i by $x_i^{m/i}$ yields the desired result.

LEMMA 3. *Let G be a group and let m and d be fixed positive integers. Then*

$$\delta_k(G^{m(d,k)}) \leqq \delta_k^{\,m}(G) \pmod{\gamma_{d+1}(G)}$$

for all $k \geqq 1$, where $m(d, k) = m^{(d-1)k}$.

Proof. It is easy to prove by induction on k that if $B, A_1, \ldots, A_k$ are normal subgroups of G, then

$$[B^{m(d,k)}, A_1, \ldots, A_k] \leqq [B, A_1, \ldots, A_k]^m \pmod{\gamma_{d+1}(G)}.$$

Now taking $B = G$ and $A_i = \delta_{i-1}(G^{m(d,k)})$ for $i = 1, \ldots, k$ yields the desired result.

LEMMA 4. *Let $\mathfrak{V}$, m, n, and t be as in Lemma 2. Then for every normal subgroup N of G contained in $\gamma_{d+1}(G)$,*

$$[N, t(k)G^{m(k)}]^{n(k)} \leqq [N, \delta_k(G)][N, \gamma_{d+1}(G)]$$

for all $k \geqq 1$, where $t(k) = t^k$, $m(k) = m^{1+(d-1)+\cdots+(d-1)^{k-1}}$ and $n(k) = n^{1+t+\cdots+t^{k-1}}$

The proof is by straightforward induction using Lemmas 2 and 3.

LEMMA 5. *Let H be a torsion-free normal nilpotent subgroup of a group G such that $[H^n, tG] = E$ for some positive integers n and t. Then $[H, tG] = E$.*

Proof. Let $\gamma_{c+1}(H) = E$ and let $w(x, t, c-k)$ be any left-normed commutator of weight at least $1 + t + c - k$ whose first entry is x and whose remaining entries contain at least $c - k$ elements of H. Then we prove by induction on $k \in \{0, \ldots, c\}$ that $w(h, t, c-k) = 1$ for all $h \in H$. When $k = 0$,

$$w(h, t, c) \in \gamma_{c+1}(H) = E.$$

Assume the result for some $k \in \{0, \ldots, c-1\}$. We have

$$1 = w(h^n, t, c-(k+1)) = w^n(h, t, c-(k+1)) \cdot u,$$

where u is a product of commutators of type $w(h, t, c-k)$. By the induction hypothesis, $u = 1$ and so $w^n(h, t, c-(k+1)) = 1$; and since H is torsion-free, it follows that $w(h, t, c-(k+1)) = 1$ as was required. In particular, $w(h, t, 0) = 1$ for all $h \in H$ and we have $[h, g_1, \ldots, g_t] = 1$ for all $h \in H$ and $g_1, \ldots, g_t \in G$.

LEMMA 6. *Let $\mathfrak{M} \not\subseteq \mathfrak{B} < \mathfrak{N}_c\mathfrak{N}_d$ and let $G \in \mathfrak{B}$ be such that $\gamma_{d+1}(G)$ is torsion-free. Then for some integer s, G^s is nilpotent.*

Proof. By Lemma 4, $[N, t(k)G^{m(k)}]^{n(k)} \leqq [N, \delta_k(G)][N, \gamma_{d+1}(G)]$ for all $k \geqq 1$. Choose an integer k such that $2^k \geqq d + 1$. Using Lemma 4 c times yields

$$[\ldots [N, t(k)G^{m(k)}]^{n(k)}, \ldots, t(k)G^{m(k)}]^{n(k)} \leqq [N, c\gamma_{d+1}(G)] = E.$$

Since N is torsion-free nilpotent, repeated applications of Lemma 5 yield $[N, c \cdot t(k)G^{m(k)}] = E$. Putting $m(k) = s$ and $N = \gamma_{d+1}(G^s)$ yields $\gamma_r(G^s) = E$, where $r = d + 1 + c \cdot t(k)$.

4. Proof of the Theorem. If $\mathfrak{A} \not\subseteq \mathfrak{B}$, then $\mathfrak{B}$ is of finite exponent. Thus we may assume that $\mathfrak{A} \subseteq \mathfrak{B}$. Let $G = F_\infty(\mathfrak{B})$. Since $\gamma_{d+1}(G)$ is nilpotent, the periodic elements of $\gamma_{d+1}(G)$ form a characteristic subgroup H of G. Put $K = G/H$, so that $\gamma_{d+1}(K)$ is torsion-free; and by Lemma 6, $\gamma_r(K^s) = E$ for some integer s and some integer $r \geqq d + 1$. In particular, $[x_1^s, \ldots, x_r^s] \in H$, where $x_1, \ldots, x_r$ are among free generators of G. Since H is periodic, $[x_1^s, \ldots, x_r^s]^l = 1$ for some integer l; and since G is relatively free, $[g_1^s, \ldots, g_r^s]^l = 1$ for all $g_1, \ldots, g_r \in G$. The nilpotency of $\gamma_{d+1}(G)$ implies that $\gamma_r(G^s)$ is of fixed exponent $e = l^{f(c)}$. Thus we conclude that $G \in \mathfrak{B}_e\mathfrak{N}_{r-1}\mathfrak{B}_s$. This completes the proof of the theorem.

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