RELATIVISTIC REDUCTION OF ASTRONOMICAL MEASUREMENTS AND REFERENCE FRAMES

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ABSTRACT

With the accuracy of modern observations the relativistic treatment of the basic astronomical reference frames only requires the consideration of comparatively simple types of metrics such as heliocentric Schwarzschild metric, geocentric Schwarzschild metric and metric of the Earth-Sun system. Dynamical (related to the motion of the bodies) and kinematical (related to the light propagation) characteristics of these metrics enable one to perform the accurate relativistic reduction of astronomical measurements. In this reduction, the choice of specific quasi-Galilean coordinates may remain arbitrary. This paper presents expressions for the main relativistic terms in coordinates of the principal planets and Moon using the PPN formalism parameters β, γ and coordinate parameter α. General formulae for the reduction of radar, radio-interferometric and astrometric observations of planets and for the interpretation of lunar laser ranging are given. For estimating the actual magnitude of relativistic effects, the ephemeris data should be expressed in terms of physically measurable quantities.

INTRODUCTION

Relativistic treatment of the problem of astronomical reference frames involves some distinctive features compared to the Newtonian treatment.

First of all, much attention is given now to the physical approach to this problem. Just as the problem of time measurement changed from astronomy to physics, the definition of three orthogonal directions in space will possibly be made in the future by laboratory means (such as gyroscopes) rather than with the use of astronomical objects. Such a laboratory measurement of time and determination of space directions provide a reference frame suitable for astronomy as well. Mathematically this system is represented by four vectors subjected to Fermi-Walker propagation along the world-line of the laboratory. They

E. M. Gaposchkin and B. Kołaczek (eds.), Reference Coordinate Systems for Earth Dynamics, 283–294.

consist of the time-like tangent to the world-line and three space-like vectors normal to it. This approach is developed in detail by Synge (1960). In spite of extensive recent investigations of the mathematical aspects of this problem, the presentation by Synge still remains the most adequate for astronomical practice.

In addition, a relativistic reduction of astronomical observations becomes increasingly important. In essence, astronomical observations reduce to measuring angles between light rays or time intervals between events marked by light signals. Relativistic treatment of these quantities is due mainly to the effect of the solar gravitational field. It is necessary to take into account both the direct influence of this field on light propagation and the effect of performing measurements in a curved space. Observations reduced in this way may be further used for determining astronomical reference frames by classical methods.

Another feature of the relativistic analysis is of a purely mathematical nature and is related with the possibility to use arbitrary quasi-Galilean coordinates for describing events in the solar system. Even though the mathematical character of the motion of bodies and the propagation of light are different in distinct quasi-Galilean coordinates, the physically measurable quantities do not depend on the choice of coordinates. It is only important to calculate dynamical characteristics and to perform relativistic reduction of kinematic data in a single coordinate system.

This paper deals with questions pertaining to the last two facts. Sections 2 and 3 are devoted to the reduction of radar, radio interferometric and astrometric observations. In extending the results of Brumberg (1979, 1981) and Brumberg and Finkelstein (1979), the geocentric Schwarzschild metric is considered as enabling us to combine, in a unified manner, both relativistic gravitation and aberration effects. In Section 4, the lunar laser ranging is considered. It is shown that, in opposition to the views of Baierlein (1967), the influence of the relativistic effects does not result in a mere multiplication of Newtonian quantities by a constant factor. The whole treatment is performed in the Parametrized Post-Newtonian approximation (PPN) formalism (Will, 1974) but in arbitrary (to the sufficient degree) quasi-Galilean coordinates.

HELIOCENTRIC SCHWARZSCHILD METRIC

Metric

Heliocentric Schwarzschild metric is of the form where $m = GM_{\odot}/c^2$

$$ds^2 = \left\{1 - 2(m/r) + 2[\beta - \alpha(r)]\,(m/r)^2\right\} c^2dt^2$$
$$- \left\{1 + 2[\gamma - \alpha(r)]\,(m/r)\right\} (d\underline{r})^2 - 2[\alpha(r) - r\alpha'(r)]\,(m/r^3)(\underline{r}d\underline{r})^2 \quad , \tag{1}$$

https://doi.org/10.1017/S0252921100081409 Published online by Cambridge University Press

β, and γ are principal constants of the PPN formalism (for GRT, $\beta=\gamma=1$), $\alpha(r)$ is an arbitrary function of r satisfying the conditions of a quasi-Galilean metric: $\alpha'(r)\to 0$, $\alpha(r)/r\to 0$ with $r\to\infty$ (the dash denotes differentiation with respect to r). Transformation

$$\underline{\tilde{r}} = \underline{r} - m\alpha(r)\underline{r}/r \tag{2}$$

converts (1) to the Eddington-Robertson metric (metric (1) with $\alpha=0$). The most widely used coordinate systems correspond to $\alpha=0$ (harmonic coordinates) or $\alpha=1$ ("standard" coordinates). Introduction of $\alpha(r)$ in a literal form has two objectives. First of all, the appearance of α in the expression of any particular quantity demonstrates its co-ordinate dependence, in other words its unobservability. Besides this, in final relations in terms of the ephemeris of physically measurable quantities, α should disappear, which would confirm the correctness of the calculations.

Motion of the Major Planets

For approximate analytical estimations, it is convenient to have the following expansions of polar orbital coordinates (radius-vector r, argument of latitude u) in powers of eccentricity e:

$$\frac{r}{a} = 1 + \frac{1}{2}\left[1 + \mu k_0(a)\right]e^2 - [1 + \mu k_1(a)]\ e \cos(\lambda - \pi)$$

$$- \frac{1}{2}\left[1 + \mu k_2(a)\right]\ e^2 \cos 2(\lambda - \pi) + \ldots \quad , \tag{3}$$

$$u = \lambda - \Omega + 2e \sin(\lambda - \pi) + \left[\frac{5}{4} + \mu\left(-3-3\gamma + \frac{5}{4}\beta\right)\right] e^2 \sin 2(\lambda - \pi) + \ldots \tag{4}$$

These expansions result from the closed expressions of the Schwarzschild problem (Brumberg, 1972). Here

$$k_0(a) = -\frac{4}{3} - \frac{4}{3}\gamma + \frac{2}{3}\beta - \alpha(a) + a\alpha'(a) + \frac{1}{2}a^2\alpha''(a) \quad ,$$

$$k_1(a) = -1 - \gamma + \beta - \alpha(a) + a\alpha'(a) \quad ,$$

$$k_2(a) = -4 - 4\gamma + 2\beta - \alpha(a) + a\alpha'(a) - \frac{1}{2}a^2 a''(a) \quad ,$$

$\mu = m/a$, λ, π, and Ω are angular arguments (mean longitude, longitude of the perihelion, longitude of the node) with Ω being constant, λ and π being linear functions of time with mean motions:

$$\dot{\lambda} = n, \qquad \dot{\pi} = \mu(2 + 2\gamma - \beta)\ n/(1 - e^2) \quad , \tag{5}$$

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$$n = \left(GM_\odot/a^3\right)^{1/2} \left\{1 + \mu\left[-\frac{1}{2}\gamma - \beta + \frac{3}{2}\alpha(a)\right]\right\} \tag{5}$$

Light Propagation

The law of light propagation under conditions

$$\underline{r}(t_0) = \underline{r}_0 \quad , \quad \dot{\underline{r}}(-\infty) = c\underline{\sigma} \quad , \quad \underline{\sigma}^2 = 1$$

is given by:

$$\underline{r}(t) = \underline{r}_0 + c(t - t_0)\,\underline{\sigma} + m\left\{(\gamma + 1)\left[\frac{\underline{\sigma} \times (\underline{r}_0 \times \underline{\sigma})}{r_0 - \underline{\sigma}\underline{r}_0} - \frac{\underline{\sigma} \times (\underline{r} \times \sigma)}{r - \underline{\sigma}\underline{r}}\right.\right.$$
$$\left.\left. - \underline{\sigma}\ln\frac{r + \underline{\sigma}\,\underline{r}}{r_0 + \underline{\sigma}\,\underline{r}_0}\right] + \frac{\alpha(r)}{r}\underline{r} - \frac{\alpha(r_0)}{r_0}\underline{r}_0\right\} \quad . \tag{6}$$

The velocity of light at the time t is obtained from:

$$\frac{1}{c}\dot{\underline{r}}(t) = \underline{\sigma} + \frac{m}{r}\left\{[\alpha(r) - \gamma - 1]\,\underline{\sigma} - [\alpha(r) - r\alpha'(r)]\frac{(\underline{\sigma}\,\underline{r})}{r^2}\underline{r}\right.$$
$$\left. - (\gamma + 1)\frac{\underline{\sigma} \times (\underline{r} \times \underline{\sigma})}{r - \underline{\sigma}\,\underline{r}}\right\} \quad . \tag{7}$$

For a solar ray,

$$\underline{\sigma} = (\underline{r}_0/r_0) = (\underline{r}/r) \quad .$$

The solution of the boundary value problem

$$\underline{r}(t_0) = \underline{r}_0 \quad , \quad \underline{r}(t_1) = \underline{r}_1$$

is determined by (6) and (7) with

$$\underline{\sigma} = \frac{\underline{D}_{01}}{D_{01}} + \frac{m}{D_{01}}\left\{(\gamma + 1)\frac{r_1 - r_0 + D_{01}}{(\underline{r}_0 \times \underline{r}_1)^2} + \frac{1}{D_{01}^2}\left[\frac{\alpha(r_0)}{r_0} - \frac{\alpha(r_1)}{r_1}\right]\right\}$$
$$\times \left[\underline{D}_{01} \times (\underline{r}_0 \times \underline{r}_1)\right] \tag{8}$$

where

$$\underline{D}_{01} = \underline{r}_1 - \underline{r}_0$$

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That time of flight will be

$$t_1 - t_0 = \frac{D_{01}}{c} + \frac{m}{c}\left\{(\gamma + 1)\ \ln \frac{r_0 + r_1 + D_{01}}{r_0 + r_1 - D_{01}} + \alpha(r_0)\ \frac{r_1^2 - r_0^2 - D_{01}^2}{2r_0 D_{01}} + \alpha(r_1)\ \frac{r_0^2 - r_1^2 - D_{01}^2}{2r_1 D_{01}}\right\} \quad . \tag{9}$$

For a solar ray (with $r_1 > r_0$) ,

$$t_1 - t_0 = \frac{1}{c}\left\{r_1 - r_0 + m\left[(\gamma + 1)\ \ln \frac{r_1}{r_0} - \alpha(r_1) + \alpha(r_0)\right]\right\} \quad . \tag{10}$$

Doppler Shift

Let ν_0 be the frequency of a light emitter at a given point $\underline{r}_0(t_0)$ and ν_1 be the frequency of the light at a receiver situated at a point $\underline{r}_1(t_1)$. Then

$$\frac{\nu_0}{\nu_1} = \frac{1 - m/r_1 - \dot{\underline{r}}_1^2/(2c^2)}{1 - m/r_0 - \dot{\underline{r}}_0^2/(2c^2)}\ \frac{dt_1}{dt_0} \quad . \tag{11}$$

GEOCENTRIC SCHWARZSCHILD METRIC

Metric

The transformation

$$\underline{r} = \underline{R}(t) + \rho \tag{12}$$

where $\underline{R}(t)$ is the heliocentric vector of the Earth transforming (1) to the geocentric Schwarzschild metric

$$\begin{aligned} ds^2 = &\Big\{1 - 2(m/r) - (\dot{\underline{R}}/c)^2 + 2\ [\beta - \alpha(r)]\ (m/r)^2 - 2(m/r)\ [\gamma - \alpha(r)] \\ &\quad \times (\dot{\underline{R}}/c)^2 \\ &- 2(m/r^3)\ [\alpha(r) - r\alpha'(r)]\ (\underline{r}\dot{\underline{R}}/c)^2\Big\}\ c^2dt^2 - 2\Big\{\dot{\underline{R}} + 2(m/r)\ [\gamma - \alpha(r)]\ \dot{\underline{R}} \\ &+ 2(m/r^3)\ [\alpha(r) - r\alpha'(r)]\ (\underline{r}\dot{\underline{R}})\underline{r}\Big\}\ d\underline{\rho}\ dt - \Big\{1 + 2(m/r)\ [\gamma - \alpha(r)]\Big\}\ d\underline{\rho}^2 \\ &- 2(m/r^3)\ [\alpha(r) - r\alpha'(r)]\ (\underline{r}d\underline{\rho})^2 \quad . \end{aligned} \tag{13}$$

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This metric describes the solar gravitational field but from the geocentric point of view. The line-element $d\ell$ of space distance of this metric is determined by

$$d\ell^2 = \left\{1 + 2(m/r)\,[\gamma - \alpha(r)]\right\} d\underline{\rho}^2 + 2(m/r^3)\,[\alpha(r) - r\alpha'(r)]\,(\underline{r}d\underline{\rho})^2 + (\dot{\underline{R}}d\underline{\rho}/c)^2 \quad . \tag{14}$$

For the case of the synchronization of clocks it results that instants t_0 at a point $\underline{\rho}$ and t_0' at a point $\underline{\rho} + d\underline{\rho}$ are simultaneous provided that

$$t_0' = t_0 + \dot{\underline{R}}d\underline{\rho}/c^2 \quad . \tag{15}$$

Lengths and Angles

If $\underline{P}$, $\underline{Q}$ are three-dimensional vectors applied at a point $\underline{r}(t)$, their scalar product and length in space (14) is

$$(\underline{PQ})_{rel} = \underline{PQ} + 2(m/r)\,[\gamma - \alpha(r)]\,(\underline{PQ}) + 2(m/r^3)\,[\alpha(r) - r\alpha'(r)] \times (\underline{Pr})(\underline{Qr}) + (\dot{\underline{R}}\underline{P})(\dot{\underline{R}}\underline{Q})/c^2 \quad , \tag{16}$$

$$P_{rel} = P\left\{1 + (m/r)\,[\gamma - \alpha(r)] + (m/r^3)\,[\alpha(r) - r\alpha'(r)]\,(\underline{Pr})^2/P^2 + (\dot{\underline{R}}\underline{P})^2/(2c^2P^2)\right\} \quad , \tag{17}$$

where $\underline{PQ}$ and $P = |\underline{P}|$ denote the scalar product and the length in Euclidean sense. For the angle ψ between vectors $\underline{P}$, $\underline{Q}$ if we define $(\underline{PQ})_{rel} = P_{rel}\,Q_{rel}\cos\psi$, one has

$$\cos\psi = (\underline{PQ})/(PQ) + \left\{(m/r^3)\,[\alpha(r) - r\alpha'(r)]\,[(\underline{Pr})(\underline{P} \times \underline{r})/P^2 - (\underline{Qr}) \times (\underline{Q} \times \underline{r})/Q^2] + [(\underline{P}\dot{\underline{R}})(\underline{P} \times \dot{\underline{R}})/P^2 - (\underline{Q}\dot{\underline{R}})(\underline{Q} \times \dot{\underline{R}})/Q^2]/(2c^2)\right\} \times (\underline{P} \times \underline{Q})/(PQ) \quad . \tag{18}$$

Geocentric Angle Between Light Rays

The direction of a light ray (7) crossing the earth in a position $\underline{R}$ at the time t may be determined by a vector $\underline{P} = \dot{\underline{\rho}}/c$ with $\dot{\underline{\rho}}$ to be computed from (12) and (7) where we set $\underline{r} = \underline{R}$. Calculating the Euclidean length P of this vector, one finds

$$\underline{P}/P = \underline{\sigma} + [\underline{\sigma} \times (\underline{\sigma} \times \dot{\underline{R}})]/c + (\underline{\sigma}\dot{\underline{R}})\,[\sigma \times (\sigma \times \dot{\underline{R}})]/c^2 - \sigma(\underline{\sigma} \times \dot{\underline{R}})^2/(2c^2)$$

$$+ \frac{m}{R}\left\{[\alpha(R) - R\alpha'(R)]\,\frac{(\underline{\sigma}\underline{R})}{R^2} + \frac{\gamma + 1}{R - \underline{\sigma}\underline{R}}\right\}[\underline{\sigma} \times (\underline{\sigma} \times \underline{R})] \quad . \tag{19}$$

Hence, from (18) there results an expression for geocentric angle ψ between two light rays having at $t = -\infty$ the directions $\underline{\sigma}_1$, $\underline{\sigma}_2$ and crossing the earth, in the position $\underline{R}$ at time t:

$$\cos\psi = \underline{\sigma}_1\underline{\sigma}_2 + \frac{1}{c}(\sigma_1\sigma_2 - 1)(\underline{\sigma}_1\dot{\underline{R}} + \underline{\sigma}_2\dot{\underline{R}}) + \frac{1}{c^2}(\underline{\sigma}_1\underline{\sigma}_2 - 1)\Big[(\underline{\sigma}_1\dot{\underline{R}})^2 + (\underline{G}_2\dot{\underline{R}})^2$$

$$+ (\underline{\sigma}_1\dot{\underline{R}})(\underline{\sigma}_2\dot{\underline{R}}) - \underline{R}^2\Big] + (\gamma + 1)\,\frac{m}{R}\left(\frac{\underline{R} \times \underline{\sigma}_1}{R - \underline{R}\underline{\sigma}_1} - \frac{\underline{R} \times \underline{\sigma}_2}{R - \underline{R}\underline{\sigma}_2}\right)(\underline{\sigma}_1 \times \underline{\sigma}_2) \quad . \tag{20}$$

Relativistic Reduction of Astronomical Measurements

For relativistic reduction, of astronomical measurements one may use relations (9) (radar ranging, radio-interferometry), (11) (Doppler's observations) and (20) (astrometry). The detailed analysis of different cases of this reduction is given in Brumberg, (1981) for the Earth at rest ($\dot{\underline{R}}$=0). Let us restrict ourselves to some basic statements.

Radar ranging. A round-trip transit reduces in essence to a double expression (9). If the coordinate dependent distance D_{01} is expressed in terms of initial measured quantities, then coordinate dependence on α in (9) disappears. The numerical value of the corresponding relativistic effect and its functional dependence on β, γ may be determined by the initial measurements (Brumberg, Finkelstein, 1979).

VLBI. Let a radio wave coming from an infinitely distant source reach station 2 on the Earth at a time t_2 and station 1 at a time t_1. At station 1, one measures a time delay $\tau = t_1 - t_2'$, t_2' being the time at station 1 simultaneous with t_2 at station 2. The expression for τ may be found from $\tau = (t_1 - t_0) - (t_2 - t_0) - (t_2' - t_2)$ (t_0 is the time of the wave emission) by using (15) for $t_2' - t_2$ and (9) for $t_i - t_0$ ($i = 1,2$) with the limit $r_0 \to \infty$. Further use of (17)-(19) permits to express τ in terms of physically measurable quantities such as the proper length of the base vector and the angle between the base vector and the direction to the radio souce (Brumberg, 1981).

Angular measurements. Relation (20) covers a great variety of types of astrometric observations. Let, for instance, $\underline{\sigma}_1$ be the direction of a light ray at $t = -\infty$ from an infinitely distant source. Then if we let $\underline{\sigma}_2 = \dot{\underline{R}}/|\dot{\underline{R}}|$ we obtain the special relativistic formula for the aberration, if we let $\underline{\sigma}_2 = \underline{R}/R$ we obtain the formula for the light deflection, and if $\underline{\sigma}_2$ is determined by (8) then we get an expression of

https://doi.org/10.1017/S0252921100081409 Published online by Cambridge University Press

the angle between an observed planet and a distant source. Choosing the last case, $\underline{\sigma}_1 = \underline{R}/R$, one gets the expression for the angular distance of a planet from the Sun. Presenting coordinates of the planet and the Sun in terms of initial measurements, one obtains coordinate independent relativistic effects determined only by measurements (Brumberg and Finkelstein, 1979).

Regarding $\underline{R}(t)$ in (12) and (13) as heliocentric vector of the point of observation on the Earth, we obtain the topocentric Schwarzschild metric enabling to take into account both the annual and diurnal motions of the Earth. In this matter we may combine the relativistic gravitational effects with those due to the special theory of relativity. In practical calculations these reductions may be separated as it was assumed in the above-mentioned papers.

LUNAR LASER RANGING

The Field of n Point Masses

Using the parameters β, γ of the PPN formalism (Will, Nordtvedt, 1972) and the coordinate parameters α, ν (Brumberg, 1972), the field of n non-rotating point masses M_i is described by the following metric

$$
\begin{aligned}
ds^2 = \Bigg\{ & 1 - 2\sum_i \frac{m_i}{\rho_i} + 2(\beta - \alpha)\left(\sum_i \frac{m_i}{\rho_i}\right)^2 + (4\beta - 2) \sum_i \frac{m_i}{\rho_i} \sum_{j\neq i} \frac{m_j}{r_{ij}} \\
& + 2\alpha \sum_i \frac{m_i}{\rho_i^3} \sum_{j\neq i} m_j \left(\frac{1}{r_{ij}} - \frac{1}{\rho_j}\right)(\underline{\rho}_i \underline{r}_{ij}) - \frac{1}{c^2}(2\gamma + 1) \sum_i \frac{m_i}{\rho_i} \dot{\underline{r}}_i^2 \\
& + \frac{1}{c^2}(\nu - 1) \frac{\partial^2}{\partial t^2} \sum_i m_i \rho_i \Bigg\} c^2 dt^2 + \frac{2}{c} \sum_i \frac{m_i}{\rho_i} \Bigg[\left(2\gamma + 2 - \alpha - \frac{\nu}{2}\right) \dot{\underline{r}}_i \\
& + \left(\alpha + \frac{\nu}{2}\right) \frac{1}{\rho_i^2} (\underline{\rho}_i \dot{\underline{r}}_i)\, \underline{\rho}_i \Bigg] d\underline{r}\, c\,dt - 2\alpha \sum_i \frac{m_i}{\rho_i^3} (\underline{\rho}_i\, d\underline{r})^2 - \Bigg[1 + 2(\gamma - \alpha) \\
& \times \sum_i \frac{m_i}{\rho_i} \Bigg] (d\underline{r})^2
\end{aligned}
\tag{21}
$$

where $\underline{\rho}_i = \underline{r} - \underline{r}_i$, $\underline{r}_{ij} = \underline{r}_i - \underline{r}_j$, $m_i = GM_i/c^2$. In contrast to (1), α is constant. In papers on PPN formalism, the coordinate system is fixed by the choice $\alpha = 0$, $\nu = 1$. In papers on GRT, harmonic system, with $\alpha = \nu = 0$ are used. Denoting harmonic coordinates by $\sim$ we have

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$$\tilde{t} = t + \frac{\nu}{2c^2}\frac{\partial}{\partial t}\sum_i m_i \rho_i \quad , \tag{22}$$

$$\underline{\tilde{r}} = \underline{r} - \alpha \sum_i \frac{m_i}{\rho_i}\underline{\rho}_i \quad , \quad \underline{\tilde{r}}_i = \underline{r}_i - \alpha \sum_{j\neq i} \frac{m_j}{r_{ij}}\underline{r}_{ij} \quad . \tag{23}$$

In the post-Newtonian approximation the parameter ν does not affect the motion of bodies and light propagation.

Main Relativistic Terms in Lunar Theory

Let indices 1 and 2 in (21) be referred to the Earth and Sun respectively. Taking into account that $m_1 \ll m_2$, put $\underline{r}_1 = R$, $\underline{r}_2 = 0$, $\underline{R}$ being the heliocentric vector of the Earth determined by (3), (4) converted to the geocentric system $\underline{r} = \underline{R} + \underline{\rho}_1$. The Moon will be considered as a probe particle moving in the field (21) of the Earth and the Sun. Then the equations of lunar motion follow from the geodesic principle and are described by the Lagrangian $L = c^2 - c(ds/dt)$, if we consider here only a point-mass the results obtained eliminate the Nordtvedt effect (breaking the principle of equivalence for massive bodies). Thus the variational inequalities turn out to be of the most importance. Neglecting the sun's eccentricity, the eccentricity and inclination of the lunar orbit and parallactic terms, one may find for the radius-vector ρ_1 and longitude v of the Moon the following expressions

$$\frac{\rho_1}{\underline{a}_0} = 1 - m^2 \cos 2D + \ldots + \mu \left\{ -\frac{2}{3}\gamma - \frac{4}{3}\beta - \frac{1}{4} + \frac{1}{2}\alpha + \frac{1}{3}(2\gamma + 1)m \right.$$
$$+ \left(\frac{1}{3}\beta - \frac{2}{3}\gamma - \frac{109}{96} + \frac{15}{16}\alpha\right)m^2 + \left[\frac{1}{4} - \frac{1}{2}\alpha + \left(\frac{1}{4} + 2\gamma + 2\beta - \frac{1}{2}\alpha\right)m^2\right]\cos 2D$$
$$\left. + \left(\frac{7}{32} - \frac{7}{16}\alpha\right)m^2 \cos 4D \right\} \quad , \tag{24}$$

$$v = nt + \varepsilon + \frac{11}{8}m^2 \sin 2D + \ldots + \mu \left\{ \left[\frac{1}{2}\alpha - \frac{1}{4} - \frac{11}{12}(\beta + 2\gamma)m^2\right] \sin 2D \right.$$
$$\left. + \left(\frac{11}{16}\alpha - \frac{11}{32}\right)m^2 \sin 4D \right\} \quad . \tag{25}$$

Here n is the lunar mean motion, N is the mean motion of the Sun related to the semimajor axis A of the Earth orbit by (5), $\mu = N^2A^2/c^2$ is consistent with values for μ as in (3), (4), $m = N/(n - N)$, D is the difference of the mean longitudes of the Moon and the Sun, $\underline{a}_0$ is the Hill's parallactic factor

$$a_0 = a_0(1 - \frac{1}{6}m^2 + \ldots), \qquad n^2 a_0^3 = GM_1 \quad .$$

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These terms are obtained up to m^2 inclusively. For $\alpha = 0$ they agree with the expressions of Finkelstein and Kreinovich (1976).

Lunar Laser Ranging

In accordance with (21), the time $t - t_0$ of light propagation between points $\underline{r}_0$ and $\underline{r}$, neglecting the motion of the bodies, is determined by

$$c(t - t_0) = |\underline{r} - \underline{r}_0| + \sum_i m_i \left\{ (\gamma + 1) \ln \frac{\rho_i + \rho_i^{(0)} + |\underline{\rho}_i - \underline{\rho}_i^{(0)}|}{\rho_i + \rho_i^{(0)} - |\underline{\rho}_i - \underline{\rho}_i^{(0)}|} + \frac{\alpha}{2\rho_i \rho_i^{(0)}} \frac{(\rho_i + \rho_i^{(0)})}{|\underline{\rho}_i - \underline{\rho}_i^{(0)}|} \left[\left(\underline{\rho}_i - \underline{\rho}_i^{(0)}\right)^2 - \left(\underline{\rho}_i - \underline{\rho}_i^{(0)}\right)^2 \right] \right\} . \quad (26)$$

Introduce the distance S between the Earth station and a reflector on the Moon, the distance d between the centers of mass of the Earth and Moon, the heliocentric distance R of the Earth, the geocentric angle H between the directions to the Sun and Moon and the radii d_e, d_m of the Earth and Moon respectively. Considering the lunar laser ranging is usually performed near the meridian of the laser station one may put in relativistic terms $S = d - d_e - d_m$ and $\rho_1^{(0)} = d_e$, $\rho_1 = d - d_m$, $\rho_2^{(0)} = R$, $\rho_2 = R - d \cos H$. Therefore, the round-trip coordinate time interval will be

$$T = \frac{2S}{c}\left[1 + \frac{m_2}{R} (\gamma + 1 - \alpha \sin^2 H)\right] + \frac{m_1}{c} (2\gamma + 2) \ln \frac{d - d_m}{d_e} . \quad (27)$$

For harmonic coordinates $\alpha = 0$ this formula was derived first by Baierlein (1967). Noticing the proportionality of the right-hand member of (27) to S/c with practically a constant factor (the last term in (27) may be multiplied within the adopted accuracy by S/d), Baierlein came to conclusion that it is impossible to reveal relativistic effects in measuring T. But the right-hand member of Baierlein's formula represents a coordinate-dependent expression valid only in harmonic coordinates. In the Newtonian, part the distance S is not a physically measurable quantity. The expression of S in terms of measurable quantities (mean motions of the Moon and Sun and gravitational parameters of the Earth and Sun) contains time-dependent relativistic corrections. In accordance with (24), Newtonian value S_N may be presented as follows

$$S_N = \underline{a}_0(1 - m^2 \cos 2D + \ldots) - d_e - d_m . \quad (28)$$

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Putting S = d in relativistic terms and using (24), there results

$$T = \frac{2S_N}{c} + \frac{m_1}{c}(2\gamma + 2)\ \ln \frac{d - d_m}{d_e} + \frac{2\mu a_0}{c}\left\{\frac{1}{3}\gamma - \frac{4}{3}\beta + \frac{3}{4} + \frac{1}{3}(2\gamma + 1)m\right.$$
$$\left. + \left(\frac{1}{3}\beta - \frac{2}{3}\gamma - \frac{109}{96}\right)m^2 + \left[\frac{1}{4} + (\gamma + 2\beta - \frac{3}{4})m^2\right] \cos 2D + \frac{7}{32}\, m^2 \cos 4D\right\} \quad . \tag{29}$$

This relation enables us to obtain a real estimate of the relativistic effects in lunar laser ranging. The most significant periodic relativistic effect with the argument, 2D (with period of 14.76 days) depends rather faintly on parameters β, γ and its amplitude is determined in fact by $\mu a_0/2$ = 2 meters ($\mu = 10^{-8}$, m = 0.08, $a_0 = 4 \cdot 10^5$ km). It may be added that according to Finkelstein and Kreinovich (1976), parameters β, γ make a contribution to the parallactic term with the argument D and a coefficient proportional to $\mu(a_0/A)/m$. But the magnitude of this term is smaller by at least one order than the magnitude of the main relativistic variational term.

The transformation to the proper time τ of the laser station is performed by

$$d\tau = \left(1 - \frac{m_1}{d_e} - \frac{1}{c^2}\,\underline{\dot{R}}\underline{V} - \frac{3}{2}\,\mu\right)\, dt \tag{30}$$

where $\underline{\dot{R}}$ is heliocentric velocity of the center of mass of the Earth, $\underline{V}$ is geocentric velocity of the laser station. Integrating the righ-hand member of (30) yields the known formula relating τ and t (Mulholland, 1972). Such a relation is necessary for an accurate computation of solar and lunar ephemerides. But in order to estimate the time delay in lunar laser ranging, one may neglect the variations of the functions appearing in (30) and consider this formula as a direct relation between proper τ and coordinate T time delays.

CONCLUSION

The problem of determining astronomical reference frames meets with difficulties even at the Newtonian level (Kovalevsky, 1975). The theory of relativity increases these difficulties. Much remains to be done to perform all part of the Newtonian theory in a relativistic basis. In this paper some questions of the relativistic reduction of astronomical measurements have been considered.

But the difficulties of the relativistic treatment of reference frames should not be exaggerated. The practical problem is to correlate results of measurements performed at different times in distinct observatories. Knowing with some accuracy the metric of the gravita-

https://doi.org/10.1017/S0252921100081409 Published online by Cambridge University Press

tional field, one may calculate the motions of observatories and reduce all measurements to one actual or fictitious laboratory at some moment of time (placed, for instance, in the center of mass of the Earth, Sun or Solar system). A discussion of such reduced measurements leads in turn to an improvement of the metric.

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