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# A DUAL VIEW OF THE CLIFFORD THEORY OF CHARACTERS OF FINITE GROUPS

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**Introduction.** Let $G$ be a finite group, $K$ a normal subgroup of $G$, $\chi$ an irreducible complex character of $G$. In the usual decomposition of $\chi|_K$, using Clifford's theorems, $G/K$ is seen to operate by conjugation on the irreducible characters of $K$ and if $\sigma$ is an irreducible component of $\chi|_K$, then $I(\sigma)$, the inertial group of $\sigma$, plays an essential role as an appropriate intermediate subgroup for the analysis. In this paper we consider the case where $G/K$ is abelian and study the action of the dual group $(G/K)\hat{}$ (of linear characters of $G/K$) on the irreducible characters of $G$ effected by multiplication. This action appears to be related in a dual way to the action of $G/K$ on the characters of $K$. We define a subgroup $J(\chi)$ of $G$ which plays a role similar to that of $I(\sigma)$ and which we call the dual inertial group of $\chi$. The dual inertial group of $J(\chi)$ equals the inertial group $I(\sigma)$ precisely in case the ramification index $e$ equals 1 and in general $[I(\sigma): J(\chi)] = e^2$. The results include some conditions that imply $e = 1$ and in general give further insight into the role of the ramification index $e$ when $G/K$ is abelian.

**1. Background.** In this paper all groups are finite and all characters are assumed to be characters of representations over the field of complex numbers. $|G|$ denotes the order of a group $G$; $[G:H]$ denotes the index of a subgroup $H$ in $G$; deg $\psi$ means the degree of the character $\psi$; and

$$(\chi, \psi) = (1/|G|)\sum_{g\in G}\chi(g)\psi(g^{-1})$$

is the usual "inner product." If $\chi$ is a character of $G$ and $K$ is a subgroup, then $\chi|_K$ denotes the restriction of $\chi$ to $K$. If $\psi$ is a character of $K$, then $\psi^G$ is the induced character of $G$. If $\psi$ and $\chi$ are characters of a subgroup $K$, then $(\psi, \chi)_K$ denotes the "inner product" computed over $K$.

Let $G$ be a group and $K$ a normal subgroup of $G$. If $\sigma$ is a character of $K$ and $g \in G$, then $\sigma^g$ is the conjugate character described by $\sigma^g(x) = \sigma(gxg^{-1})$ all $x \in K$. Let $I(\sigma) = \{g|\sigma^g = \sigma\}$, called the inertial group of $\sigma$. Since $\sigma^g = \sigma$ if $g \in K$, it is clear that any two representatives of the same coset of $G$ modulo $K$ produce the same permutation and we may consider $G/K$ to be represented in this way by permutations of the set of irreducible characters of $K$.

Let $\chi$ be an irreducible character of $G$ and let $\sigma$ be an irreducible component

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of $\chi|_K$. Then by Clifford's theorem,

$$\chi|_K = e \sum_{i=1}^{m} \sigma^{g_i},$$

where $g_1, \ldots, g_m$ are coset representatives for $G$ modulo $I(\sigma)$ and $e = e_K(\chi)$ is a positive integer called the ramification index of $\chi$ with respect to $K$ (see, for example, [**1**, Theorem 49.7, p. 345 and Ex. 1, p. 346; **2**, p. 53; **4**, Chapter 5, Theorem 17.3]). Feit in [**2**, Theorem 9.11] shows that $e_K(\chi) = e_K(\psi)$ where $\psi$ is an irreducible component of $\chi|_{I(\sigma)}$. For later reference we also quote the following result.

LEMMA 1.1. (see [**2**, Theorem 9.12]). *If $I(\sigma)/K$ is cyclic, then $e_K(\chi) = 1$.*

A number of basic properties of the character group of an abelian group will also be used in this paper. If $G$ is a finite abelian group, let $\hat{G}$ denote the group of one dimensional characters. For the following well-known properties of $\hat{G}$ see, for example, [**4**, Chapter 5, § 6].

LEMMA 1.2. (a) $\hat{G} \cong G$.

(b) *There is a one-one correspondence between the set of subgroups of $G$ and those of $\hat{G}$ defined by the mapping $K \to K^{\perp} = \{\lambda \in \hat{G} | \lambda(k) = 1$, all $k \in K\}$, for each subgroup $K$ of $G$. If $H$ is a subgroup of $\hat{G}$, then $H = K^{\perp}$ where $K = \{g \in G | \lambda(g) = 1$, all $\lambda \in H\}$. $H = (G/K)\hat{}$ in this case.*

(c) *If $K$ is a subgroup of $G$, the restriction map (restricting to $K$ the domain of the characters of $G$) is a homomorphism of $\hat{G}$ onto $\hat{K}$ with kernel $K^{\perp}$. Hence $\hat{K} \cong \hat{G}/K^{\perp}$.*

(d) $\{g | \lambda(g) = 1$, *all* $\lambda \in \hat{G}\} = \{1\}$; *i.e., the trivial subgroup of $G$ corresponds to $\hat{G}$ in the correspondence described in part* (b).

**2. The actions of $G/K$ and $(G/K)\hat{}$ on the characters.** Let $K$ be a normal subgroup of $G$. In this section we list and compare the effects of the action of $G/K$ on the characters of $K$ and the action of $(G/K)\hat{}$ on the characters of $G$ if $G/K$ is abelian. Here $(G/K)\hat{}$ denotes the group of one dimensional characters of $G/K$. These may be regarded as the one-dimensional characters of $G$ whose respective kernels include $K$. If $\lambda \in (G/K)\hat{}$ and $\chi$ is an irreducible character of $G$, then $\lambda\chi$ is also an irreducible character of $G$. Thus, $(G/K)\hat{}$ is represented by permutations on the set of irreducible characters of $G$. As described in § 1, $G/K$ is similarly represented by permutations on the set of irreducible characters of the subgroup $K$, by conjugation, whether or not $G/K$ is abelian.

THEOREM 2.1. *Let $\chi$ be an irreducible character of $G$ and let $G/K$ be abelian. Then the action of $(G/K)\hat{}$ fixes $\chi$ if and only if $e\chi = \psi^G$ where $\psi$ is an irreducible component of $\chi|_K$ and $e = e_K(\chi)$.*

*Proof.* Suppose that $e\chi = \psi^G$. $\psi^G(g) = 0$ for all $g \notin K$. Hence $\lambda\chi = \chi$ if $\lambda \in G/K\hat{}$, since $\lambda(g) = 1$ for $g \in K$ and $\chi(g) = (1/e)\psi^G(g) = 0$ if $g \notin K$.

Conversely, suppose that $\lambda\chi = \chi$ for all $\lambda \in (G/K)\hat{}$. By Lemma 1.2, part (d), if $g \notin K$, there exists $\lambda \in (G/K)\hat{}$ with $\lambda(g) \neq 1$. $\lambda(g)\chi(g) = \chi(g)$; hence $\chi(g) = 0$ for all $g \notin K$. Now let

$$\chi|_K = \sum_{i=1}^{m} e\psi^{g_i},$$

as discussed in § 1. Then

$$\begin{aligned} 1 = (\chi, \chi) &= (1/|G|) \sum_{g \in G} \chi(g)\chi(g^{-1}) \\ &= (1/|G|) \sum_{h \in K} \chi(h)\chi(h^{-1}) \\ &= (|K|/|G|)(\chi|_K, \chi|_K)_K \\ &= (|K|/|G|)(\textstyle\sum e\psi^{g_i}, e\sum \psi^{g_i})_K \\ &= (|K|/|G|)e^2 m. \end{aligned}$$

Hence $[G:K] = e^2m$. Now $\deg \psi^G = [G:K] \deg \psi = e^2m \deg \psi$, while $\deg \chi = em \deg \psi$. Hence $\deg \psi^G = e \deg \chi$. But $e = (\chi|_K, \psi)_K = (\chi, \psi^G)$ by Frobenius reciprocity, so $e\chi = \psi^G$.

COROLLARY 2.2. *If $(G/K)\hat{}$ fixes $\chi$ then $[G:K] = e^2m$ where $e = e_K(\chi)$ and $m = [G: I(\psi)]$.*

As a special case of Theorem 2.1. we have the following corollary.

COROLLARY 2.3. *Let $G/K$ be cyclic and $\chi$ be an irreducible character of $G$. Then the action of $(G/K)\hat{}$ fixes $\chi$ if and only if there exists an irreducible character $\psi$ of $K$ such that $\chi = \psi^G$.*

*Proof.* Since $G/K$ is cyclic, so is $I(\psi)/K$. Hence by Lemma 1.1., $e = e_K(\chi) = 1$.

(Note that Corollary 2.3 is [**5**, Theorem 3.1], where it is proved in a different manner.)

THEOREM 2.4. *Let $K \lhd G$ and $\chi$ be a character of $G$ which remains irreducible when restricted to $K$. Then the characters $\lambda\chi$ are distinct as $\lambda$ varies over the irreducible characters of $G/K$. Further, if $\theta$ is an irreducible character such that $\chi|_K$ is a constituent of $\theta$, then $\theta$ is of the form $\lambda\chi$ as above.*

*Proof.* See [**3**, Lemma 3.1].

Note that this theorem (with its proof) is implicitly included in [**4**, Chapter V, Theorem 17.12, p. 572].

The converse also holds for the first part of Theorem 2.4.

THEOREM 2.5. *Let $K$ be a normal subgroup of $G$ and let $\chi$ be an irreducible character such that the set of characters $\{\lambda\chi\}$ are irreducible and distinct as $\lambda$ varies over the irreducible characters of $G/K$. Then $\chi|_K$ is an irreducible character of $K$.*

*Proof.* Let $\lambda_1, \ldots, \lambda_r$ be the irreducible characters of $G/K$, where $\lambda_1$ is the 1-character, and let $\chi = \lambda_1\chi, \lambda_2\chi, \ldots, \lambda_r\chi$ be distinct and irreducible. Let

$$\chi|_K = e \sum_{i=1}^{m} \psi^{g_i},$$

where $e = e_K(\chi)$ and $m = [G: I(\psi)]$. Then $(\chi|_K, \psi) = e$ and $\lambda_i\chi|_K = n_i(\chi|_K)$ where $n_i$ is the degree of $\lambda_i$, $i = 1, 2, \ldots, r$. Hence $(\lambda_i\chi, \psi^G) = (\lambda_i\chi|_K, \psi)_K = n_ie$, by Frobenius reciprocity. Note that $\deg(\lambda_i\chi) = n_i \deg \chi$. Hence $[G:K]\deg \psi = \deg \psi^G \geqq \sum_{i=1}^{r} n_i^2 e(\deg \chi) = [G:K]e(\deg \chi) \geqq [G:K]\deg \chi \geqq [G:K]\deg \psi$, since $\psi$ is a component of $\chi|_K$. Hence $\deg \chi = \deg \psi$, $e = 1$ and $\chi|_K = \psi$.

We are primarily concerned with the above two theorems in the case $G/K$ abelian. In that case, $(G/K)\hat{}$ is said to operate faithfully on $\chi$ if $\lambda\chi = \chi$ and $\lambda \in (G/K)\hat{}$ imply that $\lambda$ is the 1 character. Restating Theorems 2.4 and 2.5 in this case we have the following corollary.

COROLLARY 2.6. *Let $G/K$ be abelian and let $\chi$ be an irreducible character of $G$. Then $\chi|_K$ is irreducible if and only if $(G/K)\hat{}$ operates faithfully on $\chi$. If $\chi|_K = \psi$ is irreducible and $\theta$ is an irreducible character of $G$ such that $\theta|_K = \psi$, then $\theta$ must be of the form $\lambda\chi$ for some $\lambda \in (G/K)\hat{}$* (i.e., *$\theta$ belongs to the orbit of $\chi$ under the action of $(G/K)\hat{}$*).

If $\psi$ is an irreducible character of $K$, $G/K$ will be said to operate faithfully on $\psi$ if $\psi^g = \psi$ implies that $g \in K$.

THEOREM 2.7. *Let $K \lhd G$ and $\psi$ be an irreducible character of $K$. Then $\psi^G$ is irreducible if and only if $G/K$ operates faithfully on $\psi$.*

*Proof.* See, for example, [**1**, Theorem 45.5, p. 329].

THEOREM 2.8. *Let $K \lhd G$ and let $\psi$ be an irreducible character of $K$. If $\psi$ extends to a character of $G$, then $G/K$ fixes $\psi$. If $G/K$ fixes $\psi$ and either $G/K$ is cyclic or $(|G/K|, |K|) = 1$, then $\psi$ extends to a character of $G$.*

*Proof.* Since a character of $G$ must be constant on $G$ conjugacy classes, it is clear that $G/K$ must fix $\psi$ if it extends to $G$. For the remainder, see [**4**, Theorem 17.12, p. 572].

The theorems in this section suggest a duality between the action of $(G/K)\hat{}$ on the irreducible characters of $G$ (when $G/K$ is abelian) and the action of $G/K$ on the characters of $K$. This relationship is most striking when $G/K$ is cyclic.

*Summary for $G/K$ cyclic.* Let $G/K$ be cyclic, $\chi$ be an irreducible character of $G$ and $\psi$ an irreducible character of $K$.

I. $(G/K)\hat{}$ operates faithfully on $\chi \Leftrightarrow \chi|_K$ is irreducible. $(G/K)\hat{}$ fixes $\chi \Leftrightarrow \chi$ is induced from a character of $K$.

II. $G/K$ operates faithfully on $\psi \Leftrightarrow \psi^G$ is irreducible. $G/K$ fixes $\psi \Leftrightarrow \psi$ is the restriction of a character of $G$.

The above summary describes only the extreme cases for $G/K$ cyclic. The general case for $G/K$ cyclic is included in the next section as part of the general treatment of the situation when $G/K$ is abelian.

**3. $G/K$ abelian: the dual inertial group.** Throughout this section $G/K$ is assumed to be abelian. We let $\chi$ be an irreducible character of $G$ and let $(G/K)\hat{}$ operate on $\chi$. $(G/K)\hat{}$ is represented transitively on the orbit of $\chi$. Let $H$ be the kernel of this representation; since $(G/K)\hat{}$ is abelian, $H$ coincides with the stabilizer of $\chi$; i.e., $H = \{\lambda \in (G/K)\hat{}|\lambda\chi = \chi\}$.

*Definition.* Let $J(\chi)$ be the subgroup of $G$ such that $(J(\chi)/K)^{\perp} = H$ in the notation of Lemma 1.2; i.e., $J(\chi) = \{g \in G|\lambda(g) = 1, \text{ all } \lambda \in H\}$. Then $J(\chi)$ is called the *dual inertial group of $\chi$ with respect to $K$*, also to be denoted in this paper as $J$, for convenience.

Note that $K \lhd J \lhd G$. In the usual use of Clifford's theorems one descends from $G$ to $K$ in two steps with the intermediate group being $I(\sigma)$, the inertial group of an irreducible component of $\chi|_K$. In this discussion, $J(\chi)$ is used as the principal intermediate subgroup.

Let

$$\chi|_J = \sum_{i=1}^{m} e\psi^{g_i},$$

with $e = e_J(\chi)$, $\psi$ an irreducible character of $J$ and $g_1, \ldots, g_m$ coset representatives of $I(\psi)$ in $G$. By Lemma 1.2, part (b), $(G/J)\hat{}$ may be identified with $H$ and since $H$ fixes $\chi$ by definition, Theorem 2.1 shows that

$$e\chi = \psi^G = (\psi^{g_i})^G, \; i = 1, 2, \ldots, m.$$

Lemma 1.2, part (c) implies that restricting the characters of $G/K$ to $J/K$ is a homomorphism of $(G/K)\hat{}$ onto $(J/K)\hat{}$ with kernel $H$.

Let $\chi = \chi_1, \chi_2, \ldots, \chi_r$ be the distinct images of $\chi$ under the operation of $(G/K)\hat{}$. Choose $\tau_1, \tau_2, \ldots, \tau_r \in (G/K)\hat{}$ with $\tau_i\chi = \chi_i$, $i = 1, 2, \ldots, r$. These are then a set of coset representatives of $(G/K)\hat{}$ modulo $H$. Let $\tau_i|_J = \lambda_i$, $i = 1, 2, \ldots, r$. Then $(J/K)\hat{} = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$.

Let $\psi = \psi_1$ be a given irreducible component of $\chi|_J$ as above and let $\lambda_j\psi = \psi_j$, $j = 1, 2, \ldots, r$. We form an $m \times r$ tableau of characters with $\psi_j^{g_i}$ appearing in the $i$th row and $j$th column ($g_1 = 1, g_2, \ldots, g_m$ being a set of coset representatives for $G$ modulo $I(\psi)$).

$$\begin{array}{rc|ccccc|l}
G\colon & \chi = & \chi_1 & \chi_2 & \cdots & \chi_j & \chi_r & \\
\cline{3-7}
 & \psi = & \psi_1 & \psi_2 & & \psi_j & \psi_r & \sigma_1 = \sigma \\
 & & \psi_1^{g_2} & \psi_2^{g_2} & & \psi_j^{g_2} & \psi_r^{g_2} & \sigma_2 \\
 & & \vdots & & & & & \\
 & & \psi_1^{g_i} & \psi_2^{g_i} & & \psi_j^{g_i} & \psi_r^{g_i} & \sigma_i \\
 & & \psi_1^{g_m} & \psi_2^{g_m} & \cdots & \psi_j^{g_m} & \psi_r^{g_m} & \sigma_m \\
\cline{3-7}
 & J & & & & & & K
\end{array}$$

THEOREM 3.1 (Properties of the tableau). *The $rm$ characters in the tableau are distinct. The elements in the $j$th column are the components of $\chi_j|_J$ while each element in row $i$ when restricted to $K$ yields the same irreducible character $\sigma_i$ and $\sigma_1, \ldots, \sigma_m$ are distinct and form one orbit under the action of $G/K$. The rows of the tableau are orbits under the faithful action of $(J/K)^\wedge$. The columns are orbits under the faithful action of $G/I$ where $I = I(\psi)$, the inertial group of $\psi$ with respect to $G$.*

*Proof.*

$$\psi_i{}^G = (\lambda_i\psi)^G = \tau_i\psi^G = e\tau_i\chi = e\chi_i \quad \text{for} \quad i = 1, 2, \ldots, r.$$

Hence the set $\psi_1, \psi_2, \ldots, \psi_r$ are distinct, $(J/K)^\wedge$ acts faithfully on $\psi$ and $\psi_1, \ldots, \psi_r$ is an orbit of $(J/K)^\wedge$. Since $G/J$ is abelian, $I(\psi)$ is normal in $G$ and

$$I(\psi) = I(\psi^{g_i}) \text{ for each } i.$$

Further, $\lambda_j\psi^g = (\lambda_j\psi)^g = \psi_j{}^g$ for each $g$ in $G$ (and each $j = 1, 2, \ldots, r$), since $\lambda_j$ is the restriction of a character of $G$. Hence

$$I = I(\psi) = I(\psi_j) = I(\psi_j{}^{g_i}).$$

Each column is thus an orbit under the faithful action of $G/I$ and contains $m$ distinct characters. Since

$$\lambda_j\psi^{g_i} = \psi_j{}^{g_i},$$

each row is an orbit under $(J/K)^\wedge$.

Now

$$\chi|_J = \sum e\psi^{g_i},$$

so

$$\chi_j|_J = \tau_j\chi|_J = \lambda_j(\sum e\psi^{g_i}) = \sum_i e\lambda_j\psi^{g_i} = \sum_i e\psi_j{}^{g_i}.$$

Thus elements in the $j$th column are the components of $\chi_j|_J$ and $e_J(\chi) = e = e_J(\chi_j)$. $\psi_j{}^{g_i}$ is an element in the $i$th column, so

$$(\psi_j{}^{g_i})^G = e\chi_j$$

(by Theorem 2.1), so two columns have no elements in common.

Hence the $rm$ characters in the tableau are distinct, since if two were equal they would need to be in the same column, and the elements in any column are distinct.

Since $(J/K)^\wedge$ acts faithfully on all the elements in the tableau, they each restrict to irreducible characters on $K$, by Corollary 2.6. For each $i = 1, \ldots, m$,

$$\psi^{g_i}|_K = (\lambda_j\psi^{g_i})|_K \text{ for all } j = 1, 2, \ldots, r$$

(since $K$ lies in the kernel of $\lambda_j$), so all the elements in a row have the same restriction to $K$. For the $i$th row let $\sigma_1$ be this character. For convenience we will often denote $\sigma_1$ merely as $\sigma$. Now if $i \neq j$, then $\sigma_i \neq \sigma_j$ by the second part of Corollary 2.6. Further,

$$\sigma_i = \psi^{g_i}|_K = (\psi|_K)^{g_i} = \sigma^{g_i}.$$

This completes the proof.

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Continuing to refer to the tableau we prove the following theorem.

THEOREM 3.2. *Let $G/K$ be abelian and $\chi$ be an irreducible character of $G$. Let $J = J(\chi)$ be the dual inertial group of $\chi$ (with respect to $K$), $\psi$ an irreducible component of $\chi|_J$ and $\sigma = \psi|_K$. Then*

(i) *$\sigma$ is irreducible and $I(\sigma) = I(\psi)$. (This subgroup will be denoted as $I$.)*
(ii) *$e_K(\chi) = e_J(\chi)$ (we denote this as "$e$" for short).*
(iii) *We have a chain of normal subgroups $K \subseteq J \subseteq I \subseteq G$ and $[I: J] = e^2$. Thus, $[G:K] = mre^2$ where $m = [G:I]$ and $r = [J:K] =$ size of orbit of $\chi$ under the action of $(G/K)\hat{}$.*
(iv) *$e = 1 \Leftrightarrow I(\sigma) = J(\chi)$.*
(v) *In general, $e^2$ divides $[I(\sigma): K]$.*
(vi) *$\sigma^G = e\sum \chi_i$ where $\{\chi_i | i = 1, 2, \ldots, r\}$ is the set of (distinct) elements in the orbit of $\chi$ under the action of $(G/K)\hat{}$.*

*Proof.* (i) The irreducibility of $\sigma$ was seen in Theorem 3.1. If $\psi^g = \psi$ then clearly $\sigma^g = \sigma$, so $I(\psi) \subseteq I(\sigma)$. If $\sigma^g = \sigma$ then $\psi^g$ is an extension of $\sigma$ to $K$; hence $\psi^g = \lambda_j\psi$ for some $\lambda_j \in (J/K)\hat{}$, by Corollary 2.6. But

$$\psi^g = \psi^{g_i} \text{ for some } g_i$$

and since the elements of the tableau are all distinct,

$$\psi^{g_i} = \lambda_j\psi$$

implies that $i = j = 1$ and $\psi^g = \psi$. Thus $I(\sigma) \subseteq I(\psi)$.

(ii)

$$\chi|_J = \sum e_J(\chi)\psi^{g_i},$$

hence

$$\chi|_K = \sum e_J(\chi)\sigma^{g_i}.$$

As shown in Theorem 3.1, the elements $\sigma^{g_i}$ are all distinct, so $e_J(\chi) = e_K(\chi)$; this is denoted as "$e$" for short.

(iii) $[G:J] = e^2m$ by Corollary 2.2, where $m = [G:I]$. Since $J \subseteq I$, we have $[I:J] = e^2$ and in the discussion of the tableau it was seen that $[J:K] = r$ was the number of distinct characters $\tau\chi$, $\tau \in (G/K)\hat{}$.

(iv) This is immediate from (iii).

(v) $[I:K] = e^2r$ follows from (iii), so $e^2$ divides $[I:K]$.

(vi) As seen in proof of (ii), $(\chi|_K, \sigma) = e$. Hence $(\chi, \sigma^G) = e$ by Frobenius reciprocity. Since $\chi|_K = \chi_i|_K$, $(\chi_i, \sigma^G) = e$. Now $\deg \sigma^G = [G:K]\deg \sigma = e^2 rm \deg \sigma$, while $\deg(e\sum \chi_i) = er \deg \chi = er(em \deg \sigma) = \deg \sigma^G$. Hence $\sigma^G = e\sum \chi_i$.

COROLLARY 3.3. *Let $G/K$ be abelian. In the action of $G/K$ on the irreducible characters of $K$, the number of orbits equals the number of orbits under the action of $(G/K)\hat{}$ on the irreducible characters of $G$.*

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*Proof.* To each orbit $\{\chi_1, \ldots, \chi_r\}$ under $(G/K)\hat{}$ the tableau construction associates an orbit $\{\sigma_1, \ldots, \sigma_m\}$ under $G/K$ consisting of precisely the irreducible components of $\chi_1|_K$. By Theorem 3.2, part (vi), the irreducible components of $\sigma_1{}^G$ form precisely one orbit under $(G/K)\hat{}$ and this was the unique orbit to which $\{\sigma_1, \ldots, \sigma_m\}$ was assigned.

The next theorem considers some conditions under which the dual inertial group equals the inertial group.

THEOREM 3.4. *Let $G/K$ be abelian. Each of the following conditions implies that $J(\chi) = I(\sigma)$.*

(a) *$I(\sigma)/J(\chi)$ is cyclic.*

(b) *$|I(\sigma)/J(\chi)|$ and $|J(\chi)|$ are relatively prime.*

(c) *$|I(\sigma)/K|$ and $|K|$ are relatively prime.*

*Proof.* $I = I(\sigma) = I(\psi)$ by Theorem 3.2. Hence by Theorem 2.8, conditions (a) and (b) each imply that the character $\psi$ of $J(\chi)$ extends to an irreducible character $\rho$ of $I$. Now $G/I$ acts faithfully on $\rho$ since it acts faithfully on $\psi = \rho|_J$. Hence $\rho^G$ is irreducible (Theorem 2.7) and

$$\rho^G|_I = \quad \rho^{g_i},$$

$\{g_1, \ldots, g_m\}$ being coset representatives of $G$ modulo $I$.

$$\rho^G|_J = (\sum \rho^{g_i})|_J = \sum \psi^{g_i}.$$

Hence $(\rho^G|_J, \psi)_J = 1 = (\rho^G, \psi^G) = (\rho^G, e\chi)$. Thus $\rho^G = \chi$ and $e = 1$. The result follows by Theorem 3.2, part (iv).

In case (c), $\sigma$ extends to an irreducible character $\rho'$ on $I(\sigma)$ by Theorem 2.8. Then $\rho'|_J = \lambda\psi$ (some $\lambda \in (J/K)\hat{}$), since these are the only extensions of $\sigma$ to $J$. Letting $\tau$ be a linear character of $G/K$ whose restriction to $J$ is $\lambda^{-1}$ (the group inverse in $(J/K)\hat{}$), then $(\tau|_I)(\rho') = \rho$, an irreducible character of $I$ which extends $\psi$. The same discussion just given for cases (a) and (b) now completes the proof.

*Remark.* Case (c) is also covered in [**2**, Theorem 9.14].

COROLLARY 3.5. *If $G/K$ is abelian, then each of the following conditions implies that $I(\sigma) = J(\chi)$.*

(i) $G/K$ is cyclic.

(ii) $I(\sigma)/K$ is cyclic.

(iii) $G/J(\chi)$ is cyclic.

*Proof.* Each of these conditions implies that $I(\sigma)/J(\chi)$ is cyclic, which is condition (a) of Theorem 3.4. Note that part (ii) also follows directly from Lemma 1.1 and Theorem 3.2, part (iv).

*Note.* Since writing and submitting this paper, the author has become aware that some of the results concerning the dual inertial group are to be found in Lemma 3.2 of G. J. Janusz's paper: *Some remarks on Clifford's theorem and the Schur index*, Pacific J. Math. 32 (1970), 119–129.

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