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PAPER

Analysis of a model describing bacterial colony expansion in radial geometry driven by chemotaxis

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Abstract

We investigate a recent model proposed in the literature elucidating patterns driven by chemotaxis, similar to viscous fingering phenomena. Notably, this model incorporates a singular advection term arising from a modified formulation of Darcy's law. It is noteworthy that this type of advection can also be well interpreted as a description of a radial fluid flow source surrounding an aggregation of cells. For the two-dimensional scenario, we establish a precise threshold delineating between blow-up and global solution existence. This threshold is contingent upon the pressure magnitude and the initial total mass of the aggregating cells.

1. Introduction and main results

Various experiments with dilute bacteria have shown that they behave differently depending on their density, given rise to collective motions, patterns, and hydrodynamic instabilities (cf. [14, 32]). In [3], the authors proposed a mathematical model to explain patterns similar to viscous finger motion for colony expansion driven by chemotaxis in radial geometry. The model has the form:

$$\partial_t n + \nabla\{n(\mathbf{u} + \chi\nabla v) - D\nabla n\} = F_1(n, v), \tag{1}$$

$$v_t = \Delta v + F_2(n, v), \tag{2}$$

where n denotes the density of the bacteria, v denotes the concentration of chemoattractant and $\mathbf{u}$ is the advection term. The parameter D is assumed positive; meanwhile, χ is a constant that can be either positive or negative. The reaction term F_1 describes cellular proliferation, while F_2 is selected based on whether there is chemoattractant consumption or production by the colony itself.

In experiments over thin films, it has been shown that describing the hydrodynamic velocity $\mathbf{u}$ accurately is difficult (cf. [4, 27]). Several studies have shown that u can vary greatly depending on the shape of the particles (cf. [282930]). For instance, Darcy's law has been demonstrated to be sufficiently precise for spherical particles. However, the motion of rod-shaped particles, such as *E. coli*, presents a different scenario altogether. Experimentation with non-Newtonian fluids and adhesive elastomers has shown that the formula:

$$\mathbf{u}(x, t) = -\frac{\nabla P}{|\nabla P|^{\zeta}}, \tag{3}$$

where ζ is a characteristic of bacterial activity and P is a normalised pressure, which describes in the colony the hydrodynamic velocity with significantly greater accuracy than the standard Darcy's law.

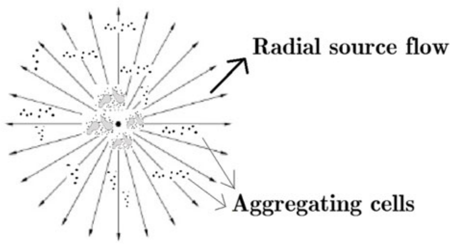

Figure 1. *Visualization of cell aggregation driven by a radially symmetric source.*

The parameter ζ is negative in shear-thinning, positive in shear-thickening solutions, and $\zeta = 0$ represents the Newtonian case. To the best of our knowledge, this generalised Darcy's law was first introduced for the case $\zeta = 2$ in reference [23]. This generalisation was further extended for general exponents ζ in references [4, 17, 27]. In this paper, we focus on the case where $\zeta > 0$.

Overall, the mathematical model (1)–(3) contributes to understanding and quantifying the physical properties of populations of rod-shaped particles which either promote cohesion or on the contrary dispersion to the colony. As a result of this, we propose to examine not only the conditions for having global solutions but also a potential blow-up.

The construction of local and global weak solutions for the system (1)–(3) proves challenging due to the integrability problems associated with the advection term expressed in (3). Our goal in this paper is to study a prototypical case with $P = -\varrho\,|x|^2$, where ϱ is a positive constant and $\zeta = 2$, yielding the

$$\mathbf{u}(x,t) = \frac{x}{\varrho\,|x|^2}.$$

We consider the case where the chemotaxis is positive, meaning the particles move towards regions of high chemical concentration. Additionally, we assume $F_1 = 0$ and $F_2 = n - v$. In other words, bacterial proliferation is neglected and the chemoattractant is produced by the organisms themselves. Consequently, we obtain the system

$$\partial_t n + \nabla \cdot \left(\frac{Qx}{|x|^2} n\right) = D\Delta n - \chi \nabla \cdot (n \nabla v)\,,$$

$$v_t = \Delta v - v + n,$$

where D, χ and Q (equal to $1/\varrho$ in this case) denote positive constants. To simplify the analysis, we also assume that the diffusion of the chemical is much faster than that of the chemoattractant. Then, a classical rescaling argument leads us to describe the dynamics of the chemical by an elliptic equation (cf. [15, 21]). This results in the following simplified version of the model:

$$\begin{array}{ll} n_t + \nabla \cdot (\frac{Qx}{|x|^2} n) = \Delta n - \chi \nabla \cdot (n \nabla v), \, x \in B, \qquad t > 0, & \\ 0 = \Delta v - \frac{\theta}{\pi} + n, \qquad \text{with} \;\; \int_B v(\cdot, t) = 0, \, x \in B, \qquad t > 0, & \end{array} \tag{4}$$

where B represents a two-dimensional ball centred at the origin with radius equal to 1, and $\theta := \int_B n(x, 0)$.

A noteworthy observation is that the system (4) can also be interpreted as a Keller–Segel-type model, where the aggregation of particles is influenced by a radial fluid flow, either inwards or outwards. The direction of the fluid is determined by the sign of the parameter Q. Figure 1 illustrates this interpretation for the case of a radial source flow.

The literature on Keller–Segel-type models describing particle aggregation in the presence of a surrounding fluid has seen significant growth in the last decade. It is beyond the scope of this paper to provide an exhaustive list of references. Interested readers are referred to [8, 9, 20, 22], and the references therein.

From this point onwards, we consider the general case where Q is a positive constant and introduce the notation:

$$\mathbf{u}(x) := \frac{Q}{|x|^2}x = Q\nabla \log |x| \,, \text{ where } x \in B \setminus \{0\} \,. \tag{5}$$

We impose no-flux boundary conditions

$$\frac{\partial v}{\partial \eta} = 0, \text{ and } \frac{\partial n}{\partial \eta} - \chi n \frac{\partial v}{\partial \eta} - n\frac{Qx}{|x|^2} \cdot \eta = 0, \quad x \in \partial B, \ t > 0, \tag{6}$$

and non-negative radial initial data in L^1 denoted by:

$$n(x, 0) = n(\,|x|\,, 0) = n_0 \geq 0. \tag{7}$$

We establish that if the initial data n_0 satisfies the condition $\int_B n_0(x)dx < \frac{4\pi(2-Q)}{\chi}$ (Theorem 1), then the solution exists globally in time. Furthermore, when $\int_B n_0(x)dx > \frac{4\pi(2-Q)}{\chi}$, then a blow-up is feasible, as demonstrated in theorem (Theorem 2). Notably, blow-up always occurs when $Q > 2$.

Our main result on global existence of weak solutions is stated as follows.

Theorem 1 (Global existence). *Let $B := B(0, 1) \subseteq \mathbb{R}^2$ be a two-dimensional ball and let η be the outward unit normal vector on ∂B. Let $Q > 0$ be a constant. Given a non-negative radial symmetrical function $n_0 \in L^\infty(B)$ satisfying*

$$\theta := \int_B n_0 dx < \frac{4\pi \ (2+Q)}{\chi} \tag{8}$$

then the problem (4)–(7) has a weak solution n in the sense of definition 4.

We also establish the possibility of blow-up in finite time:

Theorem 2 (Blow-up)**.** *Let $B = B(0, 1) \subset \mathbb{R}^2$ a ball and let η be its external normal to the boundary. Let us consider a local solution of the system (4)–(7). Let us denote $m(t) := \int_B n\,|x|^2\,dx$. If the initial mass $\theta := \int_B n_0 dx$ satisfies*

$$\theta > \frac{4\pi \ (2+Q)}{\chi} \text{ with } Q > 0.$$

and $m(0) < \frac{1}{2}\left(\theta - \frac{4\pi(2+Q)}{\chi}\right)$ then $T_{\max} \leq -\frac{\pi}{\chi\theta} \log\left(1 - \frac{m(0)}{\frac{1}{2}\left(\theta - \frac{4\pi(2+Q)}{\chi}\right)}\right)$ and

$$\limsup_{t\to T_{\max}} \|n(\cdot, t)\|_{L^q(B)} = \infty \text{ for any } q > 2. \tag{9}$$

Remark 3. *Here, it should be noticed that the case $Q = 0$ corresponds to the classical parabolic-elliptic Keller–Segel model. In this case, it is well known that the qualitative behaviour is divided basically in three cases:*

$$\int_B n_0 dx < \frac{8\pi}{\chi} \text{ (subcritical)}, \int_B n_0 dx = \frac{8\pi}{\chi} \text{ (critical)}, \int_B n_0 dx > \frac{8\pi}{\chi} \text{ (supercritical)},$$

where under appropriate conditions the corresponding solution exists globally in the subcritical case while it blows up in the supercritical case (cf. [26]). Thus, our result shows that the introduction of the hydrodynamics velocity given by (3) produce a 'shift' in the critical mass of this system. We will elucidate how the corresponding proof utilises a recent version of the Moser–Trudinger inequality capable of handling singularities (cf.[5]). Additionally, further study is required for non-radial cases and those with higher dimensions.

The structure of this paper is outlined as follows. In Section 2, we introduce regularisation to our model, leading to a version endowed with an energy functional. By subsequently passing to the limit in the regularised system, we establish the existence of local solutions for our original model. In Section 3, we employ a singular variant of the Moser–Trudinger inequality to demonstrate the existence of global

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solutions. Finally, in Section 4, we derive sufficient conditions for finite-time blow-up. Our analysis yields a novel threshold condition determining the feasibility of global existence versus blow-up.

2. Definition of weak solution

In the classical parabolic-elliptic Keller–Segel model, measure solutions are a well-established concept (cf. [31]). However, when addressing the system (4)–(7), defining a similar notion becomes more intricate due to the non-integrability of the singular flux:

$$\mathbf{u}(x) = Qx/ |x|^2 .$$

To ensure proper definition of the term $\nabla \cdot (\frac{Qx}{|x|^2}n)$ as a distribution, we observe that $\left|x/ |x|^2\right| = \frac{1}{|x|} \in L^s(B)$ for all $s \in [1, 2)$. Therefore, we will require $n \in L^q$ for some $q > 2$ to guarantee $nx/ |x|^2 \in L^1$ by virtue of Hölder's inequality. Consequently, we adopt the following definition of weak solution.

Definition 4. *Let $T > 0$ and $q > 2$ be fixed constants and $n_0 \in L^\infty(B)$. Let us define the space $V := L^\infty((0, T); L^q(B)) \cap L^2((0, T); H^1(B))$. We say that a function $n \in V$ is a weak solution to (4)–(7) if for any $\phi \in H^1(B \times (0, T))$*

$$\int_B n\phi dx - \int_0^t \int_B n\phi_\tau dxd\tau + \int_0^t \int_B \left(\nabla n - \chi n\nabla v - n\frac{Qx}{|x|^2}\right) \cdot \nabla\phi dxd\tau = \int_B n(x, 0)\phi(x, 0)dx, \quad (10)$$

and for any $\gamma \in H^1(B)$

$$\int_B \left(\nabla v \cdot \nabla\gamma + \frac{\theta}{\pi}\gamma\right) dx = \int_B n\gamma dx, \quad (11)$$

holds for a.e. $t \in (0, T)$, $v = v(\cdot, t) \in H^1(B)$ and $\int_B vdx = 0$.

To establish local existence, we initially regularise the model (4)–(7) by adapting the ideas in the reference [12] to our frame. To do so, we observe $\nabla (\log |x|) = x/ |x|^2$ for $x \neq 0$. Next, we define

$$K^\epsilon(x) := K\left(\frac{x}{\epsilon}\right),$$

where $\epsilon > 0$ and K is a radial monotone non increasing smooth function satisfying

$$K(x) := \begin{cases} -\frac{1}{2\pi}\log|x| - \frac{1}{2\pi}\log\epsilon & \text{if } |x| \geq 4, \\ 0 & \text{if } |x| \leq 1. \end{cases}$$

Additionally, we assume

$$|\nabla K(x)| \leq \frac{1}{2\pi\ |x|},\ K(x) \leq -\frac{1}{2\pi}\log|x| \text{ and } -\Delta K(x) \geq 0 \text{ for any } x \in \mathbb{R}^2. \quad (12)$$

Since $K^\epsilon(x) = K\left(\frac{x}{\epsilon}\right)$, we have

$$|\nabla K^\epsilon\ (x)| \leq \frac{1}{2\pi\ |x|} \text{ for all } x \in \mathbb{R}^2\backslash\{0\}. \quad (13)$$

Subsequently, we consider the following approximate system,

$$\begin{aligned} &n_t^\epsilon - \nabla\cdot(2\pi Qn^\epsilon\nabla K^\epsilon) = \Delta n^\epsilon - \chi\nabla\cdot(n^\epsilon\nabla v^\epsilon)\ x \in B, && t > 0, \\ &0 = \Delta v^\epsilon - \tfrac{\theta}{\pi} + n^\epsilon, \qquad \text{with } \textstyle\int_B v^\epsilon(\cdot, t) = 0, x \in B, && t > 0, \end{aligned} \quad (14)$$

accompanied by the no-flux boundary conditions

$$\frac{\partial v^\epsilon}{\partial\eta} = 0, \text{ and } \frac{\partial n^\epsilon}{\partial\eta} - \chi n^\epsilon\frac{\partial v^\epsilon}{\partial\eta} + 2\pi Qn^\epsilon\nabla K^\epsilon\cdot\eta = 0,\quad x \in \partial B,\ t > 0, \quad (15)$$

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and non-negative, radially symmetric initial data given by:

$$n^{\epsilon}(x,0) := n(x,0) = n_0 \geq 0. \tag{16}$$

The concept of weak solutions for the regularised model is defined as follows.

Definition 5. *Let $T>0$ a be fixed constant and $n_0 \in L^2(B)$. Let us define the space $V_2 := L^{\infty}((0,T);L^2(B)) \cap L^2((0,T);H^1(B))$. We say that a function $n^{\epsilon} \in V_2$ is a weak solution to (14)–(16) if for any $\phi \in H^1(B\times(0,T))$*

$$\int_B n^{\epsilon}\phi dx - \int_0^t \int_B n^{\epsilon}\phi_\tau dxd\tau + \int_0^t \int_B (\nabla n^{\epsilon} - \chi n^{\epsilon}\nabla v^{\epsilon} + 2\pi Q n^{\epsilon}\nabla K^{\epsilon}) \cdot \nabla\phi dxd\tau = \int_B n(x,0)\phi(x,0)dx, \tag{17}$$

and for any $\gamma \in H^1(B)$

$$\int_B \left(\nabla v^{\epsilon} \cdot \nabla\gamma + \frac{\theta}{\pi}\gamma\right) dx = \int_B n^{\epsilon}\gamma dx, \tag{18}$$

holds for a.e. $t \in (0,T)$, $v^{\epsilon} = v^{\epsilon}(\cdot,t) \in H^1(B)$ and $\int_B v^{\epsilon} dx = 0$.

3. Local existence of solutions for the regularised model

Let $T>0$ be a constant. Let us consider the space Y given by:

$$Y := L^4((0,T);L^2(B)),$$

whose norm

$$|\widetilde{n}|_Y := \left(\int_0^T \|\widetilde{n}(\cdot,t)\|^4_{L^2(B)} dt\right)^{1/4} = \left(\int_0^T \left(\int_B \widetilde{n}^2 dx\right)^2 dt\right)^{\frac{1}{4}}.$$

is finite. We establish the local existence of solutions for the regularised model using the Schauder fixed-point theorem. To this end, we define the convex set $B_Y(0,R) := \left\{\widetilde{n} : |\widetilde{n}|_Y \leq R,\ \int_B \widetilde{n}(\cdot,t)dx = \int_B n_0 dx =: \theta\right\}$. Next, we construct a map $\Gamma : B_Y(0,R) \to Y$ that associates each $\widetilde{n} \in B_Y(0,R)$ with a function $m := \Gamma(\widetilde{n})$, defined through the following two-step process.

1. We find the distributional solution $\widetilde{v}$ to the semicoercive homogeneous Neumann problem

$$0 = \Delta\widetilde{v} - \frac{\theta}{\pi} + \widetilde{n}, \quad \text{with} \int_B \widetilde{v}(\cdot,t) = 0,\ x \in B(0,1), \tag{19}$$

   subject to the homogeneous Neumann boundary condition $\partial\widetilde{v}/\partial\eta = 0$ on ∂B, in the trace sense. The existence of a solution to this elliptic problem is standard, as outlined in [6, Theorem 6.2.3].

2. We determine m the solution to the linear parabolic equation

$$m_t - \nabla\cdot(2\pi Q m \nabla K^{\epsilon}) = \Delta m - \chi\nabla\cdot(m\nabla\widetilde{v}), \tag{20}$$

   with initial data $m(x,0) = n_0$ and zero-flux boundary conditions. To establish the existence of solutions for this equation, we first rewrite it as:

$$m_t = \Delta m - \operatorname{div}\{m(\chi\nabla\widetilde{v} - 2\pi Q\nabla K^{\epsilon})\}. \tag{21}$$

   We then employ regularity theory for elliptic equations to obtain, for each $p \geq 1$, a constant $C_1(p)$ such that

$$|\nabla\widetilde{v}(\cdot,t)|_{L^p(B)} \leq C_1(p)\left|-\frac{\theta}{\pi} + \widetilde{n}\right|_{L^2(B)} \leq C_1\left(\theta\sqrt{\pi} + |\widetilde{n}|_{L^2(B)}\right). \tag{22}$$

   From (13) and (22), we conclude

$$\chi\nabla\widetilde{v} - 2\pi Q\nabla K^{\epsilon} \in L^4(0,T;L^p(B)) \text{ for all } p \geq 1. \tag{23}$$

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Moreover, the Sobolev embedding theorem for traces (see [1, Th. 5.36]) gives $H^1(B) \subset L^q(\partial B)$ for all $1 \le q < \infty$, yielding

$$(\chi \nabla \widetilde{v} - 2\pi Q \nabla K^\epsilon) \cdot \eta \in L^4(0, T; L^q(\partial B)) \text{ for all } q \ge 1. \tag{24}$$

Consequently, we can invoke [24, Chapter III, Theorem 5.1] to establish the existence of solutions to the linear parabolic equation (20).

Proposition 6. *Let us assume that $n_0 \in L^2(B)$. There exists $T = T(n_0) > 0$ such that the regularised model (14)–(16) has a weak solution n^ϵ in the sense of definition 5. Moreover $n_t^\epsilon \in L^2((0, T); H^{-1}(B))$.*

In the next two Lemmas, we prove that $\Gamma(B_Y(0, R)) \subseteq B_Y(0, R)$ as well as compactness of the operator Γ. Thus, the result of local existence given in Proposition 6 will follow directly from the Schauder fixed-point theorem. In order to simplify the proof, we previously introduce in the next Lemma a set of auxiliary estimates.

Lemma 7 (Estimates for the linearised problem). *Let $T > 0$ be a constant. Then the function $m := \Gamma(\widetilde{n})$ as defined above satisfies*

a) $\int_B m(x, t)dx = \int_B n(x, 0)dx$ *for all* $t \ge 0$.

b) For some constant $K_1 > 0$ independent of ϵ

$$\int_0^T \left(\int_B \widetilde{v}^2 dx \right)^2 dt + \int_0^T \left(\int_B |\nabla \widetilde{v}|^2 \, dx \right)^2 dt + \int_0^T \left(\int_B |\Delta \widetilde{v}|^2 \, dx \right)^2 dt \le K_1. \tag{25}$$

c) There exists a constant $K_2 > 0$ independent of ϵ such that

$$\int_B m^2 \Delta \widetilde{v} dx \le K_2 \left(\int_B (\Delta \widetilde{v})^2 dx + 1 \right) \left(\int_B m^2 dx \right) + \frac{1}{\chi} \int_B |\nabla m|^2 \, dx \tag{26}$$

d) There exists a constant $K_3 > 0$ independent of ϵ such that

$$\int_B m^2(x, t) dx \le K_3 \quad \textit{for } 0 < t < T. \tag{27}$$

There exists also a constant $K_4 > 0$ independent of ϵ such that

$$\|m\|_{L^2(0,T,H^1(B))} \le K_4. \tag{28}$$

Proof. To streamline our proof, we will proceed with formal computations for smooth solutions. The validity of these computations can be justified through a testing process.

a) The equation for m together with the zero-flux boundary condition give us $\frac{d}{dt} \int_B m dx = 0$.

b) Utilising the regularity theory for linear elliptic equations, we determine that $\widetilde{v} \in W^{2,2}(B)$. This allows us to derive

$$\int_B |\Delta \widetilde{v}|^2 \, dx = \int_B \left(-\frac{\theta}{\pi} + \widetilde{n} \right)^2 dx \le 2 \int_B \left(\frac{\theta}{\pi} \right)^2 dx + 2 \int_B \widetilde{n}^2 dx$$
$$\le \frac{2\theta^2}{\pi} + 2 \int_B \widetilde{n}^2 dx. \tag{29}$$

Consequently,

$$\int_0^t \left(\int_B |\Delta \widetilde{v}|^2 \, dx \right)^2 dt \le \frac{8\theta^4 T}{\pi^2} + 4 \int_0^t \left(\int_B \widetilde{n}^2 dx \right)^2 dt$$
$$\le \frac{8\theta^4 T}{\pi^2} + 4R^4. \tag{30}$$

Further, Poincaré's inequality give us a constant C_p satisfying

$$\left(\int_B v^2 dx\right)^{1/2} \leq C_p \left(\int_B |\nabla \widetilde{v}|^2\, dx\right)^{1/2}. \tag{31}$$

Using this, we deduce

$$\int_B |\nabla \widetilde{v}|^2\, dx = -\int_B v\Delta\widetilde{v} dx \leq \left(\int_B v^2 dx\right)^{1/2} \left(\int_B |\Delta\widetilde{v}|^2\, dx\right)^{1/2} \leq C_P \left(\int_B |\nabla \widetilde{v}|^2\, dx\right)^{1/2} \left(\int_B |\Delta\widetilde{v}|^2\, dx\right)^{1/2}.$$

Thus,

$$\left(\int_B |\nabla \widetilde{v}|^2\, dx\right)^{1/2} \leq C_P \left(\int_B |\Delta\widetilde{v}|^2\, dx\right)^{1/2}. \tag{32}$$

By using (30), (31) and (32), we readily arrive at (25) with

$$K_1 := \left(1 + 2C_P^4\right) \left(\frac{8\theta^4 T}{\pi^2} + 4R^4\right).$$

c) Applying Cauchy's inequality, we get

$$\int_B m^2 \Delta\widetilde{v} dx \leq \left(\int_B m^4 dx\right)^{1/2} \left(\int_B (\Delta\widetilde{v})^2 dx\right)^{1/2}. \tag{33}$$

Moreover, the Gagliardo–Nirenberg–Sobolev interpolation inequality provides constants C_1 and C_2 satisfying

$$\int_B f^4 dx \leq C_1 \left(\int_B f^2 dx\right) \left(\int_B |\nabla f|^2\, dx\right) + C_2 \left(\int_B f^2 dx\right)^2 \quad \text{for all } f \in H^1(B).$$

Applying the last inequality to $f = m$, and utilising it in combination with (33)

$$\begin{aligned}
&\int_B m^2 \Delta\widetilde{v} dx \\
&\leq \left\{C_1 \left(\int_B m^2 dx\right) \left(\int_B |\nabla m|^2\, dx\right) + C_2 \left(\int_B m^2 dx\right)^2\right\}^{1/2} \left(\int_B (\Delta\widetilde{v})^2 dx\right)^{1/2} \\
&\leq C_1^{1/2} \left(\int_B m^2 dx\right)^{1/2} \left(\int_B |\nabla m|^2\, dx\right)^{1/2} \left(\int_B (\Delta\widetilde{v})^2 dx\right)^{1/2} + C_2^{1/2} \left(\int_B m^2 dx\right) \left(\int_B (\Delta\widetilde{v})^2 dx\right)^{1/2} \\
&\leq \frac{\chi C_1}{4} \left(\int_B (\Delta\widetilde{v})^2 dx\right) \left(\int_B m^2 dx\right) + \frac{1}{\chi} \int_B |\nabla m|^2\, dx + \frac{C_2^{1/2}}{2} \left(\int_B (\Delta\widetilde{v})^2 dx + 1\right) \left(\int_B m^2 dx\right) \\
&= \left(\left(\frac{\chi C_1}{4} + \frac{C_2^{1/2}}{2}\right) \int_B (\Delta\widetilde{v})^2 dx + \frac{C_2^{1/2}}{2}\right) \left(\int_B m^2 dx\right) + \frac{1}{\chi} \int_B |\nabla m|^2\, dx.
\end{aligned}$$

The inequality (26) follows with $K_2 := \max\left\{\frac{\chi C_1}{4} + \frac{C_2^{1/2}}{2}, \frac{1}{\chi}\right\}$.

d) Employing the equation (20), we multiply it by m and integrate the product by parts to yield

$$\begin{aligned}
&\frac{d}{dt} \int_B m^2 dx \\
&\leq -2 \int_B |\nabla m|^2\, dx + 2 \int_{\partial B} m \nabla m \cdot \eta d\sigma - 2\chi \int_B m \nabla \cdot (m \nabla \widetilde{v}) dx + 2 \int_B m \nabla \cdot (2\pi Q m \nabla K^{\epsilon})\, dx.
\end{aligned}$$

Now, we use the identity

$$\int_B m \nabla \cdot (m \nabla \widetilde{v}) dx = \frac{1}{2} \int_B m^2 \Delta\widetilde{v} dx$$

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and apply item c) of this Lemma to calculate

$$\frac{d}{dt}\int_B m^2 dx \le -2\int_B |\nabla m|^2\, dx + 2\int_{\partial B} m\nabla m\cdot \eta d\sigma$$
$$+\chi\left(K_2\left(\left(\int_B (\Delta\widetilde{v})^2 dx + 1\right)\left(\int_B m^2 dx\right) + \frac{1}{\chi}\int_B |\nabla m|^2\, dx\right) + 2\int_B m\nabla\cdot(2\pi Q m\nabla K^\epsilon) dx\right.$$
$$\le -\int_B |\nabla m|^2\, dx + 2\int_{\partial B} m\nabla m\cdot\eta d\sigma$$
$$+\chi K_2\left(\|\Delta\widetilde{v}\|_2^2+1\right)\int_B m^2 dx + 2\int_{\partial B} m\,(2\pi Q m\nabla K^\epsilon)\cdot \eta d\sigma - 4\pi Q\int_B m\nabla m\cdot\nabla K^\epsilon dx. \qquad (34)$$

For the last integral, considering the positivity of Q and $\Delta K^\epsilon \le 0$, we get

$$-4\pi Q\int_B m\nabla m\cdot\nabla K^\epsilon dx = -2\pi Q\int_B \nabla m^2\cdot\nabla K^\epsilon dx$$
$$= -2\pi Q\int_{\partial B} m^2\nabla K^\epsilon\cdot\eta d\sigma + 2\pi Q\int_B m^2\Delta K^\epsilon dx \qquad (35)$$
$$\le -2\pi Q\int_{\partial B} m^2\nabla K^\epsilon\cdot\eta d\sigma.$$

Using the trace inequality $\|f\|^2_{L^2(\partial B)} \le \overline{\delta}\,\|\nabla f\|^2_{L^2(B)} + C_{\overline{\delta}}\,\|f\|^2_{L^2(B)}$, $f\in H^1(B)$, with $f=m$ and $\overline{\delta}=\frac{\delta}{Q}$ gives us

$$-2\pi Q\int_{\partial B} m^2\nabla K^\epsilon\cdot\eta d\sigma \le Q\int_{\partial B}\frac{m^2}{|x|}d\sigma = Q\int_{\partial B} m^2 d\sigma$$
$$\le \delta\,\|\nabla m\|^2_{L^2(B)} + QC_{\overline{\delta}}\,\|m\|^2_{L^2(B)}$$

Combining estimates (34) and (35) together with the zero-flux boundary conditions (15), we obtain

$$\frac{d}{dt}\int_B m^2 dx \le (-1+\delta)\int_B |\nabla m|^2\, dx + \left(\chi K_2\left(\|\Delta\widetilde{v}\|_2^2+1\right) + QC_{\overline{\delta}}\right)\int_B m^2 dx. \qquad (36)$$

Applying Gronwall's inequality and the estimate from item b).

$$\int_B m^2(x,t)dx \le \left(\int_B m_0^2 dx\right)\exp\int_0^t\left(\chi K_2\left(\|\Delta\widetilde{v}\|_2^2+1\right)+QC_{\overline{\delta}}\right)ds$$
$$\le \left(\int_B m_0^2 dx\right)\exp\int_0^T\left(\frac{\chi K_2}{2}\left(\|\Delta\widetilde{v}\|_2^4+3\right)+QC_{\overline{\delta}}\right)ds$$
$$\le \left(\int_B m_0^2 dx\right)\exp\left(\frac{\chi K_2}{2}K_1 + \frac{3\chi K_2}{2}T + QC_{\overline{\delta}}\right) =: C_3. \qquad (37)$$

Thus, we have proved (27) with $K_3 := C_3$. Finally, we proceed to prove (28). From (36) and (37),

$$(1-\delta)\int_0^t\int_B |\nabla m|^2\, dxdt$$
$$\le \int_B m^2(x,0)dx + \int_0^t\left\{\left(\chi K_2\left(\|\Delta\widetilde{v}\|_2^2+1\right)+QC_{\overline{\delta}}\right)K_3\right\}ds$$
$$\le \int_B m^2(x,0)dx + \int_0^t\left\{\left(\frac{\chi K_2}{2}\left(\|\Delta\widetilde{v}\|_2^4+3\right)+QC_{\overline{\delta}}\right)K_3\right\}ds$$
$$\le \int_B m^2(x,0)dx + \left(\frac{\chi K_2}{2}(K_1+3)+QC_{\overline{\delta}}T\right)K_3 =: C_4. \qquad (38)$$

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We conclude from (37) and (38) the validity of (28) with

$$\|m\|^2_{L^2(0,T,H^1(B))} = \int_0^T \int_B m^2 dxdt + \int_0^T \int_B |\nabla m|^2 \, dxdt$$
$$\leq C_3 T + \frac{C_4}{1-\delta} \;=: K_4.$$

□

We proceed, in the next two Lemmas, to show the existence of a radius R and a time T such that the hypotheses of the Schauder fixed-point theorem hold.

Lemma 8 (Self mapping). *There exists a time $T > 0$ independent of ϵ such that $\Gamma(B_Y(0,R)) \subset B_Y(0,R)$.*

Proof. Applying Lemma 7 (item *d*), we derive the inequality

$$|\Gamma(\widetilde{n})|_Y = \left(\int_0^T \left(\int_B m^2(x,t)dx\right)^2 dt\right)^{1/4} \leq K_3^{1/2} T^{1/4}. \tag{39}$$

Here, the constant K_3 is defined in equation (37). We conclude taking $K_3^2 T^{1/4} = R/2$. □

Lemma 9 (Compacity). Γ: $(B_Y(0,R), |\cdot|_Y) \to (B_Y(0,R), |\cdot|_Y)$ *is a compact map.*

Proof. Let us suppose that

$$\widetilde{n}_i \to \widetilde{n} \text{ in the space } Y \text{ as } i \to \infty. \tag{40}$$

According to the definition of the map Γ, we aim to show its compactness by demonstrating that the sequence of functions m_i, with $i = 1, 2, \ldots$ defined by

$$m_{it} - \nabla \cdot (2\pi Q m_i \nabla K^\epsilon) = \Delta m_i - \chi \nabla \cdot (m_i \nabla \widetilde{v}_i) \tag{41}$$
$$\Delta \widetilde{v} - \frac{\theta}{\pi} + \widetilde{n}_i = 0, \quad \text{with} \int_B \widetilde{v}_i(\cdot,t) = 0 \text{ and } \partial\widetilde{v}/\partial\eta = 0 \text{ on } \partial B,$$
$$\frac{\partial m_i}{\partial \eta} + 2\pi Q m_i \nabla K^\epsilon \cdot \eta - \chi m_i \nabla \widetilde{v} \cdot \eta = 0, \qquad x \in \partial B,$$
$$m_i(x,0) = m_0(x), \quad x \in B,$$

possesses a subsequence that converges to the solution of the linear system

$$m_t - \nabla \cdot (2\pi Q m \nabla K^\epsilon) = \Delta m - \chi \nabla \cdot (m \nabla \widetilde{v}), \tag{42}$$
$$\Delta \widetilde{v} - \frac{\theta}{\pi} + \widetilde{n} = 0, \quad \text{with} \int_B \widetilde{v}(\cdot,t) = 0 \text{ and } \partial\widetilde{v}/\partial\eta = 0 \text{ on } \partial B,$$
$$\frac{\partial m}{\partial \eta} + 2\pi Q m \nabla K^\epsilon \cdot \eta - m \nabla \widetilde{v} \cdot \eta = 0, \qquad x \in \partial B,$$
$$m_i(x,0) = m_0(x), \quad x \in B.$$

We notice that Lemma 7 (item d) provides the existence of a constant C_1 independent of i such that

$$\int_0^T \int_B |\nabla m_i|^2 \, dxdt \leq C_1. \tag{43}$$

To apply Aubin–Lions compactness lemma, we proceed to show the existence of a constant C_2 independent of the index i such that

$$\left\| \frac{dm_i}{dt} \right\|_{L^2(0,T,H^1(B)^*)} \leq C_2. \tag{44}$$

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For any $\mu \in H^1(B)$ with $\|\mu\|_{H^1(B)} \le 1$, we have

$$\begin{aligned}
&\left(\frac{dm_i}{dt}, \mu\right)_{(H^1(B)^*, H^1(B))}\\
&= \left(\Delta m_i - \chi \nabla \cdot (m_i \nabla \widetilde{v}_i) + \nabla \cdot (2\pi Q m_i \nabla K^\epsilon), \mu\right)_{(H^1(B)^*, H^1(B))}\\
&= -\left(\nabla m_i - \chi (m_i \nabla \widetilde{v}_i) + (2\pi Q m_i \nabla K^\epsilon), \nabla \mu\right)_{(H^1(B)^*, H^1(B))}\\
&= -(\nabla m_i, \nabla \mu)_{(H^1(B)^*, H^1(B))} + (m_i \nabla \widetilde{v}_i, \nabla \mu)_{(H^1(B)^*, H^1(B))} - (2\pi Q m_i \nabla K^\epsilon, \nabla \mu)_{(H^1(B)^*, H^1(B))}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left(\frac{dm_i}{dt}, \mu\right)^2_{(H^1(B)^*, H^1(B))}\\
&\le 3(\nabla m_i, \nabla \mu)^2_{(H^1(B)^*, H^1(B))} + 3(m_i \nabla \widetilde{v}_i, \nabla \mu)^2_{(H^1(B)^*, H^1(B))} + 3\,(2\pi Q m_i \nabla K^\epsilon, \nabla \mu)^2_{(H^1(B)^*, H^1(B))}.
\end{aligned} \tag{45}$$

We analyse each term separately. First, we observe

$$\begin{aligned}
&3(\nabla m_i, \nabla \mu)^2_{(H^1(B)^*, H^1(B))}\\
&= 3\left(\int_B \nabla m_i \cdot \nabla \mu dx\right)^2 \le 3\,\|\nabla m_i\|^2_{L^2(B)}\,\|\nabla \mu\|^2_{L^2(B)} \le 3\,\|\nabla m_i\|^2_{L^2(B)},
\end{aligned} \tag{46}$$

Next, we deduce for the second term in (45),

$$\begin{aligned}
(m_i \nabla \widetilde{v}_i, \nabla \mu)_{(H^1(B)^*, H^1(B))} &\le \|m_i \nabla \widetilde{v}_i\|_{L^2(B)}\,\|\nabla \mu\|_{L^2(B)} \le \|m_i \nabla \widetilde{v}_i\|_{L^2(B)} \le \|m_i\|_{L^3(B)}\,\|\nabla \widetilde{v}_i\|_{L^6(B)}\\
&\le \|m_i\|^{1/3}_{L^2(B)}\,\|m_i\|^{2/3}_{L^4(B)}\,\|\nabla \widetilde{v}_i\|_{L^6(B)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
(m_i \nabla \widetilde{v}_i, \nabla \mu)^2_{(H^1(B)^*, H^1(B))} &\le \|m_i\|^{2/3}_{L^2(B)}\,\|m_i\|^{4/3}_{L^4(B)}\,\|\nabla \widetilde{v}_i\|^2_{L^6(B)}\\
&\le \frac{1}{2}\,\|m_i\|^{4/3}_{L^2(B)}\,\|m_i\|^{8/3}_{L^4(B)} + \frac{1}{2}\,\|\nabla \widetilde{v}_i\|^4_{L^6(B)}.
\end{aligned} \tag{47}$$

To estimate the quantity $\|m_i\|^{2/3}_{L^4(B)}$ arising in the inequality (47), we use the Gagliardo–Nirengber–Sobolev inequality,

$$\int_B f^4 dx \le C_1 \left(\int_B f^2 dx\right)\left(\int_B |\nabla f|^2\, dx\right) + C_2 \left(\int_B f^2 dx\right) \quad \text{for all } f \in H^1(B),$$

with $f = m_i$ leading to

$$\int_B m_i^4 dx d\tau \le C_1 \left(\int_B m_i^2 dx\right)\left(\int_B |\nabla m_i|^2\, dx\right) + C_2 \left(\int_B m_i^2 dx\right).$$

By employing Lemma 7 (item *d*), we derive a constant C_3 such that

$$\begin{aligned}
\|m_i\|^{8/3}_{L^4(B)} = \left(\int_B m_i^4 dx\right)^{2/3} &\le C_1^{2/3}\left(\int_B m_i^2 dx\right)^{2/3}\left(\int_B |\nabla m_i|^2\, dx\right)^{2/3} + C_2^{2/3}\left(\int_B m_i^2 dx\right)^{2/3}\\
&\le C_1^{2/3} C_3^{2/3}\left(\frac{1}{3} + \frac{2}{3}\int_B |\nabla m_i|^2\, dx\right) + C_4^{2/3} C_3^{2/3}.
\end{aligned} \tag{48}$$

Combining inequalities (47) and (48) and using Lemma 7 (item *d*), we obtain a constant C_5 satisfying

$$(m_i \nabla \widetilde{v}_i, \nabla \mu)^2_{(H^1(B)^*, H^1(B))} \le C_5 \left(1 + \int_B |\nabla m|^2\, dx\right) + \frac{1}{2}\,\|\nabla \widetilde{v}_i\|^4_{L^6(B)}. \tag{49}$$

Utilising Gagliardo-Nirenberg inequality, we further establish

$$\|\nabla \widetilde{v}_i\|^4_{L^6(B)} \le C_6 \left(\|\Delta \widetilde{v}_i\|^4_{L^2(B)} + \|\widetilde{v}_i\|^4_{L^2(B)}\right). \tag{50}$$

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From equations (49) and (50), we can conclude

$$(m_i \nabla \widetilde{v}_i, \nabla \mu)^2_{(H^1(B)^*, H^1(B))} \leq C_5 \left(1 + \int_B |\nabla m|^2 \, dx\right) + \frac{1}{2} C_6 \left(\|\Delta \widetilde{v}_i\|^4_{L^2(B)} + \|\widetilde{v}_i\|^4_{L^2(B)}\right). \tag{51}$$

Finally, evaluating the last term in inequality (45)

$$\begin{aligned} 3\left(2\pi Q m_i \nabla K^\epsilon, \nabla \mu\right)^2_{(H^1(B)^*, H^1(B))} &\leq 3 \left\|2\pi Q m_i \nabla K^\epsilon\right\|^2_{L^2(B)} \|\nabla \mu\|^2_{L^2(B)} \\ &\leq 12\pi^2 Q^2 \|\nabla K^\epsilon\|^2_{L^\infty(B)} \|m_i\|^2_{L^2(B)} \|\nabla \mu\|^2_{L^2(B)} \\ &\leq 12\pi^2 Q^2 \|\nabla K^\epsilon\|^2_{L^\infty(B)} \|m_i\|^2_{L^2(B)} \\ &\leq 12\pi^2 Q^2 C_3 \|\nabla K^\epsilon\|^2_{L^\infty(B)}. \end{aligned} \tag{52}$$

Combining (45), (46), (51) and (52) along with Lemma 7, we conclude the inequality (44). In summary, Lemma 7 (item d) and inequality (44) imply the the existence of a constant C_7 such that

$$\|m_i\|_{L^2(0,T,H^1(B))} \leq C_7 \text{ as well as } \left\|\frac{dm_i}{dt}\right\|_{L^2(0,T,H^1(B)^*)} \leq C_7.$$

Utilising the embeddings

$$H^1(B) \overset{compact}{\hookrightarrow} L^2(B) \overset{continuous}{\hookrightarrow} H^1(B)^*,$$

we apply the Aubin–Lions compactness Lemma to establish the existence of a subsequence $(m_{i_j})_{j\in\mathbb{N}}$ such that

$$m_{i_j} \to m_* \text{ strong in } L^2(0, T, L^2(B)). \tag{53}$$

Let us now demonstrate that m_* satisfies equation (42). From Lemma 7 (item b), we deduce through a subsequence that

$$\nabla \widetilde{v}_{i_j} \to \nabla \widetilde{v}_* \text{ weakly in } L^2(0, T, L^2(B)), \tag{54}$$

where $\widetilde{v}_*$ fulfils the equation

$$\Delta \widetilde{v}_* - \frac{\theta}{\pi} + m_* = 0, \quad \text{with } \int_B \widetilde{v}_*(\cdot, t) = 0 \text{ and } \partial \widetilde{v}_*/\partial \eta = 0 \text{ on } \partial B.$$

in the distribution sense. Consequently, from (53) and (54), we establish the weakly convergence for the product

$$m_{i_j} \nabla \widetilde{v}_i \to m_* \nabla \widetilde{v}_* \text{ weakly in } L^2(0, T, L^2(B)).$$

Finally, by taking the limit in (41) as $j \to \infty$, we conclude that m_* corresponds to the solution of problem (42). Moreover, from Lemma 7 (item d), it follows that the sequence $\left(m_{i_j}\right)_{j\geq 1}$ as well as m_* are bounded in $L^\infty(0, T; L^2(B))$ by some constant C_7. Thus,

$$\int_0^T \left|m_{i_j} - m_*\right|^4_{L^2(B)} dt \leq 2C_7^2 \int_0^T \left|m_{i_j} - m_*\right|^2_{L^2(B)} dt \to 0 \text{ as } j \to \infty.$$

In conclusion, the operator Γ is compact. □

4. Local solution by passing to the limit in the regularised model

The problem described by equation (14) exhibits positivity-preserving behaviour, as stated below.

Proposition 10. *If the initial condition n_0 is non-negative, then the solution $n^\epsilon(x, t)$ remains non-negative for almost every x and $t \geq 0$.*

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Proof. Multiplying the first equation of system (14) by $(n^\epsilon)^- := \max\{0, -n_1^\epsilon\}$, integrating over B, and integrating by parts, we obtain

$$\begin{aligned}\frac{d}{dt}\int_B \left|(n^\epsilon)^-\right|^2 dx &= -\int_B \nabla\,(n^\epsilon)^- \cdot (\nabla n^\epsilon - \chi n^\epsilon \nabla v^\epsilon + 2\pi Q n^\epsilon \nabla K^\epsilon)\,dx \\ &\leq -\int_B \left|\nabla (n^\epsilon)^-\right|^2 dx + \chi \int_B (n^\epsilon)^-\,\nabla\,(n^\epsilon)^- \cdot \nabla v^\epsilon dx - 2\pi Q \int_B (n^\epsilon)^-\,\nabla\,(n^\epsilon)^- \cdot \nabla K^\epsilon dx. \end{aligned} \tag{55}$$

We rewrite the second integral in (55) in the form:

$$\int_B (n^\epsilon)^-\,\nabla\,(n^\epsilon)^- \cdot \nabla v^\epsilon dx = \frac{1}{2}\int_B \nabla(\,(n^\epsilon)^-\,)^2 \cdot \nabla v^\epsilon dx = -\frac{1}{2}\int_B (\,(n^\epsilon)^-\,)^2 \Delta v^\epsilon dx. \tag{56}$$

Regarding the last integral in (55), we observe

$$\begin{aligned}-2\pi Q \int_B (n^\epsilon)^-\,\nabla\,(n^\epsilon)^- \cdot \nabla K^\epsilon dx &= -\pi Q \int_B \nabla(\,(n^\epsilon)^-\,)^2 \cdot \nabla K^\epsilon dx \\ &= -\pi Q \int_{\partial B} (\,(n^\epsilon)^-\,)^2 \nabla K^\epsilon \cdot \eta dx + \pi Q \int_B (\,(n^\epsilon)^-\,)^2 \Delta K^\epsilon dx \\ &\leq -\pi Q \int_{\partial B} (\,(n^\epsilon)^-\,)^2 \nabla K^\epsilon \cdot \eta dx.\end{aligned}$$

Applying the property $|\nabla K^\epsilon| \leq \frac{1}{2\pi|x|}$ and the trace inequality $\|f\|^2_{L^2(\partial B)} \leq \overline{\delta}\,\|\nabla f\|^2_{L^2(B)} + C_{\overline{\delta}}\,\|f\|^2_{L^2(B)}$, $f \in H^1(B)$ with $f = (n^\epsilon)^-$ and $\overline{\delta} = \frac{\delta_1}{\pi Q}$ yields

$$\begin{aligned}-\pi Q \int_{\partial B} (\,(n^\epsilon)^-\,)^2 \nabla K^\epsilon \cdot \eta dx &\leq \pi Q \int_{\partial B} (\,(n^\epsilon)^-\,)^2\,|\nabla K^\epsilon|\,d\sigma \leq \pi Q \int_{\partial B} \frac{(\,(n^\epsilon)^-\,)^2}{|x|} d\sigma = \pi Q \int_{\partial B} (\,(n^\epsilon)^-\,)^2 d\sigma \\ &\leq \delta\,\left\|\nabla(\,(n^\epsilon)^-\,)\right\|^2_{L^2(B)} + \pi Q C_{\overline{\delta}}\,\left\|(n^\epsilon)^-\right\|^2_{L^2(B)}.\end{aligned} \tag{57}$$

Combining (55), (56), and (57), we arrive at

$$\frac{d}{dt}\int_B \left|(n^\epsilon)^-\right|^2 dx \leq (\,-1+\delta)\int_B \left|\nabla (n^\epsilon)^-\right|^2 dx - \frac{1}{2}\int_B (\,(n^\epsilon)^-\,)^2 \Delta v^\epsilon dx + \pi Q C_{\overline{\delta}}\,\left\|(n^\epsilon)^-\right\|^2_{L^2(B)}. \tag{58}$$

Proposition 10 yields $n^\epsilon \in V_2 := L^\infty((0,T);L^2(B))$. Thus, by the regularity theory for elliptic equations, we can assure that $v \in W^{2,2}(B)$ and

$$\begin{aligned}-\frac{1}{2}\int_B (\,(n^\epsilon)^-\,)^2 \Delta v^\epsilon dx &= \frac{1}{2}\int_B (\,(n^\epsilon)^-\,)^2 (n^\epsilon - \theta/\pi) dx \\ &= \frac{1}{2}\int_B (\,(n^\epsilon)^-\,)^3 dx - \frac{\theta}{2\pi}\int_B (\,(n^\epsilon)^-\,)^2 dx \leq \frac{\theta}{2\pi}\int_B (\,(n^\epsilon)^-\,)^3 dx.\end{aligned}$$

It follows that for any constant $\delta_2 > 0$, there exists $C_{\delta_2} > 0$ such that

$$-\frac{1}{2}\int_B (\,(n^\epsilon)^-\,)^2 \Delta v^\epsilon dx \leq \frac{\theta}{2\pi}\int_B (\,(n^\epsilon)^-\,)^2 (\,(n^\epsilon)^-\,) dx \leq \delta_2 \int_B (\,(n^\epsilon)^-\,)^4 dx + C_{\delta_2}\int_B (\,(n^\epsilon)^-\,)^2 dx. \tag{59}$$

Combining inequalities (58) and (59), we obtain

$$\frac{d}{dt}\int_B \left|(n^\epsilon)^-\right|^2 dx \leq (\,-1+\delta)\int_B \left|\nabla (n^\epsilon)^-\right|^2 dx + \delta_2 \int_B (\,(n^\epsilon)^-\,)^4 dx + \left(C_{\delta_2} + \pi Q C_{\overline{\delta}}\right)\int_B (\,(n^\epsilon)^-\,)^2 dx.$$

Using the Gagliardo–Nirenberg–Sobolev interpolation inequality (see [18, 19]), we have

$$\left\|(n^\epsilon)^-\right\|^4_{L^4(B)} \leq C^4_{GNB}\,\left\|(n^\epsilon)^-\right\|^2_{L^2(B)}\,\left\|\nabla\,(n^\epsilon)^-\right\|^2_{L^2(B)} + \left\|(n^\epsilon)^-\right\|^4_{L^2(B)}. \tag{60}$$

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Therefore,

$$\frac{d}{dt}\int_{\mathbb{R}^2}\left|(n^\epsilon)^-\right|^2 dx$$
$$\leq (-1+\delta+\delta_2 C_{GNB}^4 \left\|\left(n_1^\epsilon\right)^-\right\|_{L^2(\mathbb{R}^2)}^2)\int_B\left|\nabla(n^\epsilon)^-\right|^2 dx+\left(C_{\delta_2}+\pi Q C_{\overline{\delta}}\right)\int_B((n^\epsilon)^-)^2 dx$$
$$+\delta_2\left(\int_B((n^\epsilon)^-)^2 dx\right)^2.$$
$$\leq\left(C_{\delta_2}+\pi Q C_{\overline{\delta}}\right)\int_B((n^\epsilon)^-)^2 dx+\delta_2\left(\int_B((n^\epsilon)^-)^2 dx\right)^2.$$

Using again that $n^\epsilon\in V_2:=L^\infty((0,T);L^2(B))$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^2}\left|(n^\epsilon)^-\right|^2 dx$$
$$\leq(-1+\delta+\delta_2 C_{GNB}^4\sup_{(0,T)}\left\|(n^\epsilon)^-\right\|_{L^2(B)})\int_B\left|\nabla(n^\epsilon)^-\right|^2 dx+\left(C_{\delta_2}+\pi Q C_{\overline{\delta}}\right)\int_B((n^\epsilon)^-)^2 dx$$
$$+\delta_2\sup_{(0,T)}\left\|(n^\epsilon)^-\right\|_{L^2(B)}\left(\int_B((n^\epsilon)^-)^2 dx\right). \tag{61}$$

Choosing the parameters δ and δ_2 small enough, we conclude that

$$\frac{d}{dt}\int_{\mathbb{R}^2}\left|(n^\epsilon)^-\right|^2 dx\leq\left(C_{\delta_2}+\pi Q C_{\overline{\delta}}+\delta_2\sup_{(0,T)}\left\|(n^\epsilon)^-\right\|_{L^2(B)}\right)\int_B((n^\epsilon)^-)^2 dx.$$

Integrating the last inequality gives

$$\int_{\mathbb{R}^2}\left|(n^\epsilon)^-\right|^2 dx\leq\left(\int_{\mathbb{R}^2}\left|\left(n_1^\epsilon(x,0)\right)^-\right|^2 dx\right)e^{\left(C_{\delta_2}+\pi Q C_{\overline{\delta}}++\delta_2\sup_{(0,T)}\left\|\left(n^\epsilon\right)^-\right\|_{L^2(B)}\right)t}=0,$$

implying that $n^{\epsilon-}=0$ on $[0,T)\times\mathbb{R}^2$. Therefore, $n^\epsilon\geq 0$ on $[0,T)\times\mathbb{R}^2$. □

Lemma 11. *Let us assume that $n_0\in L^q(B)$ with $q\geq 2$. There exists $\tau\in(0,T)$ independent of ϵ such that*

$$\|n^\epsilon(\cdot,t)\|_{L^q(B)}\leq C \qquad \textit{for } 0\leq t<\tau. \tag{62}$$

Proof. We start by multiplying the first equation in (14) by $q(n^\epsilon)^{q-1}$ and integrating the resulting product by parts, yielding

$$\frac{d}{dt}\int_B(n^\epsilon)^q dx$$
$$\leq-\frac{4(q-1)}{q}\int_B\left|\nabla(n^\epsilon)^{q/2}\right|^2 dx+q\int_{\partial B}(n^\epsilon)^{q-1}\nabla n^\epsilon\cdot\eta d\sigma \tag{63}$$
$$+q\int_B(n^\epsilon)^{q-1}\nabla\cdot(2\pi Q n^\epsilon\nabla K^\epsilon)\,dx-\chi q\int_B(n^\epsilon)^{q-1}\nabla\cdot(n^\epsilon\nabla v^\epsilon)dx.$$

To estimate the last integral, we rewrite it as:

$$\int_B(n^\epsilon)^{q-1}\nabla\cdot(n^\epsilon\nabla v^\epsilon)dx=\int_{\partial B}(n^\epsilon)^q\nabla v^\epsilon\cdot\eta d\sigma-(q-1)\int_B(n^\epsilon)^{q-1}\nabla n^\epsilon\cdot\nabla v^\epsilon dx$$
$$=\int_{\partial B}(n^\epsilon)^q\nabla v^\epsilon\cdot\eta d\sigma-\frac{(q-1)}{q}\int_B\nabla(n^\epsilon)^q\cdot\nabla v^\epsilon dx$$
$$=\int_{\partial B}(n^\epsilon)^q\nabla v^\epsilon\cdot\eta d\sigma-\frac{(q-1)}{q}\int_{\partial B}(n^\epsilon)^q\nabla v^\epsilon\cdot\eta d\sigma+\frac{q-1}{q}\int_B(n^\epsilon)^q\Delta v^\epsilon dx.$$

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Since $\nabla v^\epsilon \cdot \eta = 0$ on ∂B, we obtain

$$\int_B (n^\epsilon)^{q-1} \nabla \cdot (n^\epsilon \nabla v^\epsilon) dx = \frac{q-1}{q} \int_B (n^\epsilon)^q \Delta v^\epsilon dx. \tag{64}$$

Next, we apply the Gagliardo–Nirenberg–Sobolev inequality

$$\int_B f^4 dx \le C_1 \left(\int_B f^2 dx\right) \left(\int_B |\nabla f|^2 \, dx\right) + C_2 \left(\int_B f^2 dx\right)^2 \quad \text{for all } f \in H^1(B).$$

with $f = (n^\epsilon)^{q/2}$. This yields

$$\begin{aligned}
\int_B (n^\epsilon)^q \Delta v^\epsilon dx &\le \left(\int_B (n^\epsilon)^{2q} dx\right)^{1/2} \left(\int_B (\Delta v^\epsilon)^2 dx\right)^{1/2} \\
&\le \left\{ C_1 \left(\int_B (n^\epsilon)^q dx\right) \left(\int_B \left|\nabla (n^\epsilon)^{q/2}\right|^2 dx\right) + C_2 \left(\int_B (n^\epsilon)^q dx\right)^2 \right\}^{1/2} \left(\int_B (\Delta v^\epsilon)^2 dx\right)^{1/2} \\
&\le C_1^{1/2} \left(\int_B (n^\epsilon)^q dx\right)^{1/2} \left(\int_B \left|\nabla (n^\epsilon)^{q/2}\right|^2 dx\right)^{1/2} \left(\int_B (\Delta v^\epsilon)^2 dx\right)^{1/2} \\
&+ C_2^{1/2} \left(\int_B (n^\epsilon)^q dx\right) \left(\int_B (\Delta v^\epsilon)^2 dx\right)^{1/2} \\
&\le \frac{\chi q C_1}{8} \left(\int_B (\Delta v^\epsilon)^2 dx\right) \left(\int_B (n^\epsilon)^q dx\right) + \frac{2}{\chi q} \left(\int_B \left|\nabla (n^\epsilon)^{q/2}\right|^2 dx\right) \\
&+ \frac{C_2^{1/2}}{2} \left(\int_B (n^\epsilon)^q dx\right) \left(\left(\int_B (\Delta v^\epsilon)^2 dx\right) + 1\right) \\
&= \left(\frac{\chi q C_1}{8} + \frac{C_2^{1/2}}{2}\right) \left(\int_B (\Delta v^\epsilon)^2 dx\right) \left(\int_B (n^\epsilon)^q dx\right) + \frac{2}{\chi q} \left(\int_B \left|\nabla (n^\epsilon)^{q/2}\right|^2 dx\right) \\
&+ \frac{C_2^{1/2}}{2} \left(\int_B (n^\epsilon)^q dx\right).
\end{aligned} \tag{65}$$

Substituting (64)–(65) into (63), we obtain

$$\begin{aligned}
&\frac{d}{dt} \int_B (n^\epsilon)^q dx \\
&\le -\frac{4(q-1)}{q} \int_B \left|\nabla (n^\epsilon)^{q/2}\right|^2 dx + q \int_{\partial B} (n^\epsilon)^{q-1} \nabla n^\epsilon \cdot \eta d\sigma \\
&+ q \int_B (n^\epsilon)^{q-1} \nabla \cdot (2\pi Q n^\epsilon \nabla K^\epsilon) dx - \chi q \int_{\partial B} (n^\epsilon)^q \nabla v^\epsilon \cdot \eta d\sigma \\
&+ \chi\, (q-1) \left(\left(\frac{\chi q C_1}{8} + \frac{C_2^{1/2}}{2}\right) \left(\int_B (\Delta v^\epsilon)^2 dx\right) \left(\int_B (n^\epsilon)^q dx\right) + \frac{2}{\chi q} \int_B \left|\nabla (n^\epsilon)^{q/2}\right|^2 dx \right. \\
&\left. + \frac{C_2^{1/2}}{2} \left(\int_B (n^\epsilon)^q dx\right)\right)
\end{aligned}$$

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$$
\begin{aligned}
&\leq -\frac{4(q-1)}{q}\int_B \left|\nabla (n^\epsilon)^{q/2}\right|^2 dx + q\int_{\partial B}(n^\epsilon)^{q-1}\nabla n^\epsilon\cdot\eta d\sigma \\
&+ q\int_{\partial B}(n^\epsilon)^{q-1}\left(2\pi Q n^\epsilon \nabla K^\epsilon\right)\cdot\eta d\sigma - 2\pi Q q(q-1)\int_B (n^\epsilon)^{q-1}\nabla n^\epsilon\cdot\nabla K^\epsilon dx - \chi q\int_{\partial B}(n^\epsilon)^q\nabla v^\epsilon\cdot\eta d\sigma \\
&+ \chi\left(q-1\right)\left(\left(\frac{\chi q C_1}{8}+\frac{C_2}{2}\right)\|\Delta v^\epsilon\|_2^2\int_B (n^\epsilon)^q dx + \frac{2}{\chi q}\int_B \left|\nabla\left((n^\epsilon)^{q/2}\right)\right|^2 dx + \frac{C_2^{1/2}}{2}\left(\int_B (n^\epsilon)^q dx\right)\right).
\end{aligned}
\tag{66}
$$

In order to bound the integral $-2\pi Qq(q-1)\int_B (n^\epsilon)^{q-1}\nabla n^\epsilon\cdot\nabla K^\epsilon dx$, we employ the fact that $\Delta K^\epsilon \leq 0$ to obtain

$$
\begin{aligned}
&-2\pi Qq(q-1)\int_B (n^\epsilon)^{q-1}\nabla n^\epsilon\cdot\nabla K^\epsilon dx \\
&= -2\pi Q(q-1)\int_B \nabla(n^\epsilon)^q\cdot\nabla K^\epsilon dx \\
&= -2\pi Q(q-1)\int_{\partial B}(n^\epsilon)^q\nabla K^\epsilon\cdot\eta d\sigma + 2\pi Q(q-1)\int_B (n^\epsilon)^q\Delta K^\epsilon dx \\
&\leq -2\pi Q(q-1)\int_{\partial B}(n^\epsilon)^q\left|\nabla K^\epsilon\right| d\sigma.
\end{aligned}
\tag{67}
$$

Using the property $|\nabla K^\epsilon| \leq \frac{1}{2\pi |x|}$ together with the trace inequality $\|f\|^2_{L^2(\partial B)} \leq \overline{\delta}\,\|\nabla f\|^2_{L^2(B)} + C_{\overline{\delta}}\,\|f\|^2_{L^2(B)}$, $f\in H^1(B)$, applied with $f=(n^\epsilon)^{q/2}$ and $\overline{\delta}=\frac{\delta}{Q(q-1)}$, we have

$$
\begin{aligned}
-2\pi Q(q-1)\int_{\partial B}(n^\epsilon)^q\left|\nabla K^\epsilon\right| d\sigma &\leq Q(q-1)\int_{\partial B}\frac{(n^\epsilon)^q}{|x|}d\sigma = Q(q-1)\int_{\partial B}(n^\epsilon)^q d\sigma \\
&\leq \delta\int_B \left|\nabla(n^\epsilon)^{q/2}\right|^2 dx + Q(q-1)C_{\overline{\delta}}\int_B (n^\epsilon)^q dx.
\end{aligned}
\tag{68}
$$

Combining the estimates (66)–(68) and using the zero-flux boundary conditions, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_B (n^\epsilon)^q dx \leq \left(-\frac{2(q-1)}{q}+\delta\right)\int_B \left|\nabla(n^\epsilon)^{q/2}\right|^2 dx \qquad (69)\\
&+\left(\chi\left(q-1\right)\left(\frac{\chi q C_1}{8}+\frac{C_2}{2}\right)\|\Delta v^\epsilon\|_2^2 + Q(q-1)C_{\overline{\delta}}\right)\int_B (n^\epsilon)^q dx + \frac{\chi\left(q-1\right)C_2^{1/2}}{2}\int_B (n^\epsilon)^q dx.
\end{aligned}
$$

We observe that

$$
\begin{aligned}
\int_B |\Delta v^\epsilon|^2 dx &= \int_B\left(-\frac{\theta}{\pi}+n^\epsilon\right)^2 dx \leq 2\int_B\left(\frac{\theta}{\pi}\right)^2 dx + 2\int_B (n^\epsilon)^2 dx \\
&\leq \frac{2\theta^2}{\pi} + 2\int_B (n^\epsilon)^2 dx.
\end{aligned}
\tag{70}
$$

Using that $(n^\epsilon)^2 \leq \frac{2(n^\epsilon)^q}{q}+\frac{q-2}{q}$ holds for any $q\geq 2$, we obtain from (70),

$$
\int_B |\Delta v^\epsilon|^2 dx \leq \frac{2\theta^2}{\pi} + \frac{2\left(q-2\right)\pi}{q} + \frac{4}{q}\int_B (n^\epsilon)^q\, dx.
\tag{71}
$$

Combining (69) and (71), we derive

$$
\begin{aligned}
\frac{d}{dt}\int_B (n^\epsilon)^q dx &\leq \left(-\frac{2(q-1)}{q}+\delta\right)\int_B \left|\nabla(n^\epsilon)^{q/2}\right|^2 dx \\
&+\left(\xi_1+\xi_2\int_B (n^\epsilon)^q\, dx\right)\int_B (n^\epsilon)^q dx,
\end{aligned}
$$

where

$$\xi_1 := \chi\,(q-1)\left(\frac{\chi q C_1}{8}+\frac{C_2}{2}\right)\left(\frac{2\theta^2}{\pi}+\frac{2\,(q-2)\,\pi}{q}\right)+Q(q-1)C_{\overline{\delta}}\frac{\chi\,(q-1)\,C_2^{1/2}}{2},$$

$$\xi_2 := \chi\,(q-1)\left(\frac{\chi q C_1}{8}+\frac{C_2}{2}\right)\frac{4}{q}.$$

Therefore, we get the inequality

$$\frac{d}{dt}\int_B (n^\epsilon)^q dx \le \left(\xi_1+\xi_2\int_B (n^\epsilon)^q\,dx\right)\int_B (n^\epsilon)^q dx.$$

Next, we consider the local smooth solution on the interval $[0,\tau_1)$ of the problem

$$\frac{dX}{dt}=(\xi_1+\xi_2 X)\,X,$$

$$X(0)=\int_B (n(x,0)^q dx.$$

By the comparison principle for ordinary differential equations, we obtain

$$\int_B (n^\epsilon)^q dx \le X \text{ on } [0,\tau_1).$$

Consequently, the inequality (62) holds true with $\tau=\tau_1/2$ and $C=\sup_{(0,\tau_1/2)}X(\tau)$. □

The proof of the following Lemma is an adaptation of [26, Lemma 2.1].

Lemma 12. *If $n_0\in L^\infty(B)$, then for some constant C independent of ϵ,*

$$\|n^\epsilon(\cdot,t)\|_{L^\infty}\le C\max\left\{1,\left\|n_0^\epsilon\right\|_{L^1},\|n_0\|_{L^\infty}\right\}\qquad \text{for } 0<t<\tau.$$

Furthermore, for a constant $\overline{C}$ independent of ϵ

$$\int_0^t\int_B\left|\nabla\,(n^\epsilon)^{(p+1)/2}\right|^2 dx\le\overline{C},\qquad \text{for all } p\ge 1.$$

Proof. From Lemma 11 and the elliptic regularity theory, we conclude that $v^\epsilon\in W^{2,q}(B)$ for any $q\in[2,\infty)$. Applying the Sobolev embedding theorem $W^{2,q}(B)\hookrightarrow C^{1,1-\frac{n}{q}}(B)$ with $n=2$ and $q>2$, we deduce that for a constant C_1 independent of t, we have

$$\|\nabla v^\epsilon(\cdot,t)\|_{L^\infty}\le C\quad \text{ for }\quad 0<t<\tau. \tag{72}$$

Now let $p\ge 1$. By multiplying the first equation in (14) by $(n^\epsilon)^p$ and integrating by parts, we obtain

$$\begin{aligned}\frac{1}{p+1}\frac{d}{dt}\int_B (n^\epsilon)^{p+1}\,dx &= -\int_B \nabla\,(n^\epsilon)^p\cdot(\nabla n^\epsilon-\chi n^\epsilon\nabla v^\epsilon+2\pi Q n^\epsilon\nabla K^\epsilon)\,dx\\ &\le -p\int_B (n^\epsilon)^{p-1}\,|\nabla(n^\epsilon)|^2\,dx+\chi p\int_B (n^\epsilon)^p\,\nabla n^\epsilon\cdot\nabla v^\epsilon dx-2\pi Q\int_B n^\epsilon\nabla(n^\epsilon)^p\cdot\nabla K^\epsilon dx\\ &=.-\frac{4p}{(p+1)}\int_B\left|\nabla(n^\epsilon)^{(p+1)/2}\right|^2 dx+\chi p\int_B (n^\epsilon)^p\,\nabla n^\epsilon\cdot\nabla v^\epsilon dx-2\pi pQ\\ &\int_B (n^\epsilon)^p\,\nabla n^\epsilon\cdot\nabla K^\epsilon dx.\end{aligned} \tag{73}$$

Using (72) and Hölder's inequality, we obtain

$$\begin{aligned}\chi p\int_B (n^\epsilon)^p\,\nabla n^\epsilon\cdot\nabla v^\epsilon dx\le\chi pC\int_B (n^\epsilon)^p\,|\nabla n^\epsilon|\,dx&=\frac{2\chi pC}{p+1}\int_B (n^\epsilon)^{\frac{p+1}{2}}\left|\nabla(n^\epsilon)^{\frac{p+1}{2}}\right|dx\\ &\le\frac{2p}{(p+1)^2}\int_B\left|\nabla(n^\epsilon)^{(p+1)/2}\right|^2 dx+\frac{p}{2}\chi^2C^2\int_B (n^\epsilon)^{p+1}dx.\end{aligned} \tag{74}$$

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For the last integral in (73), we notice

$$-2\pi pQ \int_B (n^\epsilon)^p \, \nabla n^\epsilon \cdot \nabla K^\epsilon dx = -\frac{2\pi pQ}{p+1} \int_B \nabla (n^\epsilon)^{p+1} \cdot \nabla K^\epsilon dx = \frac{2\pi pQ}{p+1} \int_B (n^\epsilon)^{p+1} \Delta K^\epsilon dx \le 0,$$

where we have used that $\Delta K^\epsilon \le 0$. Hence, we obtain

$$\frac{d}{dt} \int_B (n^\epsilon)^{p+1} \, dx \le -\frac{2p}{p+1} \int_B \left|\nabla \, (n^\epsilon)^{(p+1)/2}\right|^2 dx + \frac{p(p+1)}{2} C \int_B (n^\epsilon)^{p+1} \, dx, \tag{75}$$

which implies (72) by using Moser's technique (see [2]). Furthermore, rewriting the estimate (75), in the form

$$\int_B (n^\epsilon)^{p+1} \, dx - \int_B \left(n_0^\epsilon\right)^{p+1} dx \le -\frac{2p}{p+1} \int_0^t \int_B \left|\nabla \, (n^\epsilon)^{(p+1)/2}\right|^2 dx + \frac{p(p+1)}{2} C \int_0^t \int_B (n^\epsilon)^{p+1} \, dx,$$

we obtain as a byproduct of the estimate (72) that

$$\int_0^t \int_B \left|\nabla \, (n^\epsilon)^{(p+1)/2}\right|^2 dx \le \overline{C}, \text{ for all } p \ge 1,$$

where $\overline{C}$ is a constant which does not depend on ϵ. □

During the estimation (44) of the quantity $\left\| \frac{dm_i}{dt} \right\|_{L^2(0,T,H^1(B)^*)}$, we arrived at the estimate (52), which turned out to be depending on ϵ. As a consequence, we will modify this procedure completely to obtain a uniform estimate of dn^ϵ/dt. The key ingredient in our new approach is the use of an appropriate free energy functional associated with the regularised model.

4.1. *Energy functional for the regularised model*

In this section, we construct an energy functional for the regularised problem (14)–(16). To this purpose, let us assume for a moment that we are dealing with smooth solutions.

We can rewrite the equation for n^ϵ as:

$$n_t^\epsilon = \operatorname{div} \{n^\epsilon \nabla \, (\log n^\epsilon - (\chi v^\epsilon - 2\pi QK^\epsilon))\} .$$

We multiply this equation by $\log n^\epsilon - (\chi v^\epsilon - 2\pi QK^\epsilon)$, integrate the product by parts, and apply the no-flux condition (15) to obtain

$$\int_B n_t^\epsilon \, (\log n^\epsilon - (\chi v^\epsilon - 2\pi QK^\epsilon)) \, dx = -\int_B n^\epsilon \, |\nabla \, (\log n^\epsilon - (\chi v^\epsilon - 2\pi QK^\epsilon))|^2 \, dx. \tag{76}$$

We notice that the no-flux condition implies $\frac{d}{dt} \int n^\epsilon(x,t) dx = 0$ for all $t > 0$, and consequently (76) gives

$$\begin{aligned} \int_B n^\epsilon \, |\nabla \, (\log n^\epsilon - (\chi v^\epsilon - 2\pi QK^\epsilon n^\epsilon))|^2 \, dx &= \int_B n_t^\epsilon \, (\chi v^\epsilon - 2\pi QK^\epsilon) \, dx - \int_B n_t^\epsilon \log n^\epsilon dx \\ &=: I - \frac{d}{dt} \int_B n^\epsilon \log n^\epsilon dx. \end{aligned}$$

We rewrite the integral I as:

$$I = \frac{d}{dt} \int_B (\chi n^\epsilon v^\epsilon - 2\pi QK^\epsilon n^\epsilon) \, dx - \chi \int_B n^\epsilon v_t^\epsilon dx. \tag{77}$$

For the last integral in (77), we use the equation for v^ϵ to get

$$\begin{aligned} \int_B n^\epsilon v_t^\epsilon dx &= -\int_B \left(\Delta v^\epsilon - \frac{\theta}{\pi}\right) v_t^\epsilon dx = \frac{\theta}{\pi} \frac{d}{dt} \int v^\epsilon dx + \frac{1}{2} \frac{d}{dt} \, \|\nabla v^\epsilon\|_2^2 \\ &= \frac{d}{dt} \left(\frac{\theta}{\pi} \int v^\epsilon dx + \frac{1}{2} \int_B |\nabla v^\epsilon|^2 \, dx\right). \end{aligned} \tag{78}$$

Thus,

$$I = \frac{d}{dt}\int_B (\chi n^\epsilon v^\epsilon - 2\pi QK^\epsilon n^\epsilon)\,dx - \chi\frac{d}{dt}\left(\frac{\theta}{\pi}\int v^\epsilon dx - \frac{1}{2}\int_B |\nabla v^\epsilon|^2\,dx\right)$$
$$= \frac{d}{dt}\left(\int_B \chi n^\epsilon v^\epsilon - 2\pi QK^\epsilon n^\epsilon - \frac{\chi\theta}{\pi}\int v^\epsilon dx - \frac{\chi}{2}\int_B |\nabla v^\epsilon|^2\,dx\right).$$

In conclusion, we have

$$-\int_B n^\epsilon\,|\nabla(\log n^\epsilon - (\chi v^\epsilon - 2\pi QK^\epsilon n^\epsilon))|^2\,dx$$
$$= \frac{d}{dt}\left(\int_B n^\epsilon \log n^\epsilon dx - \int_B \chi n^\epsilon v^\epsilon dx + \int_B 2\pi QK^\epsilon n^\epsilon dx + \frac{\chi\theta}{\pi}\int v^\epsilon dx + \frac{\chi}{2}\int_B |\nabla v^\epsilon|^2\,dx\right),$$

or equivalently

$$\frac{d}{dt}\int_B \left(n^\epsilon \log n^\epsilon - \chi n^\epsilon v^\epsilon + 2\pi QK^\epsilon n^\epsilon + \frac{\chi\theta}{\pi}v^\epsilon + \frac{\chi}{2}|\nabla v^\epsilon|^2\right) dx$$
$$= -\int_B n^\epsilon\,|\nabla(\log n^\epsilon - (\chi v^\epsilon - 2\pi QK^\epsilon n^\epsilon))|^2\,dx. \tag{79}$$

In the framework of weak solutions, we have the following result.

Lemma 13. *Let us assume that $n_0 \in L^\infty(B)$. Let the functional W be defined by:*

$$W^\epsilon(t) := \int_B \left(n^\epsilon \log n^\epsilon - \chi n^\epsilon v^\epsilon + 2\pi QK^\epsilon n^\epsilon + \frac{\chi\theta}{\pi}v^\epsilon + \frac{\chi}{2}|\nabla v^\epsilon|^2\right) dx.$$

Then we have

$$W^\epsilon(t) - W^\epsilon(0) \le -\int_0^t\int_B n^\epsilon\,|\nabla(\log n^\epsilon - (\chi v^\epsilon - 2\pi QK^\epsilon n^\epsilon))|^2\,dx. \tag{80}$$

Proof. Let $\delta > 0$ be a constant. Consider the function $h:\mathbb{R}\to\mathbb{R}$ defined by:

$$f(z) := \log(z+\delta) \text{ for all } z \in \mathbb{R}.$$

Since f is Lipschitz continuous, by the chain rule, the composite function $\log(n^\epsilon + \delta)$ belongs to $H^1(B \times (0,T))$, as shown in [33, Theorem 2.1.11]. Similarly, the function ϕ given by $\phi := \log(n^\epsilon + \delta) - \chi v^\epsilon + 2\pi QK^\epsilon$ also belongs to the space $H^1(B\times(0,T))$. Therefore, we can use it as a test function in (17) to obtain

$$\int_B n(x,0)\phi(x,0)dx = \int_B n^\epsilon\phi dx - \int_0^t\int_B n^\epsilon\phi_\tau dxd\tau + \int_0^t\int_B (\nabla n^\epsilon - \chi n^\epsilon\nabla v^\epsilon + 2\pi Qn^\epsilon\nabla K^\epsilon)\cdot\nabla\phi dxd\tau$$
$$= I_1 - I_2 + I_3. \tag{81}$$

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We compute

$$
\begin{aligned}
I_2 &= \int_0^t \int_B n^\epsilon \left(\log\left(n^\epsilon+\delta\right) - \left(\chi v^\epsilon - 2\pi Q K^\epsilon\right)\right)_\tau dx d\tau \\
&= \int_0^t \int_B \left\{\left(n^\epsilon+\delta\right)\log\left(n^\epsilon+\delta\right)_\tau - \chi n^\epsilon v^\epsilon_\tau\right\} dx d\tau - \delta \int_0^t \int_B \log\left(n^\epsilon+\delta\right)_\tau dx \\
&= \int_0^t \int_B \left\{\left(n^\epsilon+\delta\right)\log\left(n^\epsilon+\delta\right)_\tau - \chi n^\epsilon v^\epsilon_\tau\right\} dx d\tau - \delta \int_B \log\left(n^\epsilon+\delta\right) dx + \delta \int_B \log\left(n^\epsilon(x,0)+\delta\right) dx \\
&= \int_0^t \int_B n^\epsilon_\tau dx d\tau - \chi \int_0^t \int_B n^\epsilon v^\epsilon_\tau dx d\tau - \delta \int_B \log\left(n^\epsilon+\delta\right) dx + \delta \int_B \log\left(n^\epsilon(x,0)+\delta\right) dx \\
&= -\chi \int_0^t \int_B n^\epsilon v^\epsilon_\tau dx d\tau - \delta \int_B \log\left(n^\epsilon+\delta\right) dx + \delta \int_B \log\left(n^\epsilon(x,0)+\delta\right) dx. \qquad (82)
\end{aligned}
$$

On the other hand, from (18) with $\gamma = v^\epsilon_\tau$, we have

$$
-\chi \int_0^t \int_B n^\epsilon v^\epsilon_\tau dx d\tau = -\chi \int_0^t \int_B \left(\nabla v^\epsilon \cdot \nabla v^\epsilon_\tau + \frac{\theta}{\pi} v^\epsilon_\tau\right) dx d\tau = -\chi \int_0^t \frac{d}{d\tau} \int_B \left(\frac{1}{2}\left|\nabla v^\epsilon\right|^2 + \frac{\theta}{\pi} v^\epsilon\right) dx dt. \qquad (83)
$$

From (82) and (83), we deduce

$$
\begin{aligned}
I_2 = &-\frac{\chi}{2}\int_B \left|\nabla v^\epsilon\right|^2 dx + \frac{\chi}{2}\int_B \left|\nabla v^\epsilon(x,0)\right|^2 dx - \frac{\chi\theta}{\pi}\int_B v^\epsilon dx + \frac{\chi\theta}{\pi}\int_B v^\epsilon(x,0) dx \qquad (84) \\
&- \delta \int_B \log\left(n^\epsilon+\delta\right) dx + \delta \int_B \log\left(n^\epsilon(x,0)+\delta\right) dx.
\end{aligned}
$$

For I_3, we have

$$
\begin{aligned}
I_3 &= \int_0^t \int_B \left(n^\epsilon+\delta\right) \nabla\left(\log\left(n^\epsilon+\delta\right) - \chi v^\epsilon + 2\pi Q K^\epsilon\right) \cdot \nabla\phi dx d\tau + \delta \int_0^t \int_B \left(\chi \nabla v^\epsilon - 2\pi Q \nabla K^\epsilon\right) \cdot \nabla\phi dx d\tau \\
&= \int_0^t \int_B \left(n^\epsilon+\delta\right) \left|\nabla\left(\log\left(n^\epsilon+\delta\right) - \chi v^\epsilon + 2\pi Q K^\epsilon\right)\right|^2 dx d\tau + \delta \int_0^t \int_B \left(\chi \nabla v^\epsilon - 2\pi Q \nabla K^\epsilon\right) \cdot \nabla\phi dx d\tau. \qquad (85)
\end{aligned}
$$

Combining (81), (84) and (85), and using $n^\epsilon(x,0) = n(x,0)$, we obtain

$$
\begin{aligned}
&\int_B n(x,0)\phi(x,0) dx \\
&= \int_B n^\epsilon \phi dx + \frac{\chi}{2}\int_B \left|\nabla v^\epsilon\right|^2 dx - \frac{\chi}{2}\int_B \left|\nabla v^\epsilon(x,0)\right|^2 dx + \frac{\chi\theta}{\pi}\int_B v^\epsilon dx - \frac{\chi\theta}{\pi}\int_B v^\epsilon(x,0) dx \\
&+ \delta \int_B \log\left(n^\epsilon+\delta\right) dx - \delta \int_B \log\left(n(x,0)+\delta\right) dx \\
&+ \int_0^t \int_B \left(n^\epsilon+\delta\right) \left|\nabla\left(\log\left(n^\epsilon+\delta\right) - \chi v^\epsilon + 2\pi Q K^\epsilon\right)\right|^2 dx d\tau + \delta \int_0^t \int_B \left(\chi \nabla v^\epsilon - 2\pi Q \nabla K^\epsilon\right) \cdot \nabla\phi dx d\tau. \qquad (86)
\end{aligned}
$$

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Lemma 12 give us enough control to pass to the limit as $\delta \to 0$ in the last identity, with the exception of term

$$\int_0^t \int_B (n^\epsilon + \delta) \left|\nabla \left(\log (n^\epsilon + \delta) - \chi v^\epsilon + 2\pi Q K^\epsilon\right)\right|^2 dx d\tau$$
$$= \int_0^t \int_B \frac{|\nabla (n^\epsilon + \delta)|^2}{n^\epsilon + \delta} + (n^\epsilon + \delta)\, \chi^2 |\nabla v^\epsilon|^2 + 4\pi^2 Q^2 (n^\epsilon + \delta)\, |\nabla K^\epsilon|^2$$
$$- 2\chi \nabla n^\epsilon \cdot \nabla v^\epsilon + 4\pi Q \nabla n^\epsilon \cdot \nabla K^\epsilon - 4\pi \chi Q (n^\epsilon + \delta)\, \nabla v^\epsilon \cdot \nabla K^\epsilon dx d\tau. \tag{87}$$

We rewrite the first integral in the last identity in the form

$$\int_0^t \int_B \frac{|\nabla (n^\epsilon + \delta)|^2}{n^\epsilon + \delta} dx d\tau = 4 \int_0^t \int_B \left|\nabla (n^\epsilon + \delta)^{1/2}\right|^2 dx d\tau,$$

and recalling the convexity of the functional

$$h \to \int_0^t \int_B h\, |\nabla \log h|^2 \, dx d\tau = 4 \int_0^t \int_B \left|\nabla h^{1/2}\right|^2 dx d\tau,$$

(cf. [10, Lemma 4]), we obtain by lower semicontinuity that, up to the extraction of sequence $(\delta_k)_{k\geq 1}$ which converges to 0

$$\int_0^t \int_B \left|\nabla (n^\epsilon)^{1/2}\right|^2 dx d\tau \leq \liminf_{\delta_k \to 0} \int_0^t \int_B \left|\nabla (n^\epsilon + \delta_k)^{1/2}\right|^2 dx d\tau. \tag{88}$$

In conclusion, Lemma 12 together with (88) allow us to conclude

$$W^\epsilon(0) - W^\epsilon(t) \geq \int_0^t \int_B n^\epsilon \left|\nabla \left(\log n^\epsilon - \chi v^\epsilon + 2\pi Q K^\epsilon\right)\right|^2 dx d\tau.$$

□

Lemma 14. *Let us assume that $n_0 \in L^\infty(B)$. For some constant C independent of ϵ*

$$\int_0^T \left\| \frac{dn^\epsilon}{dt} \right\|^2_{H^1(B)^*} dt \leq C.$$

Proof. We apply (12) together with Young's inequality $ab \leq a \log a + \exp(b)$ valid for all $a \geq 0$, $b \in \mathbb{R}$, to obtain

$$2\pi Q \int_B n^\epsilon K^\epsilon dx \leq -Q \int_B n^\epsilon \log |x| \, dx = Q \int_B n^\epsilon \log |x|^{-1} \, dx$$
$$\leq Q \int_B n^\epsilon \log n^\epsilon dx + Q \int_B |x|^{-1} \, dx = Q \int_B n^\epsilon \log n^\epsilon dx + 2\pi Q. \tag{89}$$

From Lemma 12, we know that for some constant constant C_0

$$\|n^\epsilon(\cdot, t)\|_{L^\infty(B)} \leq C_0 \text{ for all } t \in (0, T). \tag{90}$$

We denote $M := \max_{x\in[0,C_0]} x \log x$. The estimates (89) and (90) readily give us

$$-2\pi Q \int_B K^\epsilon n^\epsilon dx \geq -Q\pi M - 2\pi Q. \tag{91}$$

On the other hand, we recall the condition in $\int_B v^\epsilon dx = 0$, (see (4)), which allows us to apply Poincare's inequality to ensure that for a constant C_P,

$$\int_B |v^\epsilon| \, dx \leq \sqrt{\pi} \left(\int_B (v^\epsilon)^2 dx\right)^{1/2} \leq \sqrt{\pi} C_P \left(\int_B |\nabla v^\epsilon|^2 \, dx\right)^{1/2} \tag{92}$$
$$\leq \sqrt{\pi} C_P \left(\int_B \left(n^\epsilon - \frac{\theta}{\pi}\right)^2 dx\right)^{1/2} \leq \sqrt{2\pi} C_P \left(C_0^2 \pi + \frac{\theta^2}{\pi}\right)^{1/2}.$$

Thus, Lemma 12 together with (91) and (92) readily imply the existence of a constant C_1 independent of ϵ so that

$$W^\epsilon(t) \geq C_1. \tag{93}$$

On the other hand,

$$\begin{aligned}\left(\frac{dn^\epsilon}{dt}, \mu\right)_{(H^1(B)^*, H^1(B))} &= \left(\operatorname{div}\left\{n^\epsilon \nabla\left(\log n^\epsilon - (\chi v^\epsilon - 2\pi Q K^\epsilon)\right)\right\}, \mu\right)_{(H^1(B)^*, H^1(B))} \\ &= -\int_B n^\epsilon \nabla\left(\log n^\epsilon - (\chi v^\epsilon - 2\pi Q K^\epsilon)\right) \cdot \nabla \mu dx.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|\frac{dn^\epsilon}{dt}\right\|^2_{H^1(B)^*} &= \sup_{\|\mu\|_{H^1(B)} \leq 1} \left(\frac{dn^\epsilon}{dt}, \mu\right)_{(H^1(B)^*, H^1(B))} \\ &\leq \frac{1}{2}\int_B n^\epsilon \left|\nabla\left(\log n^\epsilon - (\chi v^\epsilon - 2\pi Q K^\epsilon)\right)\right|^2 dx + \frac{1}{2}\int_B n^\epsilon |\nabla \mu|^2 dx \\ &\leq \frac{1}{2}\int_B n^\epsilon \left|\nabla\left(\log n^\epsilon - (\chi v^\epsilon - 2\pi Q K^\epsilon)\right)\right|^2 dx + \frac{1}{2} C_0\end{aligned}$$

Integrating over $(0, T)$ and applying Lemma 13 together with (93),

$$\begin{aligned}&\int_0^T \left\|\frac{dn^\epsilon}{dt}\right\|^2_{L^2(0,T,H^1(B)^*)} dt \\ &\leq \frac{1}{2}\int_0^T \int_B n^\epsilon \left|\nabla\left(\log n^\epsilon - (\chi v^\epsilon - 2\pi Q K^\epsilon)\right)\right|^2 dxdt + \frac{1}{2} C_0 \\ &\leq \frac{1}{2} W^\epsilon(0) - \frac{1}{2} W^\epsilon(t) + \frac{1}{2} C_0 \leq \frac{1}{2} W^\epsilon(0) - \frac{1}{2} C_1 + \frac{1}{2} C_0.\end{aligned} \tag{94}$$

To obtain an upper estimate for $W^\epsilon(0)$, independent of ϵ, we first notice that

$$\begin{aligned}W^\epsilon(0) &:= \int_B \left(n^\epsilon(x,0) \log n^\epsilon(x,0) - \chi n^\epsilon(x,0) v^\epsilon(x,0) + 2\pi Q K^\epsilon(x) n^\epsilon(x,0) + \frac{\chi}{2}\left|\nabla v^\epsilon(x,0)\right|^2\right) dx \\ &= \int_B \left(n(x,0) \log n(x,0) - \chi n(x,0) v(x,0) + 2\pi Q K^\epsilon(x) n(x,0) + \frac{\chi}{2}\left|\nabla v(x,0)\right|^2\right) dx.\end{aligned}$$

Note that an upper bound for the integral $2\pi Q \int_B K^\epsilon(x) n(x,0) dx$ is provided by estimate (91). Thus,

$$W^\epsilon(0) \leq \int_B \left(n_0 \log n_0 - \chi n_0 v_0 + \frac{\chi}{2}|\nabla v_0|^2\right) dx + Q\pi M + 2\pi Q. \tag{95}$$

In conclusion, estimates (94) and (95) give us a constant C_2, independent of ϵ, satisfying

$$\int_0^T \left\|\frac{dn^\epsilon}{dt}\right\|^2_{H^1(B)^*} dt \leq C_2.$$

□

Theorem 15 (Local existence). *Given a non-negative initial condition $n_0 \in L^\infty(B)$, there exists $T = T(n_0) > 0$ such that the problem (4)–(7) has a weak solution in the sense of definition 4. Moreover, for any $q > 2$, there exist a constant $C(q, T)$ such that*

$$\|n(\cdot, t)\|_{L^q(B)} < C(q, T), \text{ a.e. on } 0 < t < T \tag{96}$$

If $T_{\max}$ is the maximal time of existence, the problem (4)–(7) and $T_{\max} < \infty$ then

$$\limsup_{t \to T_{\max}} \|n(\cdot, t)\|_{L^\infty(B)} = \infty. \tag{97}$$

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The functional

$$W(t) := \int_B \left(n \log n - \chi n v - Q n \log |x| + \frac{\chi \theta}{\pi} v + \frac{\chi}{2} |\nabla v|^2 \right) dx,$$

satisfies

$$W(t) \leq W(0) \text{ for all } 0 < t < T_{\max}. \tag{98}$$

Proof. Lemma (11) provides a constant C_1 and a time T independent of ϵ such that

$$\|n^\epsilon\|_{L^q((0,T)\times B)} \leq C_1.$$

Consequently, there exists $n \in L^q((0,T) \times B)$ such that

$$n^\epsilon \to n \text{ weakly in } L^q((0,T) \times B) \text{ when } \epsilon \to 0. \tag{99}$$

The weak-lower semicontinuity of the norm implies

$$\|n(\cdot, t)\|_{L^q((0,T)\times B)} < C_1, \text{ for all } 0 < t < T.$$

On the other hand, using $|\nabla K^\epsilon(x)| \leq \frac{1}{2\pi |x|} \in L^s(B)$ for all $s \in [1,2)$ together with the convergence:

$$\nabla K^\epsilon(x) \to -\frac{x}{2\pi |x|^2} \text{ a.e in } B \text{ when } \epsilon \to 0,$$

we obtain by Lebesgue dominated convergence theorem

$$\nabla K^\epsilon(x) \to -\frac{x}{2\pi |x|^2} \text{ strongly in } L^s(B) \times L^s(B).$$

In particular, for the constant $q^* = \frac{q}{q-1} \in (1,2)$,

$$\nabla K^\epsilon(x) \to -\frac{x}{2\pi |x|^2} \text{ strongly in } L^{q^*}(B) \times L^{q^*}(B). \tag{100}$$

From (99) and (100),

$$n^\epsilon \nabla K^\epsilon \to -n\frac{x}{2\pi |x|^2} \text{ weakly in } L^1((0,T) \times B) \times L^1((0,T) \times B) \text{ when } \epsilon \to 0^+. \tag{101}$$

Lemma 11 readily provides the existence of a constant C_2 satisfying

$$\|n^\epsilon\|_{L^2(0,T,H^1(B))} \leq C_2. \tag{102}$$

We also have from Lemma 14 the existence of a constant C_3, independent of ϵ such that

$$\left\| \frac{dn^\epsilon}{dt} \right\|_{L^2(0,T,H^1(B)^*)} \leq C_3. \tag{103}$$

Recalling the embeddings

$$H^1(B) \overset{compact}{\hookrightarrow} L^2(B) \overset{continuous}{\hookrightarrow} H^1(B)^*,$$

and taking into account that the constants C_2 and C_3 are independent of ϵ, we get, up to a subsequence, by the Aubin–Lions compactness Lemma

$$n^\epsilon \to n \text{ as } \epsilon \to 0, \text{ strong in } L^2(0,T,L^2(B)). \tag{104}$$

The theory of linear elliptic equations give us a constant C_4 satisfying

$$\|v^\epsilon(*,t)\|_{H^1(B)} \leq C_4 \left\| n^\epsilon(*,t) - \frac{\theta}{\pi} \right\|_{L^2(B)} \quad \text{for a.e. } t \in (0,T). \tag{105}$$

The last estimate together with Lemma 11 imply the existence of $v(*,t) \in H^1(B)$ satisfying

$$v^\epsilon(*,t) \to v(*,t) \text{ weakly in } H^1(B),$$

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as $\epsilon \to 0$, as well as

$$v^\epsilon \to v \text{ weakly in } L^2(0,T,H^1(B)). \tag{106}$$

From (104) and (106),

$$n^\epsilon \nabla v^\epsilon \to n\nabla v \text{ weakly in } L^1((0,T)\times B).$$

In conclusion, taking limits in (17) and (18) when $\epsilon \to 0$, we find that that n,v satisfy (10), (11). That is, we have proved the local existence of solutions in the interval $(0,T)$.

In order to prove the extension criteria (97), we proceed by contradiction. Let us assume $T_{\max}$ is the maximal time of existence. If (97) does not hold, then we can repeat the argument to construct solutions on $(0,T_{\max})$ to obtain a new solution with initial data in $t=T_{\max}$. Thus, we would have a solution in some interval of the form $(0,T^*)$ with $T^*>T$, which contradicts the maximality of T.

Finally, let us show that n,v satisfies (98). Using Lemma 11 along with the strong convergence result (104), we get, up to subsequence in ϵ

$$n^\epsilon(\cdot,t) \to n(\cdot,t) \text{ as } \epsilon \to 0 \ \text{ strong in } L^q(B), \text{ a.e. for } t\in(0,T). \tag{107}$$

and that the function n satisfies

$$\|n(\cdot,t)\|_{L^q(B)} \le C_5 \qquad \text{a.e. for } t\in(0,T).$$

for some constant C_5. Also from Lemma 11, estimate (105) and the Rellich–Kondrachov theorem

$$v^\epsilon(\cdot,t) \to v(\cdot,t) \text{ as } \epsilon \to 0 \ \text{ strong in } L^2(B), \text{ a.e. for } t\in(0,T), \tag{108}$$

and

$$\nabla v^\epsilon(*,t) \to \nabla v(*,t) \text{ as } \epsilon \to 0 \ \text{ weakly in } L^2(B), \text{ a.e. for } t\in(0,T).$$

Therefore,

$$n^\epsilon(\cdot,t)v^\epsilon(\cdot,t) \to n(\cdot,t)v(\cdot,t) \text{ as } \epsilon \to 0 \ \text{ strong in } L^1(B), \text{ a.e. for } t\in(0,T), \tag{109}$$

and by convexity

$$\int_B \left(n(\cdot,t)\log n(\cdot,t) + \frac{\chi}{2}\left|\nabla v(\cdot,t)\right|^2\right) \le \liminf_{\epsilon\to 0}\int_B \left(n^\epsilon(\cdot,t)\log n^\epsilon(\cdot,t) + \frac{\chi}{2}\left|\nabla v^\epsilon(\cdot,t)\right|^2\right) dx \ \text{ a.e. for } t\in(0,T). \tag{110}$$

We also note that the strong convergence (108) implies

$$v^\epsilon(\cdot,t) \to v(\cdot,t) \text{ as } \epsilon \to 0 \ \text{ strong in } L^1(B), \text{ a.e. for } t\in(0,T). \tag{111}$$

Taking into account the convergence (107), we obtain, up to a subsequence,

$$n^\epsilon \to n \text{ a.e in } B\times(0,T_{\max}) \text{ as } \epsilon\to 0. \tag{112}$$

We also have

$$K^\epsilon \to K \text{ a.e in } B \text{ as } \epsilon \to 0. \tag{113}$$

It follows from (112), (113), Lemma 12 and the estimate

$$n^\epsilon(x,\cdot)K^\epsilon(x) \le \frac{1}{2\pi}\left\|n^\epsilon(\cdot,t)\right\|_{L^\infty(B)} \left|\log|x|\right|,$$

that we can apply Lebesgue dominated convergence theorem to obtain

$$\int_B K^\epsilon n^\epsilon dx \to -\frac{1}{2\pi}\int_B n\log|x|\,dx \text{ as } \epsilon\to 0^+ \text{ for } 0\le t<T.$$

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On the other hand, we have that

$$\lim_{\epsilon\to 0^+} W^\epsilon(x,0) = \lim_{\epsilon\to 0^+}\left(\int_B\left(n_0^\epsilon \log n_0^\epsilon - \chi n_0^\epsilon v_0^\epsilon + 2\pi QK^\epsilon n_0^\epsilon + \frac{\chi\theta}{\pi}v_0^\epsilon + \frac{\chi}{2}\left|\nabla v_0^\epsilon\right|^2\right)dx\right)$$
$$= \lim_{\epsilon\to 0^+}\left(\int_B\left(n_0 \log n_0 - \chi n_0 v_0 + 2\pi QK^\epsilon n_0 + \frac{\chi\theta}{\pi}v_0 + \frac{\chi}{2}\left|\nabla v_0\right|^2\right)dx\right)$$
$$= \int_B\left(n_0 \log n_0 - \chi n_0 v_0 - Qn_0\log|x| + \frac{\chi\theta}{\pi}v_0 + \frac{\chi}{2}\left|\nabla v_0\right|^2\right)dx. \tag{114}$$

We obtain from (109), (110) and (114)

$$\int_B\left(n\log n - \chi nv - Qn\log|x| + \frac{\chi\theta}{\pi}v + \frac{\chi}{2}\left|\nabla v\right|^2\right)dx \tag{115}$$
$$\le \liminf_{\epsilon\to 0^+}\int_B\left(n^\epsilon \log n^\epsilon - \chi n^\epsilon v^\epsilon + 2\pi QK^\epsilon n^\epsilon + \frac{\chi\theta}{\pi}v^\epsilon + \frac{\chi}{2}\left|\nabla v^\epsilon\right|^2\right)dx.$$

In conclusion, an application of the energy inequality given by Lemma 13 together with the estimates (114) and (115) give us

$$W(t) = \int_B\left(n\log n - \chi nv - Qn\log|x| + \frac{\chi\theta}{\pi}v + \frac{\chi}{2}\left|\nabla v\right|^2\right)dx \le \liminf_{\epsilon\to 0^+} W^\epsilon(t)$$
$$\le \liminf_{\epsilon\to 0^+}\left(W^\epsilon(0) - \int_0^t\int_B n^\epsilon\left|\nabla\left(\log n^\epsilon - (\chi v^\epsilon - 2\pi QK^\epsilon n^\epsilon)\right)\right|^2 dx\right)$$
$$\le \liminf_{\epsilon\to 0^+} W^\epsilon(0) = \int_B\left(n_0\log n_0 - \chi n_0 v_0 - Qn_0\log|x| + \frac{\chi\theta}{\pi}v_0 + \frac{\chi}{2}\left|\nabla v_0\right|^2\right)dx = W(0).$$

□

5. Global existence

The main tools in this section are the free energy functional and a version of the Moser–Trudinger inequality involving singular weights. We recall firstly the classical Moser–Trudinger inequality.

Theorem 16 (Moser–Trudinger inequality, [25]). *Let Ω be bounded domain in $\mathbb{R}^n$ $(n\ge 2)$. Let $h\in W_0^{1,n}(\Omega)$ and*

$$\int_\Omega |\nabla h|^n\,dx \le 1.$$

Then there exists a constant κ depending only on n such that

$$\int_\Omega e^{\alpha h^{\frac{n}{n-1}}}dx \le \kappa\,|\Omega|\,,$$

where $\alpha \le n\omega_{n-1}^{1/(n-1)}$, and ω_{n-1} is the $(n-1)-$dimensional surface area of the unit sphere in $\mathbb{R}^n$.

Proposition 17. *Let $f\in W^{1,n}(B(0,L)))$ with $f(x) = f(|x|)$. Then for any $\varepsilon > 0$, there exists a constant C_ε depending on ε and $|B(0,L)|$ such that*

$$f(L) \le \varepsilon\,\|\nabla f\|_{L^n(B(0,L))} + C_\varepsilon\,\|f\|_{L^1(B(0,L))}\,, \tag{116}$$

where the left-hand side of this inequality is interpreted in sense of the trace.

Proof. Since $C^1(\overline{B(0,L)})$ is dense in $W^{1,n}(B(0,L))$, it suffices to prove the case $f\in C^1(\overline{B(0,L)})$.

Let us denote $\varepsilon_n = \varepsilon\omega_n^{1/n}$. Choose $r_0 \in [e^{-\varepsilon_n n/(n-1)}L, L)$ such that

$$f(r_0)r_0^{n-1} \le \frac{1}{(1-e^{-\varepsilon_n n/(n-1)})L}\int_{e^{-\varepsilon_n n/(n-1)}L}^{L} f(r)r^{n-1}dr.$$

Then

$$
\begin{aligned}
f(r_0) &\le r_0^{1-n}\frac{1}{(1-e^{-\varepsilon_n n/(n-1)})L\omega_{n-1}}\int_{B(0,L)} f(x)dx \\
&\le (e^{-\varepsilon_n n/(n-1)}L)^{1-n}\frac{1}{(1-e^{-\varepsilon_n n/(n-1)})L\omega_{n-1}}\int_{B(0,L)} f(x)dx \\
&= e^{n\varepsilon_n}\frac{1}{(1-e^{-\varepsilon_n n/(n-1)})L^n\omega_{n-1}}\int_{B(0,L)} f(x)dx \\
&= \frac{e^{n\varepsilon_n}}{n(1-e^{-\varepsilon_n n/(n-1)})\,|B(0,L)|}\int_{B(0,L)} f(x)dx. \qquad (117)
\end{aligned}
$$

Here, we used $|B(0,L)| = n^{-1}L^n\omega_{n-1}$. Next, we apply Hölder's inequality to obtain

$$
\begin{aligned}
f(L) &= f(r_0) + \int_{r_0}^{L} f'(r)dr \le f(r_0) + \int_{e^{-\varepsilon_n n/(n-1)}L}^{L} r^{(1-n)/n} f'(r) r^{(n-1)/n} dr \\
&\le f(r_0) + \left(\int_{e^{-\varepsilon_n n/(n-1)}L}^{L} r^{-1}dr\right)^{(n-1)/n}\left(\int_{e^{-\varepsilon_n n/(n-1)}L}^{L} \left|f'(r)\right|^n r^{n-1}dr\right)^{1/n} \\
&\le f(r_0) + \left(\int_{e^{-\varepsilon_n n/(n-1)}L}^{L} r^{-1}dr\right)^{(n-1)/n}\left(\frac{1}{\omega_n}\right)^{1/n}\|\nabla f\|_n \\
&= f(r_0) + \varepsilon_n\left(\frac{1}{\omega_n}\right)^{1/n}\|\nabla f\|_n .
\end{aligned}
$$

The last estimate, together with (117), and the definition of ε_n leads to

$$
f(L) \le \varepsilon\,\|\nabla f\|_{L^n(B(0,L))} + \frac{e^{n\varepsilon\omega_n^{-1/n}}}{n(1-e^{-\varepsilon\omega_n^{-1/n} n/(n-1)})\,|B(0,L)|}\,\|f\|_{L^1(B(0,L))}\,.
$$

In conclusion, we have proved (116) with

$$
C_\varepsilon := \frac{e^{n\varepsilon\omega_n^{-1/n}}}{n(1-e^{-\varepsilon\omega_n^{-1/n} n/(n-1)})\,|B(0,L)|}.
$$

. □

In the next result, we propose in the radial case an extension for a version of the Moser–Trudinger inequality with weight (cf. [5, Theorem 2.1] and [7, Corollary 2.5]). The main novelty is that our result allows having singularities in the weight function.

Proposition 18 (Singular Moser–trudinger inequality). *Let $P > -2$ a constant and $B := B(0,1) \subseteq \mathbb{R}^2$ a ball of radius 1 and centred at the origin. Let $f \in H_0^1(B)$ with $f(x) = f(\,|x|\,)$. Then there exists a constant $C_P = C(P, |B|\,)$ such that*

$$
\log\left(\int_B |x|^P e^{|f|}dx\right) \le \frac{1}{8\pi(2+P)}\,\|\nabla f\|_2^2 + C_P. \qquad (118)
$$

Proof. Since $C^1(\overline{B})$ is dense in $H^1(B)$, we can assume that $f \in C^1(\overline{B})$. We can also assume $f \ge 0$ because if it is not the case, we apply (118) to the function $f^* = f + \left|\sup_{\overline{B}} f\right| \ge 0$, which in turn lead us to get (118) for such a function f. Let $I := \int_B |x|^P \exp(f)dx$. In polar coordinates

$$
I = 2\pi\int_0^1 r^{P+1}\exp(f(r))dr.
$$

We look for a function $\rho = \rho(r)$ producing $\rho d\rho = r^{P+1}dr$ and hence, we take

$$
\rho := \sqrt{\frac{2}{P+2}}\,r^{\frac{P+2}{2}}. \qquad (119)
$$

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Then

$$I = 2\pi \int_0^{\sqrt{\frac{2}{P+2}}} \rho \exp f(r(\rho)) d\rho = \int_{B_P} \exp(\bar{f}) dy, \tag{120}$$

where $B_P := B\left(0, \sqrt{\frac{2}{P+2}}\right)$ and $\bar{f} := f(r(\rho))$ or equivalently,

$$\bar{f}(y) := f\left(\left(\frac{P+2}{2}\right)^{\frac{1}{P+2}} |y|^{\frac{2}{P+2}}\right) \text{ for } y \in B_P.$$

Likewise, applying (119) yields

$$\int_B |\nabla f(x)|^2 \, dx = 2\pi \int_0^1 |f_r|^2 \, r dr = 2\pi \int_0^{\sqrt{\frac{2}{P+2}}} \left|\frac{d\rho}{dr}\bar{f}_\rho\right|^2 r \frac{1}{d\rho/dr} d\rho = 2\pi \int_0^{\sqrt{\frac{2}{P+2}}} |\bar{f}_\rho|^2 \, r \frac{d\rho}{dr} d\rho$$

$$= 2\pi \frac{P+2}{2} \int_0^{\sqrt{\frac{2}{P+2}}} |\bar{f}_\rho|^2 \, \rho d\rho = \frac{P+2}{2} \int_{B_P} |\nabla \bar{f}|^2 \, dy. \tag{121}$$

Now, by applying both (120) and the Moser–Trudinger inequality (Theorem 16), we obtain

$$I = \int_{B_P} \exp(\bar{f}) dx \le \int_{B_P} \exp\left(\frac{4\pi \bar{f}^2}{\|\nabla \bar{f}\|^2_{L^2(B_P)}} + \frac{\|\nabla \bar{f}\|^2_{L^2(B_P)}}{16\pi}\right) dx$$

$$\le \exp\left(\frac{\|\nabla \bar{f}\|^2_{L^2(B_P)}}{16\pi}\right) \int_{B_P} \exp\left(\frac{4\pi \bar{f}^2}{\|\nabla \bar{f}\|^2_{L^2(B_P)}}\right) dx \le \kappa \, |B_P| \exp\left(\frac{\|\nabla \bar{f}\|^2_{L^2(B_P)}}{16\pi}\right). \tag{122}$$

We conclude from (121) and (122)

$$I \le \kappa \, |B_P| \exp\left\{\left(\frac{1}{8\pi(2+P)}\right) \|\nabla f\|^2_{L^2(B)}\right\}. \tag{123}$$

We conclude from (123) the validity of (118) with $C_P = \log(\kappa \, |B_P|) = \log(\kappa \pi P^2)$. □

Theorem 19. *Let $S > -2$ and $B := B(0,1) \subseteq \mathbb{R}^2$ a ball of radius 1. Let $g \in H^1(B)$ with $g(x) = g(|x|)$. Then for any $\delta > 0$, there exists a constant $C(\delta, S, |B|)$ such that*

$$\int_B |x|^S \exp(|g|) dx \le C(\delta, S, |B|) \exp\left\{\left(\frac{1}{8\pi(2+S)} + \delta\right) \|\nabla g\|_2^2 + \frac{2}{|B|} \|g\|_{L^1(B)}\right\}. \tag{124}$$

Proof. Recalling that $C^1(\bar{B})$ is dense in $H^1(B)$, we assume without loss of generality that $g \in C^1(\bar{B})$ and $g \ge 0$. Taking into account that $G := (g - g(1))_+ \in H_0^1(B)$ satisfies $\|\nabla G\|_2 \le \|\nabla g\|_2$, we apply Proposition 18 to obtain

$$\log\left(\int_B |x|^S e^G dx\right) \le \frac{1}{8\pi(2+S)} \|\nabla G\|_2^2 + C_S. \tag{125}$$

To estimate the left-hand side in (125) from below, we notice

$$\int_B |x|^S e^G dx \ge e^{-g(L)} \int_B |x|^S e^g dx. \tag{126}$$

Therefore, we obtain from (125) and (126)

$$\int_B |x|^S e^g dx \le C_1 \exp\left(\frac{1}{8\pi(2+S)} \|\nabla G\|_2^2 + g(1)\right), \tag{127}$$

where $C_1 := \exp(C_S)$. The term $g(1)$ can be estimated by Proposition 17 in the form

$$g(1) \le \delta \, \|\nabla g\|_2^2 + \frac{2}{|B|} \int_B g(x) dx + C_\delta, \tag{128}$$

We conclude from (127) and (128) that (124) is valid with $C(\delta, S, |B|) := C_1 \exp(C_\delta)$. □

Proof of the theorem of global existence (Theorem 1). Let us denote by b a positive parameter to be prescribed later and

$$\mu := \int_B |x|^Q e^{bv} dx.$$

By leveraging the mass conservation property for n alongside Jensen's inequality, we obtain

$$\begin{aligned} 0 &= -\log\left(\int_B \frac{|x|^Q e^{bv}}{\mu} dx\right) \le \int_B \left[-\log\left(\frac{\theta}{n}\frac{|x|^Q e^{bv}}{\mu}\right)\right]\frac{n}{\theta} dx \\ &= \frac{1}{\theta}\int_B (n\log n - bnv - n\log\theta + n\log\mu - Qn\log|x|)\, dx \\ &= \frac{1}{\theta}\int_B (n\log n - bnv)\, dx - \log\theta + \log\mu - \frac{Q}{\theta}\int_B n\log|x|\, dx. \end{aligned}$$

Consequently,

$$0 \le \int_B (n\log n - bnv)\, dx - \theta\log\theta + \theta\log\left(\int_B |x|^Q e^{bv} dx\right) - Q\int_B n\log|x|\, dx. \tag{129}$$

The singular version of the Moser–Trudinger inequality (Theorem 19) gives for any $\delta > 0$

$$\log\left(\int_B |x|^Q \exp(bv) dx\right) \le b^2\left(\frac{1}{8\pi(2+Q)} + \delta\right)\|\nabla v\|_2^2 + \frac{2b}{|B|}\|v\|_{L^1(B)} + \log C(\delta, Q, |B|). \tag{130}$$

From (129)–(130), we get

$$\begin{aligned} 0 \le &\int_B (n\log n - bnv)\, dx - \theta\log\theta \\ &+ \theta b^2\left(\frac{1}{8\pi(2+Q)} + \delta\right)\|\nabla v\|_2^2 + \frac{2b\theta}{|B|}\|v\|_{L^1(B)} \\ &+ \theta\log C(\delta, Q, |B|) - Q\int_B n\log|x|\, dx. \end{aligned} \tag{131}$$

Substituting the definition of W into (131), we get

$$\begin{aligned} 0 \le W(n, v) + (\chi - b)\int_B nvdx - \left(\frac{\chi}{2} - \frac{\theta b^2}{8\pi(2+Q)} - \delta\theta b^2\right)\|\nabla v\|_2^2 \\ + \left(\frac{2\theta b}{|B|} - \frac{\chi\theta}{\pi}\right)\|v\|_{L^1(B)} + \theta\log C(\delta, Q, |B|) - \theta\log\theta. \end{aligned} \tag{132}$$

Using the monotonicity in time of the energy functional (98), we get

$$\begin{aligned} 0 \le W(n_0, v_0) + (\chi - b)\int_B nvdx - \left(\frac{\chi}{2} - \frac{\theta b^2}{8\pi(2+Q)} - \delta\theta b^2\right)\|\nabla v\|_2^2 \\ + \left(\frac{2\theta b}{|B|} - \frac{\chi\theta}{\pi}\right)\|v\|_{L^1(B)} + \theta\log C(\delta, Q, |B|) - \theta\log\theta. \end{aligned} \tag{133}$$

Notably, the condition $\int_B vdx = 0$ permits the utilisation of Poincare's inequality, yielding

$$\|v\|_{L^2(B)}^2 \le C_P \|\nabla v\|_{L^2(B)}^2.$$

Thus, we derive the existence of a constant $C_\delta > 0$ such that

$$\left(\frac{2\theta b}{|B|} - \frac{\chi\theta}{\pi}\right)\|v\|_{L^1(B)} \le \delta\|v\|_{L^2(B)}^2 + C_{\delta_0} \le \delta C_p\|\nabla v\|_{L^2(B)}^2 + C_\delta. \tag{134}$$

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Consequently, combining (133) and (134),

$$\left(\frac{\chi}{2}-\frac{\theta b^2}{8\pi(2+Q)}-\delta\left(\theta b^2+C_P\right)\right)\|\nabla v\|_2^2+(b-\chi)\int_B nvdx \tag{135}$$
$$\leq W(n_0,v_0)+\theta\log C(\delta,Q,|B|\,)-\theta\log\theta+C_\delta.$$

Now, we look pick the b such that it satisfies

$$\frac{\chi}{2}-\frac{\theta b^2}{8\pi(2+Q)}>0 \quad \text{and} \quad b>\chi.$$

Equivalently

$$\chi^2<b^2<\frac{4\pi\chi(2+Q)}{\theta}.$$

The existence of such a b is clear since the condition (8) implies

$$\chi^2<\frac{4\pi\chi(2+Q)}{\theta}.$$

We fix one of those such a constant b and next, we choose $\delta>0$ small enough to have

$$\frac{\chi}{2}-\frac{\theta b^2}{8\pi(2+Q)}-\delta\left(\theta b^2+C_P\right)>0.$$

Therefore, we obtain from (135) that for some constant $C_1=C_1(Q,\chi,\theta)$,

$$\|v\|_{H^1}\leq C_1, \qquad \int_B nvdx\leq C_1. \tag{136}$$

To estimate the integral $\int n\log ndx$, we first notice that (132) together with (134) and (136) imply that W is lower-bounded. Let us denote by $C_2:=C_2(\chi,Q,\theta)$ a constant satisfying

$$W(n,v)\geq C_2(\chi,Q,\theta). \tag{137}$$

The function

$$\xi(s):=\frac{4\pi\ (2s+Q)}{\chi},\ \ s\in\mathbb{R},$$

satisfies $\xi(1)=\frac{4\pi(2+Q)}{\chi}>\theta$. Hence, we can assure by continuity the existence of $0<s_0<1$ such that

$$\xi(s_0)=\frac{4\pi\ (2s_0+Q)}{\chi}>\theta. \tag{138}$$

We rewrite the energy functional as:

$$W(n_0)\geq W(n(t))=(1-s_0)\int_B n\log ndx+s_0\int_B\left(n\log n-\frac{\chi}{s_0}nv-\frac{Qn\log|x|}{s_0}+\frac{\chi\theta}{\pi s_0}v+\frac{\chi}{2s_0}|\nabla v|^2\right)dx. \tag{139}$$

According to the estimate in (137), we can ensure that the second integral is lower-bounded by the constant $C_2(\chi/s_0,Q/s_0,\theta)$ as long as

$$\theta<\frac{4\pi\ (2+Q/s_0)}{\chi/s_0}=\frac{4\pi\ (2s_0+Q)}{\chi}, \tag{140}$$

what turns out to be true by definition of s_0 (cf. (138)). Consequently, we deduce from (139) that

$$\int_B n\log ndx\leq\frac{W\,(0)-s_0C_2(\chi/s_0,Q/s_0)}{1-s_0}. \tag{141}$$

Let $n_*:=nI_{n\leq 1}$, then

$$0\leq\int_B n_*\,|\log n_*|\,dx\leq\frac{|B|}{e}. \tag{142}$$

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Combining (141) and (142),

$$\int_B n\,|\log n|\,dx = \int_B n\log n dx - 2\int_B n_*\,|\log n_*|\,dx$$
$$\leq \frac{W(0) - s_0 C_2(\chi/s_0, Q/s_0)}{1-s_0} + \frac{2\,|B|}{e}. \tag{143}$$

At this juncture, we aim to control the L^r-norms of the variable n. Before proceeding, let us recall that if $n \in L^2((0,T);H^1(B))$ and $n_t \in L^2((0,T);H^{-1}(B))$, it ensures that $n \in C([0,T];L^2(B)$, as well as the absolutely continuity of the the map $t \to \|n(\cdot,t)\|_{L^2(B)}$ and the validity of the identity

$$\frac{d}{dt}\int_B n^2(x,t)dx = 2\left\langle n'(\cdot,t), n(\cdot,t)\right\rangle, \quad \text{a.e. } 0 \leq t \leq T,$$

cf. [16, chapter 5, theorem 3.]. Now, we take $\phi = n$ in (10) to obtain

$$\int_B n^2 dx - \int_0^t \int_B nn_\tau dxd\tau + \int_0^t \int_B \left(\nabla n - \chi n\nabla v - n\frac{Qx}{|x|^2}\right)\cdot \nabla ndxd\tau = \int_B n^2(x,0)dx,$$

or equivalently

$$\frac{1}{2}\int_B n^2 dx + \int_0^t \int_B \left(|\nabla n|^2 - \chi n\nabla v\cdot\nabla n - n\frac{Qx}{|x|^2}\cdot\nabla n\right) dxd\tau = \frac{1}{2}\int_B n^2(x,0)dx.$$

Rearranging and integrating by parts,

$$\frac{1}{2}\int_B n^2 dx + \int_0^t \int_B |\nabla n|^2\,dxd\tau = \frac{1}{2}\int_B n_0^2 dx + \int_0^t \int_B -\frac{\chi}{2}\Delta v n^2 dxd\tau + \int_0^t \int_B n\nabla n\cdot\frac{Qx}{|x|^2}dxd\tau. \tag{144}$$

To estimate the last integral, we rewrite it in polar coordinates

$$\int_0^t \int_B n\nabla n\cdot\frac{Qx}{|x|^2}dxd\tau = Q\int_0^t \int_0^{2\pi}\int_0^1 nn_r drd\theta d\tau$$
$$= Q\pi\int_0^t (n^2(1,\tau) - n^2(0,\tau))d\tau$$
$$\leq Q\pi\int_0^t n^2(1,\tau)d\tau. \tag{145}$$

Applying Proposition 17, we obtain that for any $\varepsilon > 0$,

$$\int_0^t n^2(1,t)d\tau \leq 2\varepsilon^2\int_0^t \int_B |\nabla n|^2\,dxd\tau + 2C_\varepsilon^2\int_0^t\left(\int_B ndx\right)^2 d\tau$$
$$= 2\varepsilon^2\int_0^t \int_B |\nabla n|^2\,dxd\tau + 2C_\varepsilon^2\left(\int_B n_0 dx\right)^2 t.$$

The last inequality together with (145) give

$$\int_0^t \int_B n\nabla n\cdot\frac{Qx}{|x|^2}dxd\tau \leq 2\varepsilon^2 Q\pi\int_0^t \int_B |\nabla n|^2\,dxd\tau + 2C_\varepsilon^2 Q\pi\left(\int_B n_0 dx\right)^2 t. \tag{146}$$

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It follows from (144) and (146) that for any $\varepsilon > 0$,

$$
\begin{aligned}
&\frac{1}{2}\int_B n^2 dx + \int_0^t \int_B |\nabla n|^2 \, dx d\tau \\
&\le \int_0^t \int_B -\frac{\chi}{2}\Delta v n^2 dx d\tau + \frac{1}{2}\int_B n_0^2 dx + 2\varepsilon^2 Q\pi \int_0^t \int_B |\nabla n|^2 \, dx d\tau + 2C_\varepsilon^2 Q\pi \left(\int_B n_0 dx\right)^2 t \\
&\le \frac{\chi}{2}\int_0^t \int_B \left(\frac{3}{2}n^3 + 3|\Delta v|^3\right) dx d\tau + \frac{1}{2}\int_B n_0^2 dx + 2\varepsilon^2 Q\pi \int_0^t \int_B |\nabla n|^2 \, dx d\tau + 2C_\varepsilon^2 Q\pi \left(\int_B n_0 dx\right)^2 t.
\end{aligned} \tag{147}
$$

Choosing ε such that $2\varepsilon^2 Q\pi = 1/2$, we get

$$
\begin{aligned}
&\frac{1}{2}\int_B n^2 dx + \frac{1}{2}\int_0^t \int_B |\nabla n|^2 \, dx d\tau \\
&\le \frac{3\chi}{4}\int_0^t \int_B n^3 dx d\tau + \frac{3\chi}{2}\int_0^t \int_B |\Delta v|^3 \, dx d\tau + C_3,
\end{aligned}
$$

where

$$
C_3 := \frac{1}{2}\int_B n_0^2 dx + 2C_\varepsilon^2 Q\pi \left(\int_B n_0 dx\right)^2 T.
$$

The theory of regularity for elliptic equations give us a constant C_4 satisfying

$$
\int_B |\Delta v|^3 \, dx \le C_4 \int_B \left| n - \frac{\theta}{\pi}\right|^3 dx \le C_5 \int_B \left(n^3 + 1\right) dx. \tag{148}
$$

It follows that

$$
\begin{aligned}
\frac{1}{2}\int_B n^2 dx + \frac{1}{2}\int_0^t \int_B |\nabla n|^2 \, dx d\tau &\le \frac{3\chi}{4}\int_0^t \int_B n^3 dx d\tau + \frac{3\chi}{2} C_5 \int_0^t \int_B \left(n^3 + 1\right) dx d\tau + C_3 \\
&\le \left(\frac{3\chi}{4} + \frac{3\chi}{2}C_5\right)\int_0^t \int_B n^3 dx d\tau + C_6.
\end{aligned}
$$

with positive constant $C_6 > 0$. Now we apply the embedding inequality (See [11]).

$$
\|n\|_p \le \overline{\varepsilon}\, \|\nabla n\|_2^{1-1/p}\, \|n \log |n|\|_1^{1/p} + k_1 \|n \log |n|\|_1 + k_2 \|n\|_1^{1/p}, \tag{149}
$$

for any $n \in H^1(B)$. Next, we apply the boundedness of the entropy (143) and the inequality (149) with $p = 3$ and $\overline{\varepsilon} > 0$ small enough to conclude that for some positive constant C_7

$$
\frac{1}{2}\int_B n^2 dx + \frac{1}{4}\int_0^t \int_B |\nabla n|^2 \, dx d\tau \le C_7. \tag{150}
$$

In order to obtain further regularity, we first use the next Gagliardo–Nirenberg inequality

$$
\|n\|_p \le C_8 \left(\|\nabla n\|_2^{1-2/p}\, \|n\|_2^{2/p} + \|n\|_2\right), \tag{151}
$$

with $p = 4$. Then using (150) and integrating (151) over $(0, t)$ gives

$$
\begin{aligned}
\int_0^t \|n\|_4^4 \, d\tau &\le C_9 \left(\int_0^t \left(\|\nabla n\|_2^2\, \|n\|_2^2 + \|n\|_2^4\right) d\tau\right) \\
&\le C_{10} \left(\int_0^t \left(\|\nabla n\|_2^2 + 1\right) d\tau\right) \le C_{11}.
\end{aligned} \tag{152}
$$

Next, we take $\phi = n^2$ in (10) and repeat the reasoning leading to (147) to deduce the control of the $L^3(B)$-norm through the estimate

$$
\frac{1}{3}\int_B n^3 dx \le C_{12}\left(\int_0^t \int_B n^4 + \int_0^t \int_B |\Delta v|^4 \, dx\right) + \frac{1}{3}\int_B n_0^3 dx \le C_{13}. \tag{153}
$$

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Hence,

$$\|v\|_{W^{2,3}(B)} \leq C_{13} \left\| n - \frac{\theta}{\pi} \right\|_{L^3(B)} \leq C_{15}.$$

We conclude from Morrey's inequality

$$\|v\|_{C^{1,1/3}(B)} \leq C_{15} \, \|v\|_{W^{2,3}(B)} \leq C_{16} C_{15} \; =: C_{17}.$$

In particular,

$$|\nabla v(\cdot, t)|_{L^\infty(B)} \leq C_{18}. \tag{154}$$

For $r > 2$, we take $\phi = n^r$ in (10) to obtain

$$\int_B n^{r+1} dx - \int_0^t \int_B n \, (n^r)_\tau \, dxd\tau + \int_0^t \int_B \left(\nabla n - \chi n \nabla v - n \frac{Qx}{|x|^2} \right) \cdot \nabla n^r dxd\tau = \int_B n^{r+1}(x, 0) dx.$$

implying that

$$\frac{1}{r+1} \int_B n^{r+1} dx + \int_0^t \int_B \left(\nabla n - \chi n \nabla v - n \frac{Qx}{|x|^2} \right) \cdot \nabla n^r dxd\tau = \frac{1}{r+1} \int_B n_0^{r+1} dx.$$

Subsequently, we utilise the identity

$$\int_B \nabla n \cdot \nabla n^r dx = r \int_0^t \int_B n^{r-1} \, |\nabla n|^2 \, dxd\tau = \frac{4r}{(r+1)^2} \int_B \left| \nabla n^{\frac{r+1}{2}} \right|^2 dx,$$

to derive

$$\begin{aligned} &\frac{1}{r+1} \int_B n^{r+1} dx \\ &= \frac{1}{r+1} \int_B n_0^{r+1} dx - \frac{4r}{(r+1)^2} \int_0^t \int_B \left| \nabla n^{\frac{r+1}{2}} \right|^2 dx + \chi \int_0^t \int_B n \nabla v \cdot \nabla n^r dxd\tau + \int_0^t \int_B n \nabla n^r \cdot \frac{Qx}{|x|^2} dxd\tau. \end{aligned} \tag{155}$$

In order to estimate the last integral, we write it in polar coordinates to obtain

$$\begin{aligned} \int_0^t \int_B n \nabla n^r \cdot \frac{Qx}{|x|^2} dxd\tau &= Q \int_0^t \int_0^{2\pi} \int_0^1 n \, (n^r)_\rho \, d\rho d\theta d\tau = Q \int_0^t \int_0^{2\pi} \int_0^1 n \left(r n^{r-1} n_\rho \right) d\rho d\theta d\tau \\ &= \frac{2rQ\pi}{r+1} \int_0^t (n^{r+1}(1, \tau) - n^{r+1}(0, \tau)) d\tau \leq \frac{2rQ\pi}{r+1} \int_0^t n^{r+1}(1, \tau) d\tau. \end{aligned} \tag{156}$$

Given any constant $\varepsilon_2 > 0$, Proposition 17 provides a constant C_{ε_2} such that

$$f^2(1, t) d\tau \leq 2\varepsilon_2^2 \int_B |\nabla f|^2 \, dxd\tau + 2C_{\varepsilon_2}^2 \left(\int_B f dx \right)^2,$$

for any $f \in W^{1,n}$. In particular for $f = n^{(r+1)/2}$

$$n^{r+1}(1, t) \leq 2\varepsilon_2^2 \int_B \left| \nabla n^{(r+1)/2} \right|^2 dxd\tau + 2C_{\varepsilon_2}^2 \left(\int_B n^{(r+1)/2} dx \right)^2. \tag{157}$$

From (156) and (157),

$$\int_0^t \int_B n \nabla n^r \cdot \frac{Qx}{|x|^2} dxd\tau \leq \frac{4rQ\pi \varepsilon_2^2}{r+1} \int_0^t \int_B \left| \nabla n^{(r+1)/2} \right|^2 dxd\tau + \frac{4rQ\pi \, C_{\varepsilon_2}^2}{r+1} \int_0^t \left(\int_B n^{(r+1)/2} dx \right)^2 d\tau.$$

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Combining this estimate with equation (155), we have

$$\frac{1}{r+1}\int_B n^{r+1}dx \le -\frac{4r}{(r+1)^2}\int_0^t\int_B \left|\nabla n^{\frac{r+1}{2}}\right|^2 dx + \chi\int_0^t\int_B n\nabla v\cdot\nabla n^r dxd\tau$$
$$+\frac{4rQ\pi\varepsilon_2^2}{r+1}\int_0^t\int_B \left|\nabla n^{(r+1)/2}\right|^2 dxd\tau$$
$$+\frac{4rQ\pi C_{\varepsilon_2}^2}{r+1}\int_0^t\left(\int_B n^{(r+1)/2}dx\right)^2 d\tau+\frac{1}{r+1}\int_B n_0^{r+1}dx. \tag{158}$$

Choosing ε_2 such that

$$\frac{4rQ\pi\varepsilon_2^2}{r+1}\le\frac{r}{(r+1)^2},$$

we get

$$\frac{1}{r+1}\int_B n^{r+1}dx \le -\frac{3r}{(r+1)^2}\int_0^t\int_B \left|\nabla n^{\frac{r+1}{2}}\right|^2 dx + \chi\int_0^t\int_B n\nabla v\cdot\nabla n^r dxd\tau$$
$$+\frac{4rQ\pi C_{\varepsilon_2}^2}{r+1}\int_0^t\left(\int_B n^{(r+1)/2}dx\right)^2 d\tau+\frac{1}{r+1}\int_B n_0^{r+1}dx. \tag{159}$$

Moreover, the boundedness of the gradient for the chemical concentration, as given by (154), implies

$$\chi\int_0^t\int_B n\nabla v\cdot\nabla n^r dxd\tau$$
$$=\chi r\int_0^t\int_B n^r\nabla v\cdot\nabla n dxd\tau\le\chi rC_{19}\int_0^t\int_B n^r\left|\nabla n\right| dxd\tau=\frac{2\chi rC_{19}}{r+1}\int_0^t\int_B n^{\frac{r+1}{2}}\left|\nabla n^{\frac{r+1}{2}}\right| dx$$
$$\le\frac{r}{(r+1)^2}\int_0^t\int_B\left|\nabla n^{\frac{r+1}{2}}\right|^2 dx+\chi^2 rC_{19}^2\int_0^t\int_B n^{r+1}dx. \tag{160}$$

From the estimates (159) and (160), we deduce

$$\frac{1}{r+1}\int_B n^{r+1}dx \le -\frac{2r}{(r+1)^2}\int_0^t\int_B \left|\nabla n^{\frac{r+1}{2}}\right|^2 dx + \chi^2 rC_{19}^2\int_0^t\int_B n^{r+1}dx$$
$$+\frac{4rQ\pi C_{\varepsilon_2}^2}{r+1}\int_0^t\left(\int_B n^{(r+1)/2}dx\right)^2 d\tau+\frac{1}{r+1}\int_B n_0^{r+1}dx.$$

Equivalently,

$$\frac{1}{2}\int_B n^{r+1}dx \le -\frac{r}{r+1}\int_0^t\int_B \left|\nabla n^{\frac{r+1}{2}}\right|^2 dx + \frac{\chi^2 r(r+1)C_{19}^2}{2}\int_0^t\int_B n^{r+1}dx$$
$$+2rQ\pi C_{\varepsilon_2}^2\int_0^t\left(\int_B n^{(r+1)/2}dx\right)^2 d\tau+\frac{1}{2}\int_B n_0^{r+1}dx.$$

Let $r=2^k-1$, $\bar{n}_1=n^{\frac{r+1}{2}}=n^{2^{k-1}}$. Thus, we establish

$$\frac{1}{2}\int_B \bar{n}^2dx\le-\frac{2^k-1}{2^k}\int_0^t\int_B\left|\nabla\bar{n}\right|^2 dxd\tau+\chi^2(2^k-1)2^{k-1}C_{19}^2\int_0^t\int_B\bar{n}^2dxd\tau$$
$$+4(2^k-1))Q\pi C_{\varepsilon_2}^2\int_0^t\left(\int_B\bar{n}_1dx\right)^2 d\tau+\frac{1}{2}\int_B\bar{n}_0^2dx. \tag{161}$$

We will leverage the inequality (161) as a foundation for applying the Moser–Alikakos technique (2) to establish the sought-after L^∞ bound of n. We demonstrate how to use the estimate (161) to derive an upper bound of the integral $\int_B u_1^{2^k}dx$ in terms of $\int_B u_1^{2^{k-1}}dx$. This step sets the stage for a recursive process. Through a recursive application of the derived estimate, we progressively obtain bounds depending solely on the bounded integral $\int_B u_1 dx = \theta_1$. Careful control of the constants involved in this iterative process allows us to gracefully transition to the limit, ultimately securing the desired L^∞ bound.

Defining $v_k = \frac{2^k-1}{2^k}$, $a_k = \chi^2(2^k-1)2^{k-1}C_{19}^2$ and $b_k = 4(2^k-1))Q\pi C_{\varepsilon_2}^2$, we derive

$$\frac{1}{2}\int_B \overline{n}^2 dx \leq -v_k \int_0^t \int_B |\nabla \overline{n}|^2\, dx d\tau + a_k \int_0^t \int_B \overline{n}^2 dx d\tau \\ + b_k \int_0^t \left(\int_B \overline{n}_1 dx\right)^2 d\tau + \frac{1}{2}\int_B \overline{n}_0^2 dx. \tag{162}$$

Recall the Nirenberg–Gagliardo interpolation inequality:

For each $1 \leq q \leq p < \infty$ and for any $f \in H^1$, there exist a constant $C > 0$ such that

$$\|f\|_{L^p} \leq C\, \|f\|_{H^1}^a\, \|f\|_{L^q}^{1-a}$$

where $a = 1 - \frac{q}{p}$.

Applying this inequality with $p = 2$ and $q = 1$, we obtain

$$\|f\|_{L^2} \leq C\, \|f\|_{H^1}^{1/2}\, \|f\|_{L^1}^{1/2}.$$

By utilising Young's inequality, the equality $\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2$ and choosing $0 < \varepsilon < \frac{1}{2}$ it follows that

$$\frac{1}{2}\,\|f\|_{L^2}^2 < (1-\varepsilon)\,\|f\|_{L^2}^2 \leq \varepsilon\,\|\nabla f\|_{L^2}^2 + \frac{C^2}{4\varepsilon}\,\|f\|_{L^1}^2$$

Substituting $f = \overline{n}$ and $\varepsilon = \varepsilon_k$, we derive

$$-\int_B \overline{n}^2 dx + \frac{C^2}{2\varepsilon_k}\left(\int_B \overline{n} dx\right)^2 \geq -2\varepsilon_k \int_B |\nabla \overline{n}|^2\, dx.$$

Multiplying both sides by $a_k + \varepsilon_k$, we get

$$-(a_k+\varepsilon_k)\int_B \overline{n}^2 dx + \frac{C^2(a_k+\varepsilon_k)}{2\varepsilon_k}\left(\int_B \overline{n} dx\right)^2 \geq -2\varepsilon_k(a_k+\varepsilon_k)\int_B |\nabla \overline{n}|^2\, dx. \tag{163}$$

Choosing ε_k such that $2\varepsilon_k(a_k+\varepsilon_k) \leq v_k$, we obtain from (162) via (163),

$$\begin{aligned}\frac{1}{2}\int_B \overline{n}^2 dx &\leq -v_k \int_0^t \int_B |\nabla \overline{n}|^2\, dx d\tau + a_k \int_0^t \int_B \overline{n}^2 dx d\tau \\ &+ b_k \int_0^t \left(\int_B \overline{n} dx\right)^2 d\tau + \frac{1}{2}\int_B \overline{n}_0^2 dx \\ &\leq -2\varepsilon_k(a_k+\varepsilon_k)\int_0^t \int_B |\nabla \overline{n}|^2\, dx d\tau + a_k \int_0^t \int_B \overline{n}^2 dx d\tau \\ &+ b_k \int_0^t \left(\int_B \overline{n} dx\right)^2 d\tau + \frac{1}{2}\int_B \overline{n}_0^2 dx \\ &\leq -(a_k+\varepsilon_k)\int_0^t \int_B \overline{n}^2 dx d\tau + \frac{C^2(a_k+\varepsilon_k)}{2\varepsilon_k}\int_0^t \left(\int_B \overline{n} dx\right)^2 d\tau + a_k \int_0^t \int_B \overline{n}^2 dx d\tau \\ &+ b_k \int_0^t \left(\int_B \overline{n} dx\right)^2 d\tau + \frac{1}{2}\int_B \overline{n}_0^2 dx \\ &\leq -\varepsilon_k \int_0^t \int_B \overline{n}^2 dx d\tau + \left(\frac{C^2(a_k+\varepsilon_k)}{2\varepsilon_k} + b_k\right)\int_0^t \left(\int_B \overline{n} dx\right)^2 d\tau + \frac{1}{2}\int_B \overline{n}_0^2 dx \\ &\leq \left(\frac{C^2(a_k+\varepsilon_k)}{2\varepsilon_k} + b_k\right)\left(\sup_{t\geq 0}\int_B \overline{n} dx\right)^2 t + \frac{1}{2}\int_B \overline{n}_0^2 dx.\end{aligned}$$

Therefore,

$$\int_B \overline{n}^2 dx \le 2\max\left\{\left(\frac{C^2(a_k+\varepsilon_k)}{\varepsilon_k}+2b_k\right)\left(\sup_{t\ge 0}\int_B \overline{n}dx\right)^2 T,\int_B \overline{n}_0^2 dx\right\}$$
$$\le 2\max\left\{\left(\frac{C^2(a_k+\varepsilon_k)}{\varepsilon_k}+2b_k\right)\left(\sup_{t\ge 0}\int_B \overline{n}dx\right)^2 T,\pi\,\|n_0\|_{L^\infty}^{2^k}\right\}.$$

Thus, we obtain the recursive inequality

$$\int_B n^{2^k} dx \le \max\left\{\left(\frac{2C^2(a_k+\varepsilon_k)}{\varepsilon_k}+4b_k\right)\left(\sup_{t\ge 0}\int_B n^{2^{k-1}}dx\right)^2 T, 2\pi\,\|n_0\|_{L^\infty}^{2^k}\right\}. \tag{164}$$

Similarly

$$\int_B n^{2^{k-1}} dx \le \max\left\{\left(\frac{2C^2(a_{k-1}+\varepsilon_{k-1})}{\varepsilon_{k-1}}+4b_{k-1}\right)\left(\sup_{t\ge 0}\int_B n^{2^{k-2}}dx\right)^2 T, 2\pi\,\|n_0\|_{L^\infty}^{2^{k-1}}\right\}.$$

Then

$$\begin{aligned}&\left(\tfrac{2C^2(a_k+\varepsilon_k)}{\varepsilon_k}+4b_k\right)\left(\sup_{t\ge 0}\textstyle\int_B n^{2^{k-1}}dx\right)^2\\ &\le \max\left\{\left(\tfrac{2C^2(a_k+\varepsilon_k)}{\varepsilon_k}+4b_k\right)\left(\tfrac{2C^2(a_{k-1}+\varepsilon_{k-1})}{\varepsilon_{k-1}}+4b_{k-1}\right)^2\left(\sup_{t\ge 0}\textstyle\int_B n^{2^{k-2}}dx\right)^{2^2} T^2,\right.\\ &\left.\left(\tfrac{2C^2(a_k+\varepsilon_k)}{\varepsilon_k}+4b_k\right)2^2\pi^2\,\|n_0\|_{L^\infty}^{2^k}\right\}.\end{aligned} \tag{165}$$

We choose ε_k such that $\frac{2C^2(a_k+\varepsilon_k)}{\varepsilon_k^2}+4b_k\ge 1$, $k=1,2,\ldots$. From (164) and (165), we conclude that,

$$\begin{aligned}&\int_B n^{2^k}dx\\ &\le \max\left\{\left(\frac{2C^2(a_k+\varepsilon_k)}{\varepsilon_k}+4b_k\right)\left(\frac{2C^2(a_{k-1}+\varepsilon_{k-1})}{\varepsilon_{k-1}}+4b_{k-1}\right)^2\left(\sup_{t\ge 0}\int_B n^{2^{k-2}}dx\right)^{2^2} T^3,\right.\\ &\left.\left(\frac{2C^2(a_k+\varepsilon_k)}{\varepsilon_k}+4b_k\right)2^2\pi^2\,\|n_0\|_{L^\infty}^{2^k}\right\}.\end{aligned} \tag{166}$$

To simplify further, we impose the additional condition on ε_k,

$$4b_k \le \frac{2C^2(a_k+\varepsilon_k)}{\varepsilon_k}\ \text{for } k=1,2,\ldots$$

Therefore,

$$\begin{aligned}&\int_B n^{2^k}dx\\ &\le \max\left\{\left(\frac{4C^2(a_k+\varepsilon_k)}{\varepsilon_k}\right)\left(\frac{4C^2(a_{k-1}+\varepsilon_{k-1})}{\varepsilon_{k-1}}\right)^2\left(\sup_{t\ge 0}\int_B n^{2^{k-2}}dx\right)^{2^2} T^3, \left(\frac{4C^2(a_k+\varepsilon_k)}{\varepsilon_k}\right)2^2\pi^2\,\|n_0\|_{L^\infty}^{2^k}\right\}.\end{aligned}$$

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Continuing this process

$$\int_B n^{2^k} dx$$
$$\leq \max\left\{ \frac{4C^2(a_k+\varepsilon_k)}{\varepsilon_k}\left(\frac{4C^2(a_{k-1}+\varepsilon_{k-1})}{\varepsilon_{k-1}}\right)^2 \cdots \left(\frac{4C^2(a_1+\varepsilon_1)}{\varepsilon_1}\right)^{2^{k-1}} \left(\sup_{t\geq 0}\int_B n dx\right)^{2^k} T^{2^k-1}, \right.$$
$$\left. \left(\frac{4C^2(a_k+\varepsilon_k)}{\varepsilon_k}\right)\left(\frac{4C^2(a_{k-1}+\varepsilon_{k-1})}{\varepsilon_{k-1}}\right) \cdots \left(\frac{4C^2(a_1+\varepsilon_1)}{\varepsilon_1}\right) 2^{2^k}\pi^{2^k} \|n_0\|_{L^\infty}^{2^k} \right\}.$$

With $K = \max\{1, \|n_0\|_{L^\infty}, \|n_0\|_{L^1}, T\}$, this last inequality implies

$$\int_B n^{2^k} dx \leq \frac{4C^2(a_k+\varepsilon_k)}{\varepsilon_k}\left(\frac{4C^2(a_{k-1}+\varepsilon_{k-1})}{\varepsilon_{k-1}}\right)^2 \cdots \left(\frac{4C^2(a_1+\varepsilon_1)}{\varepsilon_1}\right)^{2^{k-1}} 2^{2^k}\pi^{2^k} K^{2^k}. \tag{167}$$

Now, let's demonstrate that the right-hand side of the last inequality behaves like a constant to the power of 2^k. By taking the $1/2^k$ power of both sides, we can transition to the limit and derive the L^∞ estimate.

First, let us estimate ε_k. With $2\varepsilon_k(a_k+\varepsilon_k) \leq v_k$, we find that

$$\chi^2(2^k-1)2^{k-1}C_{19}^2\varepsilon_k + \varepsilon_k^2 \leq \frac{1}{2}\left(1-\frac{1}{2^k}\right).$$

So, it is enough to find ε_k such that

$$\chi^2(2^k-1)2^{k-1}C_{19}^2\varepsilon_k + \varepsilon_k \leq \frac{1}{2}\left(1-\frac{1}{2^k}\right),$$

or

$$\left((2^k-1)2^{k-1}C_{19}^2\chi^2+1\right)\varepsilon_k \leq \frac{1}{2}\left(1-\frac{1}{2^k}\right).$$

This implies $\varepsilon_k \leq \frac{1}{2\left((2^k-1)2^{k-1}C_{19}^2\chi^2+1\right)}\left(1-\frac{1}{2^k}\right)$. Now,

$$\frac{1}{2\left((2^k-1)2^{k-1}C_{19}^2\chi^2+1\right)}\left(1-\frac{1}{2^k}\right) \geq \frac{1}{2\left((2^k)2^{k-1}C_{19}^2\chi^2+1\right)}\left(1-\frac{1}{2}\right)$$
$$\geq \frac{1}{4\left(2^{2k-1}C_{19}^2\chi^2+1\right)}.$$

By setting $\varepsilon_k = \frac{1}{4\left(2^{2k-1}C_{19}^2\chi^2+1\right)}$, we find that

$$\frac{4C^2(a_k+\varepsilon_k)}{\varepsilon_k} = \frac{4C^2\varepsilon_k(a_k+\varepsilon_k)}{\varepsilon_k^2} \leq 2C^2\frac{v_k}{\varepsilon_k^2}$$
$$= 2C^2\frac{1-\frac{1}{2^k}}{\left(\frac{1}{4\left(2^{2k-1}C_{19}^2\chi^2+1\right)}\right)^2} \leq 2C^2(2^{2k+1}C_{19}^2\chi^2+4)^2.$$

Thus, for every k up to finite number, we conclude

$$2^{2k+1}C_{19}^2\chi^2 \geq 4.$$

Consequently,

$$\frac{4C^2(a_k+\varepsilon_k)}{\varepsilon_k} \leq 2^{4k}a,$$

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for some constant value a. Thus, we get from (167) that

$$\int_B n^{2^k}\,dx \le 2^{4k}a\left(2^{4(k-1)}a\right)^2\left(2^{4(k-2)}a\right)^{2^2}\cdots\left(2^{4(k-(k-1))}a\right)^{2^{k-1}}2^{2^k}\pi^{2^k}K^{2^k}$$
$$= a^{2^k-1}2^{4(2^{k+1}-k-2)}2^{2^k}\pi^{2^k}K^{2^k}.$$

Taking the limit $k\to\infty$ for the $1/2^k$-th power of both sides, we obtain

$$\|n\|_{L^\infty} \le \lim_{k\to\infty} a^{(2^k-1)/2^k}2^{4(2^{k+1}-k-2)/2^k}2\pi K = 2^9 a\pi K.$$

6. Blow-up

The following result is an adaptation of the classical moments technique for the Keller–Segel system (cf. [26]).

Proof of the result of blow-up (Theorem 2). We formally multiply the equation for n in (4) by $|x|^2$ and integrate to obtain

$$\frac{d}{dt}\int_B n\,|x|^2\,dx = -\int_{\partial B}|x|^2\,n(\mathbf{u}\cdot\eta)d\sigma + 2\int_B(x\cdot\mathbf{u})\,ndx + 4\int_B ndx + \int_{\partial B}|x|^2\,\frac{\partial n}{\partial\eta}d\sigma - \int_{\partial B} n\frac{\partial\,|x|^2}{\partial\eta}d\sigma$$
$$-\chi\int_{\partial B}|x|^2\,\frac{\partial v}{\partial\eta}d\sigma + 2\chi\int_B n\,(x\cdot\nabla v)\,dx.$$

Using that $|x|=1$ on ∂B, applying the zero-flux boundary condition (6), and noting $\int_{\partial B} n\frac{\partial|x|^2}{\partial\eta}d\sigma = 2\int_{\partial B} nd\sigma \ge 0$, we get

$$\frac{d}{dt}\int_B n\,|x|^2\,dx \le 2\int_B(x\cdot\mathbf{u})\,ndx + 4\theta + 2\chi\int_B n\,(x\cdot\nabla v)\,dx. \tag{168}$$

On the other hand, multiplying the equation of the chemical concentration

$$0 = \frac{1}{r}\partial_r(rnv_r) - \frac{\theta}{\pi} + n$$

by r and integrating over the interval $(0,r)$ yields

$$0 = r\frac{\partial v}{\partial r} + \int_0^r \rho n d\rho - \frac{\theta}{\pi}\int_0^r \rho d\rho. \tag{169}$$

Denoting the cumulative mass by $M(r,t) := \int_{B(0,r)} ndx = 2\pi\int_0^r n\rho d\rho$, we obtain

$$\frac{\partial v}{\partial r} = -\frac{M}{2\pi r} + \frac{\theta}{\pi r}\int_0^r \rho d\rho \le -\frac{M}{2\pi r} + \frac{\theta r}{2\pi}. \tag{170}$$

Hence, applying (170) and using the identity $x\cdot\nabla v = r\frac{\partial v}{\partial r}$, we obtain

$$\int_B n(x\cdot\nabla v)dx = 2\pi\int_0^1 n\rho\frac{\partial v}{\partial\rho}\rho d\rho \le 2\pi\int_0^1 n\rho\left(-\frac{M}{2\pi\rho} + \frac{\theta\rho}{2\pi}\right)\rho d\rho$$
$$= -\int_0^1 Mn\rho d\rho + \theta\int_0^1 n\rho^3 d\rho$$
$$= -\frac{1}{2\pi}\int_0^1 M\frac{\partial M}{\partial\rho}d\rho + \frac{\theta}{2\pi}\int_B n\,|x|^2\,dx. \tag{171}$$

Let $m(t) := \int_B n\,|x|^2\,dx$. From (171), we have

$$\int_B n(x\cdot\nabla v)dx \le -\frac{\theta^2}{4\pi} + \frac{\theta}{2\pi}m(t). \tag{172}$$

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Returning to (168) and using $x \cdot \mathbf{u} = Q$ together with (172) and the estimate (172), we obtain

$$\frac{dm(t)}{dt} \leq 2Q\theta + 4\theta + 2\chi \int_B n\,(x \cdot \nabla v)\, dx$$
$$\leq 2\,(2+Q)\,\theta - \frac{\chi\theta^2}{2\pi} + \frac{\chi\theta}{\pi} m(t). \tag{173}$$

Solving the last differential inequality, we get

$$m(t) \leq \mathrm{e}^{\frac{\chi\theta}{\pi}t} \left\{ m(0) + \frac{1}{2}\left(\frac{4\pi\,(2+Q)}{\chi} - \theta\right)\left(1 - \mathrm{e}^{-\frac{\chi\theta}{\pi}t}\right) \right\}. \tag{174}$$

We notice that the right-hand side of inequality (174) vanishes at the time t^* defined by:

$$t^* := -\frac{\pi}{\chi\theta} \log\left(1 - \frac{m(0)}{\frac{1}{2}\left(\theta - \frac{4\pi(2+Q)}{\chi}\right)}\right),$$

and therefore

$$m(t) < 0 \text{ for all } t > t^*.$$

This last inequality contradicts the positivity of m. We conclude that that $T_{\max} \leq t^*$. The result (9) follows from the extensibility criterion in Theorem 15.

Competing interest. The author declare no competing interests.

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