



RESEARCH ARTICLE

# Every complex Hénon map is exponentially mixing of all orders and satisfies the CLT

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Received: 5 June 2023; Revised: 18 September 2023; Accepted: 8 October 2023

2020 Mathematics Subject Classification: Primary – 37F80; Secondary – 32U05, 32H50, 37A25, 60F05

*Dedicated to Professor Jun-Muk Hwang in occasion of his 60th birthday*

## Abstract

We show that the measure of maximal entropy of every complex Hénon map is exponentially mixing of all orders for Hölder observables. As a consequence, the Central Limit Theorem holds for all Hölder observables.

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**Notation.** The pairing  $\langle \cdot, \cdot \rangle$  is used for the integral of a function with respect to a measure or, more generally, the value of a current at a test form. By  $(p, p)$ -currents we mean currents of bi-degree  $(p, p)$ . Given  $k \geq 1$ , we denote by  $\omega_{\text{FS}}$  the Fubini-Study form on  $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$ . The mass of a positive closed  $(p, p)$ -current  $R$  on  $\mathbb{P}^k$  is equal to  $\langle R, \omega_{\text{FS}}^{k-p} \rangle$  and is denoted by  $\|R\|$ . The notations  $\lesssim$  and  $\gtrsim$  stand for inequalities up to a multiplicative constant. If  $R$  and  $S$  are two real currents of the same bi-degree, we write  $|R| \leq S$  when  $S \pm R \geq 0$ . Observe that this forces  $S$  to be positive.

## 1. Introduction

Hénon maps are among the most studied dynamical systems that exhibit interesting chaotic behaviour. The main goal of this work is to prove that the measure of maximal entropy of any complex Hénon map is *exponentially mixing of all orders* with respect to Hölder observables. As a consequence, we also solve a long-standing question proving the Central Limit Theorem for all Hölder observables with respect to the maximal entropy measures of complex Hénon maps.

The reader can find in the work of Bedford, Fornæss, Lyubich, Sibony, Smillie and the second author fundamental dynamical properties of these systems; see [1, 3, 4, 18, 27, 28, 31] and the references therein. It is shown in [1] that the measure of maximal entropy  $\mu$  is Bernoulli. In particular, it is

mixing of all orders [9]. Namely, for every  $\kappa \in \mathbb{N}$ , observables  $g_0, \dots, g_\kappa \in L^{\kappa+1}(\mu)$  and integers  $0 =: n_0 \leq n_1 \leq \dots \leq n_\kappa$ , we have

$$\langle \mu, g_0(g_1 \circ f^{n_1}) \dots (g_\kappa \circ f^{n_\kappa}) \rangle - \prod_{j=0}^\kappa \langle \mu, g_j \rangle \rightarrow 0.$$

However, the control of the speed of mixing (i.e., the rate of the above convergence) for general dynamical systems and for regular enough observables is a challenging problem, and usually one can obtain it only under strong hyperbolicity assumptions on the system.

Let us recall the following general definition.

**Definition 1.1.** Let  $(X, f)$  be a dynamical system and  $\nu$  an  $f$ -invariant measure. Let  $(E, \|\cdot\|_E)$  be a normed space of real functions on  $X$  with  $\|\cdot\|_{L^p(\nu)} \lesssim \|\cdot\|_E$  for all  $1 \leq p < \infty$ . We say that  $\nu$  is *exponentially mixing of order  $\kappa \in \mathbb{N}^*$*  for observables in  $E$  if there exist constants  $C_\kappa > 0$  and  $0 < \theta_\kappa < 1$  such that, for all  $g_0, \dots, g_\kappa$  in  $E$  and integers  $0 =: n_0 \leq n_1 \leq \dots \leq n_\kappa$ , we have

$$\left| \langle \nu, g_0(g_1 \circ f^{n_1}) \dots (g_\kappa \circ f^{n_\kappa}) \rangle - \prod_{j=0}^\kappa \langle \nu, g_j \rangle \right| \leq C_\kappa \cdot \left( \prod_{j=0}^\kappa \|g_j\|_E \right) \cdot \theta_\kappa^{\min_{0 \leq j \leq \kappa-1} (n_{j+1} - n_j)}.$$

We say that  $\nu$  is *exponentially mixing of all orders* for observables in  $E$  if it is exponentially mixing of order  $\kappa$  for every  $\kappa \in \mathbb{N}$ .

A recent major result by Dolgopyat, Kanigowski and Rodriguez-Hertz [22] ensures that, under suitable assumptions on the system, the exponential mixing of order 1 implies that the system is Bernoulli. In particular, it implies the mixing of all orders (with no control on the rate of decay of correlation). It is a main open question whether the exponential mixing of order 1 implies the exponential mixing of all orders; see, for instance, [22, Question 1.5].

Let now  $f$  be a complex Hénon map on  $\mathbb{C}^2$ . It is a polynomial diffeomorphism of  $\mathbb{C}^2$ . We can associate to  $f$  its unique measure of maximal entropy  $\mu$  [1, 3, 4, 31]; see Section 2 for details. It was established by the second author in [13] that such measure if exponential mixing of order 1 for Hölder observables; see also Vigny [35] and Wu [36]. Similar results were obtained by Liverani [29] in the case of uniformly hyperbolic diffeomorphisms and Dolgopyat [20] for Anosov flows.

**Theorem 1.2.** *Let  $f$  be a complex Hénon map and  $\mu$  its measure of maximal entropy. Then, for every  $\kappa \in \mathbb{N}^*$ ,  $\mu$  is exponential mixing of order  $\kappa$  as in Definition 1.1 for  $\mathcal{C}^\gamma$  observables ( $0 < \gamma \leq 2$ ), with  $\theta_\kappa = d^{-(\gamma/2)\kappa+1/2}$ .*

For endomorphisms of  $\mathbb{P}^k(\mathbb{C})$ , the exponential mixing for all orders for the measure of maximal entropy and Hölder observables was established in [15]. We recently proved such property for a large class of invariant measures with strictly positive Lyapunov exponents [6]. This was done by constructing a suitable (semi-)norm on functions that turns the so-called Ruelle-Perron-Frobenius operator (suitably normalized) into a contraction. As far as we know, the present paper gives the first instance where the exponential mixing of all orders is established for holomorphic dynamical systems with both positive and negative Lyapunov exponents.

The exponential mixing of all orders is one of the strongest properties in dynamics. It was recently shown to imply a number of statistical properties; see, for instance, [8, 21]. As an example, a consequence of Theorem 1.2 is the following result. Take  $u \in L^1(\mu)$ . As  $\mu$  is ergodic, Birkhoff's ergodic theorem states that

$$n^{-1}S_n(u) := n^{-1}(u(x) + u \circ f(x) + \dots + u \circ f^{n-1}(x)) \rightarrow \langle \mu, u \rangle \quad \text{for } \mu - \text{a.e. } x \in X.$$

We say that  $u$  satisfies the *Central Limit Theorem (CLT)* with variance  $\sigma^2 \geq 0$  with respect to  $\mu$  if  $n^{-1/2}(S_n(u) - n\langle \mu, u \rangle) \rightarrow \mathcal{N}(0, \sigma^2)$  in law, where  $\mathcal{N}(0, \sigma^2)$  denotes the (possibly degenerate, for  $\sigma = 0$ ) Gaussian distribution with mean 0 and variance  $\sigma^2$ ; that is, for any interval  $I \subset \mathbb{R}$ , we have

$$\lim_{n \rightarrow \infty} \mu \left\{ \frac{S_n(u) - n\langle \mu, u \rangle}{\sqrt{n}} \in I \right\} = \begin{cases} 1 & \text{when } I \text{ is of the form } I = (-\delta, \delta) \quad \text{if } \sigma^2 = 0, \\ \frac{1}{\sqrt{2\pi\sigma^2}} \int_I e^{-t^2/(2\sigma^2)} dt & \text{if } \sigma^2 > 0. \end{cases}$$

By a result of Björklund and Gorodnik [8], the following is then a consequence of Theorem 1.2. We refer to [6, 12, 16, 24, 30, 32, 33] for other cases where the CLT for Hölder observables was established in holomorphic dynamics. As is the case for Theorem 1.2, this is the first time that this is done for systems with both positive and negative Lyapunov exponents.

**Corollary 1.3.** *Let  $f$  be a complex Hénon map and  $\mu$  its measure of maximal entropy. Then all Hölder observables  $u$  satisfy the Central Limit Theorem with respect to  $\mu$  with*

$$\sigma^2 = \sum_{n \in \mathbb{Z}} \langle \mu, \tilde{u}(\tilde{u} \circ f^n) \rangle = \lim_{n \rightarrow \infty} \frac{1}{n} \int_X (\tilde{u} + \tilde{u} \circ f + \dots + \tilde{u} \circ f^{n-1})^2 d\mu,$$

where  $\tilde{u} := u - \langle \mu, u \rangle$ .

Theorem 1.2 and Corollary 1.3, in particular, apply to any real Hénon map of maximal entropy [5] (i.e., complex Hénon maps with real coefficients and whose measure of maximal entropy is supported in  $\mathbb{R}^2$ ). By Friedland-Milnor [26], they apply to all automorphisms of  $\mathbb{C}^2$  which are not conjugated to a map preserving a fibration. They hold also in the larger settings of Hénon-Sibony automorphisms (sometimes called regular, or regular in the sense of Sibony) of  $\mathbb{C}^k$  in any dimension [31] (see Definition 2.1 and Remark 3.2) and invertible horizontal-like maps in any dimension [14, 17]; see Remark 3.3. We postpone the case of automorphisms of compact Kähler manifolds to the forthcoming paper [7]; see Remark 3.4.

Our method to prove Theorem 1.2 relies on pluripotential theory and on the theory of positive closed currents. The idea is as follows. Using the classical theory of interpolation [34], we can reduce the problem to the case  $\gamma = 2$ . For simplicity, assume that  $\|g_j\|_{C^2} \leq 1$  for all  $j$ . The measure of maximal entropy  $\mu$  of a Hénon map  $f$  of  $\mathbb{C}^2$  of algebraic degree  $d \geq 2$  is the intersection  $\mu = T_+ \wedge T_-$  of the two Green currents  $T_+$  and  $T_-$  of  $f$  [3, 31]. If we identify  $\mathbb{C}^2$  to an affine chart of  $\mathbb{P}^2$  in the standard way, these currents are the unique positive closed  $(1, 1)$ -currents of mass 1 on  $\mathbb{P}^2$ , without mass at infinity, satisfying  $f^*T_+ = dT_+$  and  $f_*T_- = dT_-$ .

Consider the automorphism  $F$  of  $\mathbb{C}^4$  given by  $F := (f, f^{-1})$ . Such an automorphism also admits Green currents  $\mathbb{T}_+ = T_+ \otimes T_-$  and  $\mathbb{T}_- = T_- \otimes T_+$ . These currents satisfy  $(F^n)^*\mathbb{T}_+ = d^{2n}\mathbb{T}_+$  and  $(F^n)_*\mathbb{T}_- = d^{2n}\mathbb{T}_-$ . Under mild assumptions on their support, other positive closed  $(2, 2)$ -currents  $S$  of mass 1 of  $\mathbb{P}^4$  satisfy the estimate

$$|\langle d^{-2n}(F^n)_*(S) - \mathbb{T}_-, \Phi \rangle| \leq c_{S,\Phi} d^{-n} \tag{1.1}$$

when  $\Phi$  is a sufficiently smooth test form. Here,  $c_{S,\Phi}$  is a constant depending on  $S$  and  $\Phi$ .

We show that proving the exponential mixing for  $\kappa + 1$  observables  $g_0, \dots, g_\kappa$  with  $\|g_j\|_{C^2} \leq 1$  can be reduced to proving the convergence (we assume that  $n_1$  is even for simplicity)

$$|\langle d^{-n_1}(F^{n_1/2})_*[\Delta] - \mathbb{T}_-, \Theta_{\{g_j\}, \{n_j\}} \rangle| \lesssim d^{-\min_{0 \leq j \leq \kappa-1} (n_{j+1} - n_j)/2}, \tag{1.2}$$

where

$$\Theta_{\{g_j\}, \{n_j\}} := g_0(w)g_1(z)(g_2 \circ f^{n_2 - n_1}(z)) \dots (g_\kappa \circ f^{n_\kappa - n_1}(z))\mathbb{T}_+,$$

$[\Delta]$  denotes the current of integration on the diagonal  $\Delta$  of  $\mathbb{C}^2 \times \mathbb{C}^2$ , and  $(z, w)$  denote the coordinates on  $\mathbb{C}^2 \times \mathbb{C}^2$ . A crucial point here is that the estimate should not only be uniform in the  $g_j$ 's but also in the  $n_j$ 's. Note also that the current  $[\Delta]$  is singular and the dependence of the constant  $c_{S,\Phi}$  in (1.1) from  $S$  makes it difficult to employ regularization techniques to deduce the convergence (1.2) from (1.1).

The key point here is to notice that, when  $i\partial\bar{\partial}\Phi \geq 0$  (on a suitable open set), one can also get the following variation of (1.1):

$$\langle d^{-2n}(F^n)_*(S) - \mathbb{T}_-, \Phi \rangle \leq c_\Phi d^{-n}. \tag{1.3}$$

With respect to (1.1), only the bound from above is present, but the constant  $c_\Phi$  is now independent of  $S$ . This permits to regularize  $\Delta$  and work as if this current were smooth. Note also that, although  $\Theta_{\{g_j\}, \{n_j\}}$  is not smooth, we can handle it using a similar regularization.

Working by induction, we show that it is possible to replace both  $\Theta_{\{g_j\}, \{n_j\}}$  and  $-\Theta_{\{g_j\}, \{n_j\}}$  in (1.2) with currents  $\Theta^\pm$  satisfying  $i\partial\bar{\partial}\Theta^\pm \geq 0$ . This permits to deduce the estimate (1.2) from two upper bounds given by (1.3) for  $\Theta^\pm$ , completing the proof.

## 2. Hénon-Sibony automorphisms of $\mathbb{C}^k$ and convergence towards Green currents

Let  $F$  be a polynomial automorphism of  $\mathbb{C}^k$ . We still denote by  $F$  its extension as a birational map of  $\mathbb{P}^k$ . Denote by  $\mathbb{H}_\infty := \mathbb{P}^k \setminus \mathbb{C}^k$  the hyperplane at infinity and by  $\mathbb{I}^+, \mathbb{I}^-$  the indeterminacy sets of  $F$  and  $F^{-1}$ , respectively. They are analytic sets strictly contained in  $\mathbb{H}_\infty$ . If  $\mathbb{I}^+ = \emptyset$  or  $\mathbb{I}^- = \emptyset$ , then both of them are empty and  $F$  is given by a linear map, and its dynamics are easy to describe. Hence, we assume that  $\mathbb{I}^\pm \neq \emptyset$ . The following definition, under the name of *regular automorphism*, was introduced by Sibony [31].

**Definition 2.1.** We say that  $F$  is a *Hénon-Sibony automorphism of  $\mathbb{C}^k$*  if  $\mathbb{I}^\pm \neq \emptyset$  and  $\mathbb{I}^+ \cap \mathbb{I}^- = \emptyset$ .

Given  $F$  a Hénon-Sibony automorphism of  $\mathbb{C}^k$ , it is clear that  $F^{-1}$  is also Hénon-Sibony. We denote by  $d_+(F)$  and  $d_-(F)$  the algebraic degrees of  $F$  and  $F^{-1}$ , respectively. Observe that  $d_\pm(F) \geq 2$ ,  $d_+(F) = d_-(F^{-1})$  and  $d_-(F) = d_+(F^{-1})$ . Later, we will drop the letter  $F$  and just write  $d_\pm$  instead of  $d_\pm(F)$  for simplicity. We will recall here some basic properties of  $F$  and refer the reader to [3, 18, 27, 28, 31] for details.

**Proposition 2.2.** *Let  $F$  be a Hénon-Sibony automorphism of  $\mathbb{C}^k$  as above.*

- (i) *There exists an integer  $1 \leq p \leq k - 1$  such that  $\dim \mathbb{I}^+ = k - p - 1$ ,  $\dim \mathbb{I}^- = p - 1$  and  $d_+(F)^p = d_-(F)^{k-p}$ .*
- (ii) *The analytic sets  $\mathbb{I}^\pm$  are irreducible, and we have*

$$F(\mathbb{H}_\infty \setminus \mathbb{I}^+) = F(\mathbb{I}^-) = \mathbb{I}^- \quad \text{and} \quad F^{-1}(\mathbb{H}_\infty \setminus \mathbb{I}^-) = F^{-1}(\mathbb{I}^+) = \mathbb{I}^+.$$

- (iii) *For every  $n \geq 1$ , both  $F^n$  and  $F^{-n}$  are Hénon-Sibony automorphisms of  $\mathbb{C}^k$ , of algebraic degrees  $d_+(F)^n$  and  $d_-(F)^n$ , and indeterminacy sets  $\mathbb{I}^+$  and  $\mathbb{I}^-$ , respectively.*

**Example 2.3** (Generalized). Hénon maps on  $\mathbb{C}^2$  correspond to the case  $k = 2$  in Definition 2.1. In this case, we have  $p = k - p = 1$  and  $d_+ = d_- = d$ , the algebraic degree of the map; see [3, 26, 31].

The set  $\mathbb{I}^+$  (resp.  $\mathbb{I}^-$ ) is attracting for  $F^{-1}$  (resp.  $F$ ). Let  $\widetilde{W}^\pm$  be the basin of attraction of  $\mathbb{I}^\pm$ . Set  $W^\pm := \widetilde{W}^\pm \cap \mathbb{C}^k$ . Then the sets  $\mathbb{K}^+ := \mathbb{C}^k \setminus W^-$  and  $\mathbb{K}^- := \mathbb{C}^k \setminus W^+$  are the sets of points (in  $\mathbb{C}^k$ ) with bounded orbit for  $F$  and  $F^{-1}$ , respectively. We have  $\overline{\mathbb{K}^+} = \mathbb{K}^+ \cup \mathbb{I}^+$  and  $\overline{\mathbb{K}^-} = \mathbb{K}^- \cup \mathbb{I}^-$  where the closures are taken in  $\mathbb{P}^k$ . We also define  $\mathbb{K} := \overline{\mathbb{K}^+} \cap \overline{\mathbb{K}^-}$  which is a compact subset of  $\mathbb{C}^k$ .

In the terminology of [18], the set  $\overline{\mathbb{K}^+}$  (resp.  $\overline{\mathbb{K}^-}$ ) is  $p$ -rigid (resp.  $(k - p)$ -rigid): it supports a unique positive closed  $(p, p)$ -current (resp.  $(k - p, k - p)$ -current) of mass 1 that we denote by  $\mathbb{T}_+$  (resp.  $\mathbb{T}_-$ ). The currents  $\mathbb{T}_\pm$  have no mass on  $\mathbb{H}_\infty$  and satisfy the invariance relations

$$F^*(\mathbb{T}_+) = d_+^p \mathbb{T}_+ \quad \text{and} \quad F_*(\mathbb{T}_-) = d_-^{k-p} \mathbb{T}_-$$

as currents on  $\mathbb{C}^k$  or  $\mathbb{P}^k$ . We call them *the main Green currents of  $F$* . They can be obtained as intersections of positive closed  $(1, 1)$ -currents with local Hölder continuous potentials in  $\mathbb{C}^k$ . Therefore, the measure

$\mathbb{T}_+ \wedge \mathbb{T}_-$  is well-defined and supported in the compact set  $\mathbb{K}$ . This is the unique invariant probability measure of maximal entropy [10, 31] and describes the distribution of saddle periodic points [19]; see also [1, 2, 3, 4, 23] for the case of dimension  $k = 2$ .

Using the above description of the dynamics of  $F$ , we can fix neighbourhoods  $U_1, U_2$  of  $\overline{\mathbb{K}^+}$  and  $V_1, V_2$  of  $\overline{\mathbb{K}^-}$  such that  $F^{-1}(U_i) \Subset U_i, U_1 \Subset U_2 \Subset \mathbb{Q}^k \setminus \mathbb{I}^-, F(V_i) \Subset V_i, V_1 \Subset V_2 \Subset \mathbb{Q}^k \setminus \mathbb{I}^+$  and  $U_2 \cap V_2 \Subset \mathbb{C}^k$ . Let  $\Omega$  be a real  $(p + 1, p + 1)$ -current with compact support in  $U_1$ . Assume that there exists a positive closed  $(p + 1, p + 1)$ -current  $\Omega'$  with compact support in  $U_1$  such that  $|\Omega| \leq \Omega'$ . Define the norm  $\|\Omega\|_{*,U_1}$  of  $\Omega$  as

$$\|\Omega\|_{*,U_1} := \inf\{\|\Omega'\| : |\Omega| \leq \Omega'\},$$

where the infimum is taken over all  $\Omega'$  as above. Observe that when  $\Omega$  is a  $d$ -exact current, we can write  $\Omega = \Omega' - (\Omega' - \Omega)$ , which is the difference of two positive closed currents in the same cohomology class in  $H^{p+1,p+1}(\mathbb{Q}^k, \mathbb{R})$ . Therefore, the norm  $\|\cdot\|_{*,U_1}$  is equivalent to the norm given by  $\inf\|\Omega^\pm\|$ , where  $\Omega^\pm$  are positive closed currents with compact support in  $U_1$  such that  $\Omega = \Omega^+ - \Omega^-$ . Note that  $\Omega^+$  and  $\Omega^-$  have the same mass as they belong to the same cohomology class.

The following property was obtained by the second author; see [13, Proposition 2.1].

**Proposition 2.4.** *Let  $R$  be a positive closed  $(k - p, k - p)$ -current of mass 1 with compact support in  $V_1$  and smooth on  $\mathbb{C}^k$ . Let  $\Phi$  be a real-valued  $(p, p)$ -form of class  $\mathcal{C}^2$  with compact support in  $U_1 \cap \mathbb{C}^k$ . Assume that  $i\partial\bar{\partial}\Phi \geq 0$  on  $V_2$ . Then there exists a constant  $c > 0$  independent of  $R$  and  $\Phi$  such that*

$$\langle d_-^{-(k-p)n}(F^n)_*(R) - \mathbb{T}_-, \Phi \rangle \leq c d_-^{n} \|i\partial\bar{\partial}\Phi\|_{*,U_1} \quad \text{for all } n \geq 0.$$

Note that in what follows, since  $\mathbb{T}_-$  is an intersection of positive closed  $(1, 1)$ -currents with local continuous potentials [31], the intersections  $R \wedge \mathbb{T}_-$  and  $\mathbb{T}_+ \wedge \mathbb{T}_-$  are well-defined and the former depends continuously on  $R$ . In particular, the pairing in the next statement is meaningful and depends continuously on  $R$ .

**Corollary 2.5.** *Let  $R$  be a positive closed  $(k - p, k - p)$ -current of mass 1 supported in  $V_1$ . Let  $\phi$  be a  $\mathcal{C}^2$  function with compact support in  $\mathbb{C}^k$  such that  $i\partial\bar{\partial}\phi \geq 0$  in a neighbourhood of  $\mathbb{K}_+ \cap V_2$ . Then there exists a constant  $c > 0$  independent of  $R$  and  $\phi$  such that*

$$\langle d_-^{-(k-p)n}(F^n)_*(R) - \mathbb{T}_-, \phi\mathbb{T}_+ \rangle \leq c d_-^{n} \|i\partial\bar{\partial}\phi \wedge \mathbb{T}_+\|_{*,U_1} \quad \text{for all } n \geq 0. \tag{2.1}$$

*Proof.* As  $\mathbb{P}^k$  is homogeneous, we will use the group  $\text{PGL}(k + 1, \mathbb{C})$  of automorphisms of  $\mathbb{P}^k$  and suitable convolutions in order to regularize the currents  $R$  and  $\phi\mathbb{T}_+$  and deduce the result from Proposition 2.4. Choose local coordinates centered at the identity  $\text{id} \in \text{PGL}(k + 1, \mathbb{C})$  so that a small neighbourhood of  $\text{id}$  in  $\text{PGL}(k + 1, \mathbb{C})$  is identified to the unit ball  $\mathbb{B}$  of  $\mathbb{C}^{k^2+2k}$ . Here, a point of coordinates  $\epsilon$  represents an automorphism of  $\mathbb{P}^k$  that we denote by  $\tau_\epsilon$ . Thus,  $\tau_0 = \text{id}$ .

Consider a smooth non-negative function  $\rho$  with compact support in  $\mathbb{B}$  and of integral 1 with respect to the Lebesgue measure and, for  $0 < r \leq 1$ , define  $\rho_r(\epsilon) := r^{-2k^2-4k} \rho(r^{-1}\epsilon)$ , which is supported in  $\{|\epsilon| \leq r\}$ . This function allows us to define an approximation of the Dirac mass at  $0 \in \mathbb{B}$  when  $r \rightarrow 0$ . We define  $\Psi := \phi\mathbb{T}_+$  and consider the following regularized currents

$$R_r := \int \rho_r(\epsilon)(\tau_\epsilon)^*(R) \quad \text{and} \quad \Psi_r := \int \rho_r(\epsilon)(\tau_\epsilon)^*(\Psi) = \int \rho_r(\epsilon)(\phi \circ \tau_\epsilon)(\tau_\epsilon)^*(\mathbb{T}_+),$$

where the integrals are with respect to the Lebesgue measure on  $\epsilon \in \mathbb{B}$ .

When  $r$  is small enough and goes to 0, the current  $R_r$  is smooth, positive, closed and with compact support in  $V_1$ , and it converges to  $R$ . Since the RHS of (2.1) depends continuously on  $R$ , we can replace

$R$  by  $R_r$  and assume that  $R$  is smooth. When  $\epsilon$  goes to 0,  $\phi \circ \tau_\epsilon$  converges uniformly to  $\phi$  and  $(\tau_\epsilon)^*(\mathbb{T}_+)$  converges to  $\mathbb{T}_+$ . Using that  $R$  is smooth and  $\mathbb{T}_-$  is a product of  $(1, 1)$ -currents with continuous potentials, we deduce that the LHS of (2.1) is equal to

$$\lim_{r \rightarrow 0} \langle d_-^{-(k-p)n} (F^n)_*(R) - \mathbb{T}_-, \Psi_r \rangle.$$

Since  $\mathbb{T}_+$  is supported in  $\overline{\mathbb{K}_+}$  and we have  $i\partial\bar{\partial}\phi \geq 0$  on a neighbourhood of  $\mathbb{K}_+ \cap V_2$ , we deduce that  $i\partial\bar{\partial}\Psi \geq 0$  on  $V_2$ . By reducing slightly  $V_2$ , we still have  $i\partial\bar{\partial}\Psi_r \geq 0$  on  $V_2$  for  $r$  small enough. We will use the last limit and Proposition 2.4 for  $\Psi_r$  instead of  $\Phi$  and  $U_2$  instead of  $U_1$ . Observe that for  $\epsilon$  small enough, since  $U_1 \Subset U_2$ , we have  $\|(\tau_\epsilon)^*(i\partial\bar{\partial}\Psi)\|_{*,U_2} \leq \|i\partial\bar{\partial}\Psi\|_{*,U_1}$ . We deduce that the LHS of (2.1) is smaller than or equal to

$$\lim_{r \rightarrow 0} c d_-^n \|i\partial\bar{\partial}\Psi_r\|_{*,U_2} \leq c d_-^n \|i\partial\bar{\partial}\Psi\|_{*,U_1} = c d_-^n \|i\partial\bar{\partial}\phi \wedge \mathbb{T}_+\|_{*,U_1}.$$

This completes the proof of the corollary. □

In order to use the above corollary, we will need the following lemmas.

**Lemma 2.6.** *Let  $\kappa \geq 1$  be an integer and  $g_0, \dots, g_\kappa$  functions with compact support in  $\mathbb{C}^k$  with  $\|g_j\|_{C^2} \leq 1$ . Then there is a constant  $c_\kappa > 0$  independent of the  $g_j$ 's such that for all  $\ell_0, \dots, \ell_\kappa \geq 0$ , we have*

$$\|i\partial\bar{\partial}((g_0 \circ F^{\ell_0}) \dots (g_\kappa \circ F^{\ell_\kappa})) \wedge \mathbb{T}_+\|_{*,U_1} \leq c_\kappa.$$

*Proof.* Set  $\tilde{g}_j := g_j \circ f^{\ell_j}$  for simplicity. We have

$$i\partial\bar{\partial}(\tilde{g}_0 \dots \tilde{g}_\kappa) = \sum_{j=0}^\kappa i\partial\bar{\partial}\tilde{g}_j \prod_{l \neq j} \tilde{g}_l + \sum_{0 \leq j \neq l \leq \kappa} i\partial\tilde{g}_j \wedge \bar{\partial}\tilde{g}_l \prod_{m \neq j, l} \tilde{g}_m.$$

Since  $\|g_j\|_{C^2} \leq 1$ , we have  $|g_j| \leq 1$ . Denote by  $\omega_{FS}$  the Fubini-Study form on  $\mathbb{C}^k$ . Then

$$\left| \sum_{j=0}^\kappa i\partial\bar{\partial}\tilde{g}_j \prod_{l \neq j} \tilde{g}_l \right| \lesssim \sum_{j=0}^\kappa (F^{\ell_j})^* \omega_{FS}$$

and an application of Cauchy-Schwarz inequality gives

$$\begin{aligned} \left| \sum_{0 \leq j \neq l \leq \kappa} i\partial\tilde{g}_j \wedge \bar{\partial}\tilde{g}_l \prod_{m \neq j, l} \tilde{g}_m \right| &\lesssim \sum_{j=0}^\kappa i\partial\tilde{g}_j \wedge \bar{\partial}\tilde{g}_j \\ &= \sum_{j=0}^\kappa (F^{\ell_j})^*(i\partial g_j \wedge \bar{\partial} g_j) \\ &\lesssim \sum_{j=0}^\kappa (F^{\ell_j})^*(\omega_{FS}). \end{aligned}$$

As we have  $i\partial\bar{\partial}(\tilde{g}_0 \dots \tilde{g}_\kappa) = 0$  near  $\mathbb{H}_\infty$ , its intersection with  $\mathbb{T}_+$  can be computed on  $\mathbb{C}^k$ . We deduce from the above inequalities and  $d_-^{k-p} = d_+^p$  that

$$\begin{aligned}
 |i\bar{\partial}\bar{\partial}((g_0 \circ F^{\ell_0}) \dots (g_\kappa \circ F^{\ell_\kappa})) \wedge \mathbb{T}_+| &\lesssim \sum_{j=0}^{\kappa} (F^{\ell_j})^*(\omega_{\text{FS}}) \wedge \mathbb{T}_+ \\
 &= \sum_{j=0}^{\kappa} (F^{\ell_j})^*(\omega_{\text{FS}}) \wedge d_+^{-p\ell_j} (F^{\ell_j})^*\mathbb{T}_+ \tag{2.2} \\
 &= \sum_{j=0}^{\kappa} d_+^{-(k-p)\ell_j} (F^{\ell_j})^*(\omega_{\text{FS}} \wedge \mathbb{T}_+).
 \end{aligned}$$

We will use that the  $(p + 1, p + 1)$ -current  $\omega_{\text{FS}} \wedge \mathbb{T}_+$  is positive, closed and of mass 1, and its support is contained in  $\mathbb{K}^+ \subset U_1$ . We have

$$\|(F^{\ell_j})^*(\omega_{\text{FS}} \wedge \mathbb{T}_+)\| = \langle (F^{\ell_j})^*(\omega_{\text{FS}} \wedge \mathbb{T}_+), \omega_{\text{FS}}^{k-p-1} \rangle = \langle \omega_{\text{FS}} \wedge \mathbb{T}_+, (F^{-\ell_j})^*(\omega_{\text{FS}}^{k-p-1}) \rangle,$$

where the last form is positive closed and smooth outside  $\mathbb{I}^-$ . The last pairing only depends on the cohomology classes of  $\omega_{\text{FS}}, \mathbb{T}_+$ , and  $(F^{-\ell_j})^*(\omega_{\text{FS}}^{k-p-1})$ . Hence, it is equal to the mass of  $(F^{-\ell_j})^*(\omega_{\text{FS}}^{k-p-1})$ , which is equal to  $d_+^{-(k-p-1)\ell_j}$ ; see [31]. It follows that each term in the last sum in (2.2) is bounded by 1, which implies that the sum is bounded by  $\kappa + 1$ . The lemma follows.  $\square$

**Lemma 2.7.** *Let  $D \Subset D'$  be two bounded domains in  $\mathbb{C}^k$ . Let  $g$  be a function with compact support in  $D$  and such that  $\|g\|_{\mathcal{C}^2} \leq 1$ . Then there are a constant  $A > 0$  independent of  $g$  and functions  $g^\pm$  with compact supports in  $D'$  and  $\|g^\pm\|_{\mathcal{C}^2} \leq 1$  such that*

$$g = A(g^+ - g^-), \quad i\bar{\partial}g^+ \wedge \bar{\partial}g^+ \leq i\bar{\partial}\bar{\partial}g^+ \text{ on } D \quad \text{and} \quad i\bar{\partial}g^- \wedge \bar{\partial}g^- \leq i\bar{\partial}\bar{\partial}g^- \text{ on } D.$$

*Proof.* Let  $\rho$  be a smooth non-negative function with compact support in  $D'$  and equal to 1 in a neighbourhood of  $\bar{D}$ . Observe that  $\rho g = g$ . We denote by  $z$  the coordinates of  $\mathbb{C}^k$ . Since  $\|g\|_{\mathcal{C}^2} \leq 1$  and  $g$  has compact support in  $D$ , there exists a constant  $A_1 > 0$  independent of  $g$  such that  $|i\bar{\partial}\bar{\partial}g| \leq A_1 i\bar{\partial}\bar{\partial}(\|z\|^2)$ . Set  $g^+ := A^{-1}\rho(g + 2A_1\|z\|^2)$  and  $g^- := 2A^{-1}A_1\rho\|z\|^2$  for some constant  $A > 0$ . It is not difficult to check that we have  $i\bar{\partial}\bar{\partial}g^\pm \geq A^{-1}A_1 i\bar{\partial}\bar{\partial}(\|z\|^2)$  on  $D$ ,  $\|i\bar{\partial}g^\pm \wedge \bar{\partial}g^\pm\|_\infty = O(A^{-2})$  on  $D$ ,  $g = A(g^+ - g^-)$  and  $\|g^\pm\|_{\mathcal{C}^2} = O(A^{-1})$ . Taking  $A$  large enough gives the lemma.  $\square$

### 3. Exponential mixing of all orders for Hénon maps and further remarks

Throughout this section (except for Remarks 3.2 and 3.3),  $f$  denotes a Hénon map on  $\mathbb{C}^2$  of algebraic degree  $d = d_+ = d_- \geq 2$ . Define  $F := (f, f^{-1})$ . It is not difficult to check that  $F$  is a Hénon-Sibony automorphism of  $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$ ; see, for instance, [13, Lemma 3.2]. We will use the notations and the results of Section 2 with  $k = 4$  and  $p = 2$ . We denote in this section by  $\mathbb{T}_\pm$  the Green  $(1, 1)$ -currents of  $f$  and reserve the notation  $\mathbb{T}_\pm$  for the main Green currents of  $F$ . Observe that  $\mathbb{T}_+ = T_+ \otimes T_-$  and  $\mathbb{T}_- = T_- \otimes T_+$ ; see [25, Section 4.1.8] for the tensor (or cartesian) product of currents. We denote by  $K^\pm$  the sets of points of bounded orbit for  $f^{\pm 1}$ . The wedge product  $\mu := T_+ \wedge T_-$  is well-defined and is the measure of maximal entropy of  $f$  [1, 3, 4, 31]. Its support is contained in the compact set  $K = K^+ \cap K^-$ . We have  $\mathbb{K}^+ = K^+ \times K^-$  and  $\mathbb{K}^- = K^- \times K^+$ . Note also that the diagonal  $\Delta$  of  $\mathbb{C}^2 \times \mathbb{C}^2$  satisfies  $\bar{\Delta} \cap \mathbb{I}^+ = \emptyset$  and  $\bar{\Delta} \cap \mathbb{I}^- = \emptyset$  in  $\mathbb{P}^4$ ; see also [13].

We now prove Theorem 1.2. By a standard interpolation [34] (see, for instance, [13, pp. 262-263] and [20, Corollary 1] for similar occurrences), it is enough to prove the statement for  $\gamma = 2$  (i.e., in the case where all the functions  $g_j$  are of class  $\mathcal{C}^2$ ). The statement is clear for  $\kappa = 0$  (i.e., for one test function). By induction, we can assume that the statement holds for up to  $\kappa$  test functions and prove it for  $\kappa + 1 \geq 1$  test functions – that is, show that

$$\left| \langle \mu, g_0(g_1 \circ f^{n_1}) \dots (g_\kappa \circ f^{n_\kappa}) \rangle - \prod_{j=0}^{\kappa} \langle \mu, g_j \rangle \right| \lesssim \left( \prod_{j=0}^{\kappa} \|g_j\|_{\mathcal{C}^2} \right) \cdot d^{-\min_{0 \leq j \leq \kappa-1} (n_{j+1} - n_j)/2}.$$

Recall that  $n_0 = 0$ . The induction assumption implies that we are allowed to modify each  $g_j$  by adding a constant. Moreover, using the invariance of  $\mu$ , the desired estimate does not change if we replace  $n_j$  by  $n_j - 1$  for  $1 \leq j \leq \kappa$  and  $g_0$  by  $g_0 \circ f^{-1}$ . Therefore, we can for convenience assume that  $n_1$  is even.

We fix a large bounded domain  $B \subset \mathbb{C}^2$  satisfying

$$K \subset B, \quad K^- \cap B \subset f(B), \quad \text{and} \quad K^+ \cap B \subset f^{-1}(B).$$

By induction, the inclusions above imply that

$$K \subset B, \quad K^- \cap B \subset f^n(B), \quad \text{and} \quad K^+ \cap B \subset f^{-n}(B) \quad \text{for all } n \geq 1. \tag{3.1}$$

Because of Lemma 2.7 and the fact that we are only interested in the values of the  $g_j$ 's on the support of  $\mu$ , we can assume that all the  $g_j$ 's are compactly supported in  $\mathbb{C}^2$  and satisfy

$$\|g_j\|_{C^2} \leq 1 \text{ on } \mathbb{C}^2 \quad \text{and} \quad i\partial g_j \wedge \bar{\partial} g_j \leq i\partial\bar{\partial} g_j \text{ on } B. \tag{3.2}$$

For simplicity, write  $h := g_1(g_2 \circ f^{n_2-n_1}) \dots (g_\kappa \circ f^{n_\kappa-n_1})$ . We need to prove that

$$|\langle \mu, g_0(h \circ f^{n_1}) \rangle - \langle \mu, g_0 \rangle \cdot \langle \mu, h \rangle| \lesssim d^{-\min_{0 \leq j \leq \kappa-1} (n_{j+1}-n_j)/2}$$

since this estimate, together with the induction assumption applied to  $\langle \mu, h \rangle$ , would imply the desired statement. In order to obtain the result, we will prove separately the two estimates

$$\langle \mu, g_0(h \circ f^{n_1}) \rangle - \langle \mu, g_0 \rangle \cdot \langle \mu, h \rangle \lesssim d^{-\min_{0 \leq j \leq \kappa-1} (n_{j+1}-n_j)/2} \tag{3.3}$$

and

$$-\langle \mu, g_0(h \circ f^{n_1}) \rangle + \langle \mu, g_0 \rangle \cdot \langle \mu, h \rangle \lesssim d^{-\min_{0 \leq j \leq \kappa-1} (n_{j+1}-n_j)/2}. \tag{3.4}$$

Set  $M := 10\kappa$  and fix a smooth function  $\chi$  with compact support in  $\mathbb{C}^2$  and equal to 1 in a neighbourhood of  $\bar{B}$  (we will not need any information on the value of  $\chi$  outside of  $B$ ). Consider the following four functions, which will later allow us to produce some p.s.h. test functions:

$$g_0^+ := \chi \cdot (g_0 + M) \quad \text{and} \quad h^+ := \chi \cdot (g_1 + M)(g_2 \circ f^{n_2-n_1} + M) \dots (g_\kappa \circ f^{n_\kappa-n_1} + M)$$

and

$$g_0^- := \chi \cdot (M - g_0) \quad \text{and} \quad h^- := \chi \cdot ((g_1 + M)(g_2 \circ f^{n_2-n_1} + M) \dots (g_\kappa \circ f^{n_\kappa-n_1} + M) - 2(M + 1)^\kappa).$$

Recall that  $n_0 = 0$ . To prove (3.3) and (3.4), it is enough to show that

$$\langle \mu, g_0^+(h^+ \circ f^{n_1}) \rangle - \langle \mu, g_0^+ \rangle \cdot \langle \mu, h^+ \rangle \lesssim d^{-n_1/2} \tag{3.5}$$

and

$$\langle \mu, g_0^-(h^- \circ f^{n_1}) \rangle - \langle \mu, g_0^- \rangle \cdot \langle \mu, h^- \rangle \lesssim d^{-n_1/2}. \tag{3.6}$$

Indeed, we observe that  $\chi$  does not play any role in (3.5) and (3.6). Hence, the difference between the LHS of (3.5) and the one of (3.3) (resp. of (3.6) and of (3.4)) is a finite combination of expressions involving no more than  $\kappa$  functions among  $g_0, \dots, g_\kappa$  that we can estimate using the induction hypothesis on the mixing of order up to  $\kappa - 1$ . It remains to prove the two inequalities (3.5) and (3.6).



Denote by  $(z, w)$  the coordinates on  $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$  and define

$$\phi^\pm(z, w) := g_0^\pm(w) h^\pm(z).$$

We have the following lemma for a fixed domain  $U_1$  as in Section 2.

**Lemma 3.1.** *The functions  $\phi^\pm$  satisfy*

- (i)  $i\partial\bar{\partial}\phi^\pm \wedge \mathbb{T}_+ \geq 0$  on  $B \times B$ ;
- (ii)  $\|i\partial\bar{\partial}\phi^\pm \wedge \mathbb{T}_+\|_{*,U_1} \leq c_\kappa$ ,

where  $c_\kappa$  is a positive constant depending on  $\kappa$ , but not on the  $g_j$ 's and the  $n_j$ 's.

*Proof.* (i) For simplicity, we set  $\ell_0 = \ell_1 := 0$  and  $\ell_j := n_j - n_1$ . Define also  $\tilde{g}_j := g_j \circ f^{\ell_j}$ . In what follows,  $\tilde{g}_0$  depends on  $w$  and  $\tilde{g}_j$  depends on  $z$  when  $j \geq 1$ . Observe that by the invariance property of  $K^+ \cap B$  in (3.1) and the constraints in (3.2), the following inequalities hold in a neighbourhood of  $K^+ \cap B$ :

$$i\partial\tilde{g}_j \wedge \bar{\partial}\tilde{g}_j = (f^{\ell_j})^*(i\partial g_j \wedge \bar{\partial}g_j) \leq (f^{\ell_j})^*(i\partial\bar{\partial}g_j) = i\partial\bar{\partial}\tilde{g}_j. \tag{3.7}$$

In particular, we have  $i\partial\bar{\partial}\tilde{g}_j \geq 0$  in a neighbourhood of  $K^+ \cap B$ . Note that for  $\tilde{g}_0 = g_0$ , the properties hold on  $B$ , which contains  $K^- \cap B$ .

Now, since  $\mathbb{T}_+$  is closed, positive and supported in  $\overline{\mathbb{K}^+} = \overline{K^+ \times K^-}$ , in order to prove the first assertion, it is enough to show that  $i\partial\bar{\partial}\phi^\pm \geq 0$  on a neighbourhood of  $(K^+ \cap B) \times (K^- \cap B)$  in  $\mathbb{C}^4$  where  $\chi = 1$ . In what follows, we only work on such a neighbourhood. Recall that  $i\partial\bar{\partial} = \frac{i}{\pi}\partial\bar{\partial}$ . We have

$$i\partial\bar{\partial}\phi^+ = \sum_{j=0}^\kappa i\partial\bar{\partial}\tilde{g}_j \prod_{l \neq j} (\tilde{g}_l + M) + \sum_{0 \leq j \neq l \leq \kappa} i\partial\tilde{g}_j \wedge \bar{\partial}\tilde{g}_l \prod_{m \neq j, l} (\tilde{g}_m + M),$$

where we recall that  $\tilde{g}_0$  is  $\tilde{g}_0(w)$  and the other  $\tilde{g}_j$ 's are  $\tilde{g}_j(z)$  for  $1 \leq j \leq \kappa$ . For the first term in the RHS of the last expression, we have

$$\sum_{j=0}^\kappa i\partial\bar{\partial}\tilde{g}_j \prod_{l \neq j} (\tilde{g}_l + M) \geq (M - 1)^\kappa \sum_{j=0}^\kappa i\partial\bar{\partial}\tilde{g}_j.$$

For the second term, an application of the Cauchy-Schwarz inequality and (3.7) give

$$\begin{aligned} \left| \sum_{0 \leq j \neq l \leq \kappa} i\partial\tilde{g}_j \wedge \bar{\partial}\tilde{g}_l \prod_{m \neq j, l} (\tilde{g}_m + M) \right| &\leq \kappa(M + 1)^{\kappa-1} \sum_{j=0}^\kappa i\partial\tilde{g}_j \wedge \bar{\partial}\tilde{g}_j \\ &\leq \kappa(M + 1)^{\kappa-1} \sum_{j=0}^\kappa i\partial\bar{\partial}\tilde{g}_j. \end{aligned}$$

It follows that

$$\begin{aligned} i\partial\bar{\partial}\phi^+ &\geq (M - 1)^\kappa \sum_{j=0}^\kappa i\partial\bar{\partial}\tilde{g}_j - \kappa(M + 1)^{\kappa-1} \sum_{j=0}^\kappa i\partial\bar{\partial}\tilde{g}_j \\ &= (M - 1)^\kappa \left[ 1 - \frac{\kappa}{M + 1} \left( 1 + \frac{2}{M - 1} \right)^\kappa \right] \sum_{j=0}^\kappa i\partial\bar{\partial}\tilde{g}_j, \end{aligned}$$

which gives  $i\partial\bar{\partial}\phi^+ \geq 0$  since the choice  $M = 10\kappa$  implies that  $(1 + \frac{2}{M-1})^\kappa < (1 + \frac{1}{\kappa})^\kappa < 3$ . Similarly, in the same way, we also have

$$i\partial\bar{\partial}\phi^- \geq (M-1)^\kappa \sum_{j=0}^\kappa i\partial\bar{\partial}\tilde{g}_j - (M+1)^{\kappa-1} \sum_{0 \leq j \neq l \leq \kappa} |i\partial\bar{\partial}\tilde{g}_j \wedge \bar{\partial}\tilde{g}_l|,$$

which gives  $i\partial\bar{\partial}\phi^- \geq 0$ . This concludes the proof of the first assertion of the lemma.

(ii) The second assertion of the lemma is a consequence of Lemma 2.6. □

*End of the proof of Theorem 1.2.* Recall that it remains to prove (3.5) and (3.6). Since  $\mu$  is invariant and  $n_1$  is even, we have

$$\langle \mu, g_0^\pm \cdot (h^\pm \circ f^{n_1}) \rangle = \langle \mu, (g_0^\pm \circ f^{-n_1/2})(h^\pm \circ f^{n_1/2}) \rangle.$$

Recall that we have  $\mu = T_+ \wedge T_-$  and that the currents  $T_\pm$  have local continuous potentials in  $\mathbb{C}^2$ . It follows that the intersections of  $T_\pm$  with positive closed currents on  $\mathbb{C}^4$  are meaningful. Moreover, the invariance of  $T_\pm$  implies that  $(F^{n_1/2})_*(T_+) = d^{-n_1}T_+$  on  $\mathbb{C}^4$ . Thanks to the above identities, using the coordinates  $(z, w)$  on  $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$  and denoting by  $\Delta := \{(z, w) : z = w\}$  the diagonal of  $\mathbb{C}^2 \times \mathbb{C}^2$ , we have

$$\begin{aligned} \langle \mu, g_0^\pm \cdot (h^\pm \circ f^{n_1}) \rangle &= \langle T_+ \wedge T_-, (g_0^\pm \circ f^{-n_1/2})(h^\pm \circ f^{n_1/2}) \rangle \\ &= \langle (T_+ \otimes T_-) \wedge [\Delta], (g_0^\pm \circ f^{-n_1/2}(w))(h^\pm \circ f^{n_1/2}(z)) \rangle \\ &= \langle T_+ \wedge [\Delta], (F^{n_1/2})^*(\phi^\pm) \rangle \\ &= \langle d^{-n_1}T_+ \wedge (F^{n_1/2})_*[\Delta], \phi^\pm \rangle \\ &= \langle d^{-n_1}(F^{n_1/2})_*[\Delta], \phi^\pm T_+ \rangle. \end{aligned} \tag{3.8}$$

We apply Corollary 2.5 with the functions  $\phi^\pm = g_0^\pm(w) \cdot h^\pm(z)$  instead of  $\phi$  and the current  $[\Delta]$  instead of  $R$ . For this purpose, since  $\bar{\Delta} \cap \mathbb{I}^+ = \emptyset$ , we can choose a suitable open set  $V_1$  containing  $\Delta$ . We also fix an open set  $V_2$  as in Section 2. Since  $B$  is large enough, Lemma 3.1 implies that  $i\partial\bar{\partial}\phi^\pm \geq 0$  on a neighbourhood of  $\mathbb{K}^+ \cap V_2$ . Thus, we obtain from Corollary 2.5 that

$$\langle d^{-n_1}(F^{n_1/2})_*[\Delta] - T_-, \phi^\pm T_+ \rangle \lesssim d^{-n_1/2},$$

or equivalently,

$$\langle d^{-n_1}(F^{n_1/2})_*[\Delta], \phi^\pm T_+ \rangle - \langle T_-, \phi^\pm T_+ \rangle \lesssim d^{-n_1/2}. \tag{3.9}$$

Together, (3.8), (3.9) and the fact that

$$\langle T_-, \phi^\pm T_+ \rangle = \langle T_+ \wedge T_-, \phi^\pm \rangle = \langle \mu \otimes \mu, g_0^\pm(w) \cdot h^\pm(z) \rangle = \langle \mu, g_0^\pm \rangle \cdot \langle \mu, h^\pm \rangle$$

give the desired estimates (3.5) and (3.6). The proof of Theorem 1.2 is complete.

**Remark 3.2.** When  $f$  is a Hénon-Sibony automorphism of  $\mathbb{C}^k$  with  $k$  even and  $p = k/2$ , the map  $(f, f^{-1})$  is a Hénon-Sibony automorphism of  $\mathbb{C}^{2k}$ . The same proof as above gives us the exponential mixing of all orders and the CLT for  $f$ . The results still hold for every Hénon-Sibony automorphism, but the proof requires some extra technical arguments that we choose not to present here for simplicity; see, for instance, de Thélin and Vigny [11, 35].

**Remark 3.3.** When  $f$  is a horizontal-like map such that the main dynamical degree is larger than the other dynamical degrees, the same strategy gives the exponential mixing of all orders and the CLT; see the papers [14, 17] by Nguyen, Sibony and the second author, and, in particular, [14] for the necessary estimates. In particular, these results hold for all Hénon-like maps in dimension 2; see also Dujardin [23].

**Remark 3.4.** In the companion paper [7], we explain how to adapt our strategy to get the exponential mixing of all orders and the Central Limit Theorem for automorphisms of compact Kähler manifolds

with simple action on cohomology. As the proof in that case requires the theory of super-potentials, which is not needed for Hénon maps, we choose not to present it here.

**Acknowledgements.** We would like to thank the National University of Singapore, the Institut de Mathématiques de Jussieu-Paris Rive Gauche and Xiaonan Ma for the warm welcome and the excellent work conditions. We also thank Romain Dujardin, Livio Flaminio and Giulio Tiozzo for very useful remarks and discussions.

**Competing interest.** The authors have no competing interest to declare.

**Funding statement.** This project has received funding from the French government through the Programme Investissement d'Avenir (LabEx CEMPI /ANR-11-LABX-0007-01, ANR QuaSiDy /ANR-21-CE40-0016, ANR PADAWAN /ANR-21-CE40-0012-01) and the NUS and MOE through the grants A-0004285-00-00 and MOE-T2EP20120-0010.

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