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SOME COMMENTS ON SCALAR DIFFERENTIATION OF VECTOR-VALUED FUNCTIONS

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Abstract

We make some comments on the existence, uniqueness and integrability of the scalar derivatives and approximate scalar derivatives of vector-valued functions. We are particularly interested in the connection between scalar differentiation and the weak Radon–Nikodým property.

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1. Introduction

A deep result of Dilworth and Girardi [8] shows that for an arbitrary infinite-dimensional Banach space X, there exist Bochner measurable X-valued functions defined on the unit interval of the real line whose indefinite Pettis integrals are not weakly differentiable anywhere. Therefore, studying more general notions of function differentiability, rather than that of weak differentiability, may be of some interest in the vector-valued setting. The notion of a *scalar* derivative, which seems more appropriate for vector-valued integration since in general a scalar derivative need not be Bochner measurable, was first defined by Pettis [20] and termed *pseudo-derivative*. However, we believe that the term *scalar derivative* [10, 11, 19] better indicates the nature of this concept of derivative. It should also be noted that scalar differentiation was used in [10, Theorem 2.19] and independently in [19, Theorem 5.1] to give a descriptive definition of the Pettis integral [16, 26]. The purpose of this note is to extend several important theorems that involve the scalar derivative.

In 1939, Pettis established the following relationship between differentiability almost everywhere and scalar differentiability within the function class of strong bounded variation: the Pettis differentiability theorem states that if a function of strong bounded variation on $[a, b]$ has a *separably*-valued scalar derivative on $[a, b]$, then the function is differentiable almost everywhere on $[a, b]$ and its derivative is Bochner integrable on $[a, b]$ [21, Theorem 2.8]. In Theorem 3.2 we have been able to prove the Pettis integrability of a scalar derivative of a function of strong bounded variation without assuming the separability of the range of the scalar derivative in question.

In 1967, Solomon studied the existence and uniqueness of a scalar derivative in the case in which the range space has a *countable* norming set [23]. Corollary 3.4 extends Solomon's uniqueness theorem [23, Theorem 1] by replacing his condition on the range space with a weaker one involving the density character of the dual space equipped with the w^*-topology. In Theorem 3.6 we are able to remove from Solomon's existence theorem [23, Theorem 2] his assumption that the range space has a countable norming set.

A classical result of Dunford and Morse [9] states that if each Lipschitz function mapping $[0, 1]$ into a Banach space X is differentiable almost everywhere on $[0, 1]$, then each X-valued function of strong bounded variation mapping $[0, 1]$ into X is necessarily differentiable almost everywhere on $[0, 1]$; in other words, X has the so-called *Radon–Nikodým property* (RNP) [1, 5–7, 16]. On the other hand, if X lacks the RNP, then there exists an X-valued Lipschitz function defined on $[0, 1]$ that is not differentiable at any point of $[0, 1]$ [25, Theorem 1]. In 1987, Gordon carried out a comprehensive treatment of integration and differentiation in Banach spaces [10]. In particular, he considered functions of generalised strong bounded variation ($sVBG$) and proved that each $sVBG$ ($sVBG_*$) function with values in an RNP space is approximately differentiable (differentiable) almost everywhere [10, Theorems 6.24 and 6.26]. We should remark that Gordon's theorem concerning the differentiability of $sVBG_*$ functions with values in an RNP space was independently obtained by using a different technique and published recently in [2]. The key result of [2] is a characterisation of the RNP in terms of the differentiability properties of several vector-valued function classes closely related to the *variational Henstock integral* [2, Theorem 3.6]. In [3, Theorem 4.5], the same group of authors as in [2] applied the notion of a scalar derivative in order to obtain an analogue of the above mentioned result to functions with values in a Banach space that has the *weak Radon–Nikodým property* (WRNP) [7, 15, 16, 26]. We complete the main result of [3] from two opposite angles: Theorems 3.7 and 3.8 generalise the Dunford–Morse theorem and Gordon's theorem concerning the approximate differentiability of $sVBG$ RNP-space-valued functions, respectively, to functions that assume values in a WRNP space.

2. Definitions and preliminary facts

Before stating our main results, it is necessary to introduce some notation and terminology. Throughout this note $[a, b]$ will denote a fixed nondegenerate interval of the real line and I its closed nondegenerate subinterval. X denotes a real Banach space and X^* its dual. Given $F : [a, b] \to X$, $\Delta F(I)$ denotes the *increment* of F on I. Let E be a set; then $\overline{E}$, χ_E, and $\mu(E)$ will denote the *closure* of E, the *characteristic function* (or *indicator*) of E, and *Lebesgue measure* of E, respectively. For ease of notation, we will drop the adjective Lebesgue and refer to *measurable* and *negligible* sets.

We first define the notions of *strong bounded variation* (sVB) and of *generalised strong bounded variation* ($sVBG$).

DEFINITION 2.1. Let $F : [a,b] \to X$ and let $E \subset [a,b]$.

(a) The function F is sVB on E if

$$\sup\Big\{\sum_k \|\Delta F(I_k)\|\Big\} = \mathrm{V}(F,E) < \infty$$

where the supremum is taken over all finite collections $\{I_k\}$ of nonoverlapping intervals that have endpoints in E.

(b) The function F is $sVBG$ on E if E can be expressed as a countable union of sets on each of which F is sVB.

The adjective *strong* serves as an indication that the norm is inside the sum. It is an easy exercise to prove that for each c in $[a,b]$, $\mathrm{V}(F,[a,b]) = \mathrm{V}(F,[a,c]) + \mathrm{V}(F,[c,b])$ whenever either side is defined.

We now define the derivatives, approximate derivatives, scalar derivatives, and approximate scalar derivatives of vector-valued functions.

DEFINITION 2.2. Let $F : [a,b] \to X$ and let $t \in [a,b]$.

(a) A vector w in X is the *derivative* of F at t if

$$\lim_{s\to t} \frac{F(s) - F(t)}{s - t} = w$$

in the norm topology. We will write $F'(t) = w$.

(b) A vector w in X is the *approximate derivative* of F at t if there exists a measurable set $E \subset [a,b]$ that has t as a point of density such that

$$\lim_{E\ni s\to t} \frac{F(s) - F(t)}{s - t} = w$$

in the norm topology. We will write $F'_{\mathrm{ap}}(t) = w$.

(c) Let $E \subset [a,b]$. The function $f : E \to X$ is a *scalar derivative* of F on E if for each x^* in X^* the real-valued function x^*F is differentiable almost everywhere on E and $(x^*F)' = x^*f$ almost everywhere on E.

(d) Let $E \subset [a,b]$. The function $f : E \to X$ is an *approximate scalar derivative* of F on E if for each x^* in X^* the real-valued function x^*F is approximately differentiable almost everywhere on E and $(x^*F)'_{\mathrm{ap}} = x^*f$ almost everywhere on E.

Some comments on the above definition are appropriate here. While (a) is standard, in (b) we require the set through which the limit exists to be *measurable*, but we feel that including measurability as part of the definition of an approximate derivative is not limiting. In connection with (c) and (d), we underscore that the exceptional set may vary with x^*.

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If $(T, \mathscr{T})$ is a topological space, then $\text{dens}(T, \mathscr{T})$ denotes the smallest cardinal for which there is a dense set of that cardinality. This cardinal is called the *density character* of $(T, \mathscr{T})$. Each cardinal number is identified with the first ordinal number of that cardinality.

The cardinal number $\varkappa(\mu)$ is defined to be the minimal cardinal number $\varkappa$ such that there exists the union of $\varkappa$ negligible sets of positive outer Lebesgue measure. As is usually done, we call *Axiom M* the statement that $\varkappa(\mu) = \mathfrak{c}$.

The reader who is not familiar with the classical theories of vector-valued integration such as those of Bochner and Pettis may wish to consult [6, 16, 26]. These sources also contain an interesting collection of more advanced results.

3. Existence, uniqueness and integrability of the scalar derivative

For the reader's convenience, we recall that

$$\int_a^b F' \le F(b) - F(a)$$

whenever F is a monotone nondecreasing real-valued function defined on $[a, b]$ (see, for example, [13, Theorem 4.10] for the details). The proof of the first new theorem of this note showing that a scalar derivative of a function of strong bounded variation is coordinatewise majorised in absolute value by a *summable* function relies on the same idea. This theorem lays the basis for our result regarding the Pettis integrability of a scalar derivative.

Theorem 3.1. *Let $F : [a, b] \to X$. If F is sVB on $[a, b]$ and f is a scalar derivative of F on $[a, b]$, then there exists a nonnegative Lebesgue integrable function φ defined on $[a, b]$ such that $|x^* f| \le \varphi$ almost everywhere on $[a, b]$ (the exceptional set may vary with x^*) and*

$$\int_a^b |x^* f| \le \int_a^b \varphi \le \mathrm{V}(F, [a, b])$$

for each x^* in X^* with $\|x^*\| \le 1$.

Proof. Select integers p and q such that $p - 1 < a \le p$ and $q - 1 \le b < q$. Then extend the function F to $[p - 1, q]$ by setting $F(t) = F(a)$ on $[p - 1, a)$ and $F(t) = F(b)$ on $(b, q]$. Define a sequence of step functions (a *step function* is a linear combination of characteristic functions of intervals) by

$$f_n = \sum_{k=(p-1)2^n+1}^{q2^n} \Big\{F\Big(\frac{k}{2^n}\Big) - F\Big(\frac{k-1}{2^n}\Big)\Big\} \cdot 2^n \chi_{((k-1)/2^n,\, k/2^n]}.$$

Fix $x^* \in X^*$ such that $\|x^*\| \le 1$ in the remainder of this proof. We will prove that the sequence $\{x^* f_n\}$ converges to $x^* f$ almost everywhere on $[a, b]$. Set

$$E^{x^*} = \{t \in (a, b) : t \text{ is not a dyadic rational and } (x^* F)'(t) = x^* f(t)\}.$$

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Then $\mu(E^{x^*}) = b - a$. Fix t in E^{x^*}. For each positive integer n choose an integer k_n such that $t \in (a_n, b_n)$ where $a_n = (k_n - 1)2^{-n}$ and $b_n = k_n 2^{-n}$. Then $\lim_n a_n = \lim_n b_n = t$ and

$$\begin{aligned}\lim_n \{x^* f_n(t) - x^* f(t)\} &= \lim_n \Big\{ \frac{x^* F(b_n) - x^* F(a_n)}{b_n - a_n} - x^* f(t) \Big\} \\ &= \lim_n \Big\{ \frac{b_n - t}{b_n - a_n} \cdot \Big(\frac{x^* F(b_n) - x^* F(t)}{b_n - t} - x^* f(t) \Big) \\ &\qquad + \frac{t - a_n}{b_n - a_n} \cdot \Big(\frac{x^* F(t) - x^* F(a_n)}{t - a_n} - x^* f(t) \Big) \Big\} = 0.\end{aligned}$$

Hence for all t in E^{x^*} we obtain the result that $\lim_n x^* f_n(t) = x^* f(t)$.

Note that for each positive integer n,

$$\int_a^b \|f_n\| \le \int_{p-1}^q \|f_n\| \le \mathrm{V}(F, [a, b]).$$

Define $\varphi(t) = \liminf_n \|f_n(t)\|$ for each t in $[a, b]$. Then for each t in E^{x^*},

$$|x^* f(t)| = \lim_n |x^* f_n(t)| = \liminf_n |x^* f_n(t)| \le \varphi(t).$$

By Fatou's lemma it follows that

$$\int_a^b \varphi \le \liminf_n \int_a^b \|f_n\| \le \mathrm{V}(F, [a, b]).$$

The proof is complete. □

We are now in a position to prove our generalisation of the Pettis differentiability theorem.

Theorem 3.2. *Let $F : [a, b] \to X$. If F is sVB on $[a, b]$ and f is a scalar derivative of F on $[a, b]$, then f is Pettis integrable on $[a, b]$ and*

$$\mathrm{V}\Big(\int_a^{\cdot} f, [a, b] \Big) \le \mathrm{V}(F, [a, b]).$$

Proof. Let the nonnegative function φ be as in the preceding theorem. We first make note of the fact that

$$\sup_{x^* \in X^* : \|x^*\| \le 1} \int_a^b |x^* f| \le \int_a^b \varphi \le \mathrm{V}(F, [a, b]).$$

Fix $\varepsilon > 0$. Then there exists $\eta > 0$ such that $\int_E \varphi < \varepsilon$ whenever $\mu(E) < \eta$. Consequently,

$$\sup_{x^* \in X^* : \|x^*\| \le 1} \Big| \int_E x^* f \Big| \le \sup_{x^* \in X^* : \|x^*\| \le 1} \int_E |x^* f| \le \int_E \varphi < \varepsilon$$

whenever $\mu(E) < \eta$. Define $T_f : X^* \to L^1[a, b]$ by $T_f x^* = x^* f$ for each x^* in X^*. By the Dunford–Pettis theorem (see, for example, [14, Theorem 3.24]), the operator T_f is weakly compact.

Let the function sequence $\{f_n\}$ be as in the proof of the preceding theorem. Denote the closed linear span of $\bigcup_n f_n([a,b])$ by Y and note that Y is *separable*. Then, for each x^* in X^* such that $x^*|_Y = 0$, the relation

$$x^* f = \lim_n x^* f_n = 0$$

holds almost everywhere on $[a,b]$. Now it follows from [24, Theorem 2.8] that the function f is Pettis integrable on $[a,b]$. Hence, for each finite collection $\{I_k\}$ of nonoverlapping intervals,

$$\sum_k \left\| \int_{I_k} f \right\| = \sum_k \sup_{x^* \in X^* : \|x^*\| \le 1} \left| \int_{I_k} x^* f \right|$$
$$\le \sum_k \sup_{x^* \in X^* : \|x^*\| \le 1} \int_{I_k} |x^* f| \le \sum_k \int_{I_k} \varphi \le \int_a^b \varphi \le \mathrm{V}(F,[a,b]),$$

which is what we desired. □

A fairly simple example can be given to demonstrate that two scalar derivatives of a single function may differ on a set of full Lebesgue measure [23, Section 2]. However, we have a sufficient condition, Corollary 3.4, for a function to have no more than one approximate scalar derivative (compare [23, Theorem 1]). We first need the following familiar fact, which we have been unable to find in print (although similar questions have received some attention by Rodríguez [22]).

Theorem 3.3. *Suppose that* $\mathrm{dens}(X^*, w^*) < \varkappa(\mu)$. *Let* $f : [a,b] \to X$ *and let* $E \subset [a,b]$. *If, for each* x^* *in* X^*, $x^* f = 0$ *almost everywhere on* E *(the exceptional set may vary with* x^**), then* $f = 0$ *almost everywhere on* E.

Proof. Let λ denote $\mathrm{dens}(X^*, w^*)$. By [14, Fact 4.10], there exists a set $\{x^*_\alpha\}_{\alpha<\lambda} \subset X^*$ with $\|x^*_\alpha\| \le 1$ for each $\alpha < \lambda$ that *separates* the points of X. For each $\alpha < \lambda$ set

$$E_\alpha = \{t \in E : x^*_\alpha f(t) = 0\}.$$

Define

$$N = \bigcup_{\alpha<\lambda} E \backslash E_\alpha$$

and note that N is negligible. Then for each $\alpha < \lambda$ and for each t in $E\backslash N$ we have $x^*_\alpha f(t) = 0$. Since $\{x^*_\alpha\}_{\alpha<\lambda}$ separates the points of X, it follows that $f = 0$ on $E\backslash N$. The proof is complete. □

Corollary 3.4. *Suppose that* $\mathrm{dens}(X^*, w^*) < \varkappa(\mu)$. *Let* $F : [a,b] \to X$ *and let* $E \subset [a,b]$. *If* f *and* g *are approximate scalar derivatives of* F *on* E, *then* $f = g$ *almost everywhere on* E.

We should note at this point that if we assume Axiom M, then the above corollary is valid provided that $\mathrm{dens}(X^*, w^*) < \mathfrak{c}$.

The conclusion of the next lemma is not surprising, but we have not found the details in the literature.

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LEMMA 3.5. *Let $F : [a,b] \to X$ and let $E \subset [a,b]$. If E can be written as a countable union of sets on each of which F has a scalar derivative (an approximate scalar derivative), then F is scalarly differentiable (approximately scalarly differentiable) on E.*

PROOF. We will prove the approximate scalar derivative case. Suppose that $E = \bigcup_n E_n$ and F is approximately scalarly differentiable to $f_n : E_n \to X$ on E_n for each n. Set $H_1 = E_1$ and

$$H_n = E_n \Big\backslash \bigcup_{k=1}^{n-1} E_k \quad \text{for each } n > 2.$$

Define $f : E \to X$ by $f(t) = f_n(t)$ for each t in H_n. Fix x^* in X^*. For each n, set $N_n^{x^*} = H_n \backslash \{t \in H_n : (x^* F)'_{\mathrm{ap}}(t) = x^* f_n(t)\}$. Since each $N_n^{x^*}$ is negligible, it follows that $(x^* F)'_{\mathrm{ap}} = x^* f$ almost everywhere on E. Hence F is approximately scalarly differentiable to f on E. The proof is complete. □

The next theorem relates the notion of scalar differentiability on a closed set to a local scalar differentiability condition. The key notion for this theorem is that of a portion of a set. To be clear, recall that a *portion* of a set $E \subset [a,b]$ is a nonempty set of the form $(c,d) \cap E$.

THEOREM 3.6. *Let E be a nonempty closed subset of $[a,b]$ and let $F : [a,b] \to X$. Then each of the following two statements about the function F implies the other:*

(i) *F has a scalar derivative (an approximate scalar derivative) on E;*
(ii) *given any perfect set $P \subset E$, there is a portion Q of P such that F has a scalar derivative (an approximate scalar derivative) on Q.*

PROOF. We will prove the approximate scalar derivative case.

Obviously (i) implies (ii).

Now assume that (ii) holds and suppose if possible that F has no approximate scalar derivative on E. Set

$$\mathscr{F} = \{(c,d) : F \text{ has an approximate scalar derivative on } (c,d) \cap E\}$$

and note that $\mathscr{F} \neq \emptyset$. Then there is no loss of generality in supposing that $\mathscr{F} = \{(c_n, d_n)\}$ without affecting the set of points of E covered by $\mathscr{F}$. Next, if

$$P = E \Big\backslash \bigcup_n (c_n, d_n),$$

then P is *perfect* and, by Lemma 3.5, P is *nonempty*. This is impossible since it follows from the definition of the family $\mathscr{F}$ that F has no approximate scalar derivative on any portion of P. □

The last two theorems of this note apply to spaces with the WRNP. Recall that X has the WRNP if each function of strong bounded variation mapping $[a,b]$ into X has a scalar derivative on $[a,b]$ (see [3] for the details behind this definition). Also,

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the reader should compare our Theorem 3.8 with [3, Theorem 4.5] to realise how closely the $sVBG$ function class parallels the $sVBG_*$ function class in terms of its differentiability properties.

THEOREM 3.7. *Any one of the following statements about a Banach space X implies all the other statements:*

(i) *X has the WRNP;*

(ii) *each Lipschitz function mapping $[a, b]$ into X has a scalar derivative on $[a, b]$;*

(iii) *each Lipschitz function mapping $[a, b]$ into X has a scalar derivative on some portion of any perfect set $P \subset [a, b]$.*

PROOF. Obviously (i) implies (ii). Theorem 3.6 applies to yield the equivalence of (ii) and (iii).

To see that (ii) implies (i), let $F : [a, b] \to X$ be sVB on $[a, b]$. We now apply a classical argument [9, Section 4] to show that F has a scalar derivative on $[a, b]$. Define a strictly monotone increasing real-valued function on $[a, b]$ by

$$\sigma(t) = t + \mathrm{V}(F, [a, t]) \quad \text{for each } t \text{ in } [a, b],$$

and let $E = \sigma([a, b])$. Let $\tau : E \to [a, b]$ be the inverse of σ. Define $G(s) = F(\tau(s))$ for each s in E. Then for any two points $s < s'$ in E we obtain the result that

$$\begin{aligned} \|G(s) - G(s')\| &\le \mathrm{V}(F, [\tau(s), \tau(s')]) \le \tau(s') - \tau(s) + \mathrm{V}(F, [\tau(s), \tau(s')]) \\ &= \tau(s') - \tau(s) + \mathrm{V}(F, [a, \tau(s')]) - \mathrm{V}(F, [a, \tau(s)]) \\ &= \sigma(\tau(s')) - \sigma(\tau(s)) = s' - s. \end{aligned}$$

We first extend the domain of G to $\overline{E}$ in the natural way. We further let G denote the function defined on the interval $[a, b + \mathrm{V}(F, [a, b])]$ that is linear on the intervals contiguous to $\overline{E}$ with respect to $[a, b + \mathrm{V}(F, [a, b])]$. It is easy to see that for any two points s and s' in $[a, b + \mathrm{V}(F, [a, b])]$ we obtain the same Lipschitz condition $\|G(s) - G(s')\| \le |s' - s|$. Thus G has a scalar derivative, g say, on $[a, b + \mathrm{V}(F, [a, b])]$. Fix x^* in X^* and set $N^{x^*} = E \setminus \{s \in E : (x^*G)'(s) = x^*g(s)\}$. Since τ satisfies the Lipschitz condition $|\tau(s) - \tau(s')| \le |s' - s|$ on E and the set N^{x^*} is negligible, we see that the set $\tau(N^{x^*})$ is negligible. That is, we have the relation

$$(x^*G)'(\sigma(t)) = x^*g(\sigma(t)) \quad \text{for each } t \text{ in } [a, b] \setminus \tau(N^{x^*}).$$

Set

$$N = \{t \in [a, b] : \sigma \text{ is not differentiable at } t\}.$$

Since σ is increasing in a strictly monotone manner on $[a, b]$, it follows that the set N is negligible. Thus, for each t in $[a, b] \setminus (N^{x^*} \cup N)$,

$$\begin{aligned} \lim_{h \to 0} \frac{x^*F(t + h) - x^*F(t)}{h} &= \lim_{h \to 0} \frac{x^*G(\sigma(t + h)) - x^*G(\sigma(t))}{\sigma(t + h) - \sigma(t)} \cdot \lim_{h \to 0} \frac{\sigma(t + h) - \sigma(t)}{h} \\ &= x^*g(\sigma(t)) \cdot \sigma'(t), \end{aligned}$$

so that $f = g \circ \sigma \cdot \sigma'$ is a scalar derivative of the function F on $[a, b]$. The proof is complete. □

Theorem 3.8. *Suppose that X has the WRNP and let $F : [a, b] \to X$ be Bochner measurable on $[a, b]$. If F is sVBG on a set $E \subset [a, b]$, then F has an approximate scalar derivative on E.*

Proof. Our proof is patterned after a proof in Gordon [10, Theorem 6.24].

Let $E = \bigcup_n E_n$ where F is sVB on each E_n. Since F is Bochner measurable on $[a, b]$, for each n we use a result of Gordon [11] (see [17, Theorem 2] for a published version of the proof) to choose a measurable set $A_n \subset [a, b]$ such that F is sVB on A_n. By Lemma 3.5, it is sufficient to prove that F has an approximate scalar derivative on each A_n.

Fix n and let $\varepsilon > 0$. Choose a closed set $H \subset A_n$ such that $\mu(A_n \backslash H) < \varepsilon$ and let $G : [a, b] \to X$ be the function that equals F on H and is linear on the intervals contiguous to H. Then G is sVB on $[a, b]$ by [12, Theorem 3]. Since X has the WRNP, the function G has a scalar derivative, g say, on $[a, b]$.

Fix x^* in X^*. Let $H_1^{x^*}$ be the set of all points t in H such that t is a point of density of H and $(x^*G)'(t)$ exists. Then the set $H \backslash H_1^{x^*}$ is negligible and, for each t in $H_1^{x^*}$,

$$(x^*G)'(t) = \lim_{H \ni s \to t} \frac{x^*G(s) - x^*G(t)}{s - t} = \lim_{H \ni s \to t} \frac{x^*F(s) - x^*F(t)}{s - t}.$$

This shows that $(x^*F)'_{\mathrm{ap}}(t) = x^*g(t)$ for each t in $H_1^{x^*}$. Hence, the function F is approximately scalarly differentiable to g on the set H and $\mu(A_n \backslash H) < \varepsilon$. Once again, since $\varepsilon > 0$ was arbitrary, it follows from Lemma 3.5 that F has an approximate scalar derivative on A_n. This completes the proof. □

Our discussion reveals the following two potentially interesting questions.

Question 3.9. Under the hypothesises of Theorem 3.8, can the approximate scalar derivative be written as a sum $g + h$, where g is Bochner measurable and, for each x^* in X^*, $x^*h = 0$ almost everywhere on E (the exceptional set may vary with x^*)?

Question 3.10. Which proper subclasses of the Lipschitz function class may replace the latter within the hypothesis of a Dunford–Morse type theorem for either the RNP or the WRNP or both these properties?

We wish to make an observation about the nature of the second question. *Indefinite Riemann integrals* make up a function class that is a proper subclass of the Lipschitz function class (we refer to a recent paper by Thomson [27] for a thorough discussion of real-valued indefinite Riemann integrals). Surprisingly, the indefinite Riemann integral function class is *not* sufficient to resolve our question in the affirmative even for separable spaces: a rather delicate counterexample is the Bourgain–Rosenthal subspace of L^1 denoted by E in their original 1980 paper [4]. Indeed, the Bourgain–Rosenthal space E has the *strong Schur property* but lacks the RNP (see [1, 5] for better adapted details of the Bourgain–Rosenthal construction). Since E is a Schur subspace

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of L^1, E has the *Lebesgue property* (see [18] and the references therein). As a result, each E-valued indefinite Riemann integral must be differentiable almost everywhere.

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