J. Austral. Math. Soc. 21 (Series B) (1980), 402–417

THE STRUCTURE OF RI-INVARIANT TWELVE-TONE ROWS

JAMES A. FILL and ALAN J. IZENMAN

(Received 13 November 1978)

Abstract

This paper presents an efficient method for generating the class of all twelve-tone rows which are transpositions of their own retrograde-inversions. It is shown here that the members of this class can be obtained from a subclass of those rows whose first six notes are ascending and whose first note is C. The number of twelve-tone rows in this subclass is 192, and a complete listing is given in an appendix to this paper. The theory as developed here can be applied to tone rows having any even number of notes.

1. Introduction

Since Schoenberg's introduction of twelve-tone row composition in the early part of the twentieth century, his ideas have been examined and re-examined at great length in the music literature, and much ink has been spilled in attempts to extract structure from his highly controversial method. Despite (or, perhaps, because of) this controversy, the "method" has produced acknowledged masterpieces by composers Schoenberg, Berg, Webern, Stockhausen, Boulez, Stravinsky and Babbitt. Many twelve-tone works are available today and several important studies of the music-analytic properties of the twelve-tone row system have appeared in recent years. Research on this topic has ranged from a greater understanding of specific twelve-tone works to the development of a suitable nomenclature with which to describe succinctly the compositional techniques involved in twelve-tone row composition. The generation (usually by computer) of all twelve-tone rows possessing some desirable property is also of importance.

It is the aim of this paper to explore the structure of twelve-tone rows that are transpositions of their own retrograde-inversions. Following Babbitt [1], we call such rows *RI-invariant*. In this paper we shall determine the exact number of RI-invariant twelve-tone rows and also find an efficient method for generating them. Our initial introduction to this subject was through Professor Easley Blackwood, who conjectured that the members of the class of all RI-invariant twelve-tone rows could be obtained (somehow) from the subclass of rows whose first six notes are ascending and whose first note is C. This paper demonstrates the validity of Blackwood's conjecture through the use of a novel method involving selective partitionings of a circle. This method produces an algorithm which both generates and counts all RI-invariant twelve-tone rows without resorting to a computer search of all 12! twelve-tone rows. The algorithm is extremely efficient and can be readily extended to the generation of RI-invariant tone rows possessing a given *even* number of notes. The reader may also be interested in the companion paper [2].

2. Preliminaries

In order that the overall structure of RI-invariant twelve-tone rows be rendered visible to the reader, we shall analyze the twelve-tone row system within the more general framework of *m-tone rows*, where m is an arbitrary integer (greater than 2). The elements of such an m-tone row are some permutation of the notes of an m-tone scale. The advantages of such a generalization are not only mathematical; with the advent of electronic music produced by a suitably programmed computer or synthesizer, the composition and study of m-tone (or microtonal) works is now more feasible than previously.

For notational simplicity and technical ease, we shall substitute the integers 0 through $m-1$ for the ordered succession of m ascending notes that constitute the m-tone scale in question. Such a representation is standard for twelve-tone row theory (see Perle [3]), and is probably necessary for the theory of microtonal composition.

Let $m > 2$ be an integer and let

$$\Pi = \begin{pmatrix} 0 & 1 & 2 & \dots & m-1 \\ \pi_0 & \pi_1 & \pi_2 & \dots & \pi_{m-1} \end{pmatrix} \tag{1}$$

denote a permutation of $\bar{Z}_m = \{0, 1, 2, \dots, m-1\}$. Then Π is called an *m-tone row* (or *m-phonic sequence*). The set of all m-tone rows is denoted by $\mathscr{P}_m$. For example, the twelve-tone row

$$\Pi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 9 & 10 & 3 & 11 & 4 & 6 & 0 & 1 & 7 & 8 & 2 & 5 \end{pmatrix} \tag{2}$$

forms the basis for Schoenberg's *Violin Concerto*, Opus 36.

We now define three operations on Π. Let α be the cyclic permutation $(0\ 1\ 2\ \dots\ m-1)$, which corresponds to raising each note by one unit. Using the convention that multiplication is carried out from left to right (that is, functions

https://doi.org/10.1017/S0334270000002101 Published online by Cambridge University Press

are written to the right of the argument), the *transposition* of Π by an integer a is then the m-tone row $\Pi\alpha^a$ (that is, $(i)\pi\alpha^a = (\pi_i)\alpha^a = (\pi_i + a) \bmod m$). Thus, the transposition of Π in (2) above by 3 yields the twelve-tone row

$$\Pi\alpha^3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 0 & 1 & 6 & 2 & 7 & 9 & 3 & 4 & 10 & 11 & 5 & 8 \end{pmatrix}.$$

Next let β be the permutation $(0\ m-1)(1\ m-2)(2\ m-3)\ldots$, which corresponds to reversing the order of the elements. The *retrograde* (or *crab*) of Π is then the m-tone row $\beta\Pi$. Thus the retrograde of Π in (2) above is

$$\beta\Pi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & 2 & 8 & 7 & 1 & 0 & 6 & 4 & 11 & 3 & 10 & 9 \end{pmatrix}.$$

Note that $\beta^2 = 1$. Finally, let γ be the permutation $(1\ m-1)(2\ m-2)(3\ m-3)\ldots$. Then the *negative* of Π is the m-tone row $\Pi\gamma$. The negative of Π in (2) above is, therefore,

$$\Pi\gamma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 2 & 9 & 1 & 8 & 6 & 0 & 11 & 5 & 4 & 10 & 7 \end{pmatrix}.$$

Note that $\alpha\gamma = \gamma\alpha^{-1}$. The *inversion* of an m-tone row Π is then the m-tone row $\Pi\alpha^{-2\pi_0}\gamma\,(=\Pi\gamma\alpha^{2\pi_0})$. For the Π in (2) above the inversion is

$$\Pi\alpha^6\gamma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 9 & 8 & 3 & 7 & 2 & 0 & 6 & 5 & 11 & 10 & 4 & 1 \end{pmatrix}.$$

It is worth noting that this last definition is not the usual way in which inversion is defined. Perle [8], p. 3, for example, calls our negative of a tone row Π the inversion of Π. The *retrograde-inversion* (or *crab-inversion*) of Π is the retrograde of the inversion of Π and is the m-tone row $\beta\Pi\alpha^{-2\pi_0}\gamma$. Thus, the retrograde-inversion of Π in (2) above is

$$\beta\Pi\alpha^6\gamma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 4 & 10 & 11 & 5 & 6 & 0 & 2 & 7 & 3 & 8 & 9 \end{pmatrix}.$$

It will be convenient to use the following abbreviated version of an m-tone row Π in the discussion below. Instead of writing Π in the form (1) above, we shall henceforth write

$$\Pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_{m-1}),$$

the permutational ordering of the notes being understood. Furthermore, for the remainder of this paper, m will be an *even* integer (greater than 2), and we shall sometimes write $2k$ for m (so that $k > 1$).

In this notation then, an *RI-invariant 2k-tone row* Π is defined to be a $2k$-tone row for which

$$\Pi = \beta\Pi\alpha^{-2\pi_0}\gamma\alpha^a = \beta\Pi\gamma\alpha^b, \quad b = a+2\pi_0,$$

for some $a \in \bar{Z}_{2k}$. (Necessarily, $b = a+2\pi_0 = \pi_{k-1}+\pi_k$.) The *generating sequence* for Π is the sequence of integers denoted by

$$GS(\Pi) = (g_1, g_2, \ldots, g_{k-1}),$$

or simply GS, where $g_i = (\pi_i - \pi_{i-1}) \pmod{2k}$, $1 \leqslant i \leqslant k-1$. The quantity

$$X(\Pi) = (\pi_k - \pi_{k-1}) \mod 2k$$

is called the *rotation factor* of Π and is sometimes written X. As an illustration of these definitions, consider the twelve-tone row

$$\Pi = (3, 7, 2, 9, 8, 0, 1, 5, 4, 11, 6, 10),$$

which is also RI-invariant. Its generating sequence is $GS(\Pi) = (4, 7, 7, 11, 4)$ and its rotation factor is $X(\Pi) = 1$.

In order to generate (and count) all such RI-invariant tone rows, it will first be necessary to examine in detail the following special type of RI-invariant tone row. We call $S = (s_0, s_1, s_2, \ldots, s_{2k-1})$ a *canonical RI-invariant 2k-tone row* if:

(i) S is RI-invariant;

(ii) the first note of S is 0; and

(iii) the first k notes of S are ascending.

In other words, S has to satisfy the two conditions:

(1*) $S = \beta S\gamma\alpha^a$;

(2*) $0 = s_0 < s_1 < s_2 < \ldots < s_{k-1}$,

for some $a \in \bar{Z}_{2k}$. (Necessarily, $a = s_{k-1}+s_k$.) The *interval sequence* corresponding to such an S is denoted by $IS(S)$, or simply IS, and is the sequence defined by

$$IS(S) = (GS(S), X(S), GS^{\text{rev}}(S)) = (g_1, g_2, \ldots, g_{k-1}, X, g_{k-1}, \ldots, g_2, g_1),$$

where $GS^{\text{rev}}(S)$ is the reversed generating sequence $(g_{k-1}, \ldots, g_2, g_1)$. Thus the twelve-tone row

$$S = (0, 1, 4, 5, 8, 9, 2, 3, 6, 7, 10, 11),$$

which forms the basis for Schoenberg's *Suite*, Opus 29, is a canonical RI-invariant twelve-tone row with generating sequence, rotation factor and interval sequence given by

$$GS(S) = (1, 3, 1, 3, 1), \quad X(S) = 5,$$

$$IS(S) = (1, 3, 1, 3, 1, 5, 1, 3, 1, 3, 1),$$

respectively.

3. Canonical diagrams

We next present a convenient (and novel) way of representing canonical RI-invariant $2k$-tone rows. Without any loss of generality, we take $k=6$, the arguments of this section extending in a straightforward manner to $2k$-tone rows with $k>1$.

Let S be a canonical RI-invariant twelve-tone row and let $t(=s_{11})\in\bar{Z}_{12}$ be the final note of this row. The interval sequence IS, by definition, must be of the form

$$IS(S)=(g_1,g_2,g_3,g_4,g_5,X,g_5,g_4,g_3,g_2,g_1),$$

where g_1, g_2, g_3, g_4, and g_5 are positive integers whose sum is less than 12. From the form of this interval sequence, we see that the individual notes of S can be written as

$$S=(s_0,s_1,,s_2,\dots,s_{11}),$$

where

$$s_0=0,\quad s_1=g_1,\quad s_2=g_1+g_2,\quad s_3=g_1+g_2+g_3,\quad s_4=g_1+g_2+g_3+g_4,$$
$$s_5=g_1+g_2+g_3+g_4+g_5,\quad s_6=t-s_5,\quad s_7=t-s_4,\quad s_8=t-s_3,\quad s_9=t-s_2,$$
$$s_{10}=t-s_1,\quad s_{11}=t-s_0=t,$$

and addition and subtraction are understood to be taken modulo 12. Note that t and X must both be odd integers. That t has to be odd can be seen from the fact that $\sum_{i=0}^{11}s_i\equiv 6\,(\text{mod}\,12)$, and that the special structure of S also gives $\sum_{i=0}^{11}s_i\equiv 6t\,(\text{mod}\,12)$. Summing now over the first half of the tone row S yields the relation $2s_5+X\equiv t\,(\text{mod}\,12)$; since t is odd, this forces X to be odd.

Now the function $f(s)=t-s$, which characterizes the last six notes of S, corresponds to reflection across a diameter of the circle in Fig. 1, one of whose

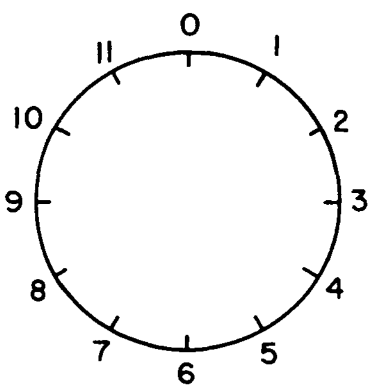

Fig. 1. Basic diagram.

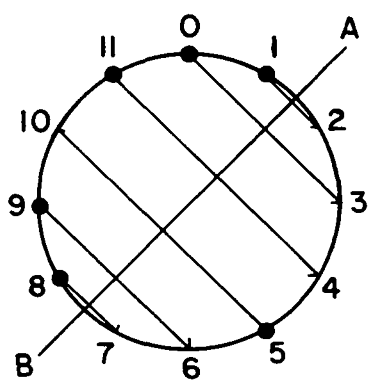

Fig. 2. Diagram for the canonical RI-invariant twelve-tone row $S=(0, 1, 5, 8, 9, 11, 4, 6, 7, 10, 2, 3)$.

https://doi.org/10.1017/S0334270000002101 Published online by Cambridge University Press

end-points lies half-way between $(t-1)/2$ and $(t+1)/2$. For example, consider the canonical RI-invariant twelve-tone row S given by

$$S = (0, 1, 5, 8, 9, 11, 4, 6, 7, 10, 2, 3).$$

This row has generating sequence $GS(S) = (1, 4, 3, 1, 2)$, rotation factor $X(S) = 5$, and $t = 3$. So s_i is the reflection of $s_{2k-1-i} = s_{11-i}$ in the diameter with one end-point half-way between $(t-1)/2 = 1$ and $(t+1)/2 = 2$; see Fig. 2. The dots in this diagram represents the first six elements of S (in this case, 0, 1, 5, 8, 9 and 11), while the remaining positions represent the last six. The *cross-lines* (or *line-system*) of the diagram show the correspondence $s_i \leftrightarrow s_{11-i}$. Clearly, a canonical RI-invariant twelve-tone row corresponds *uniquely* to such a diagram.

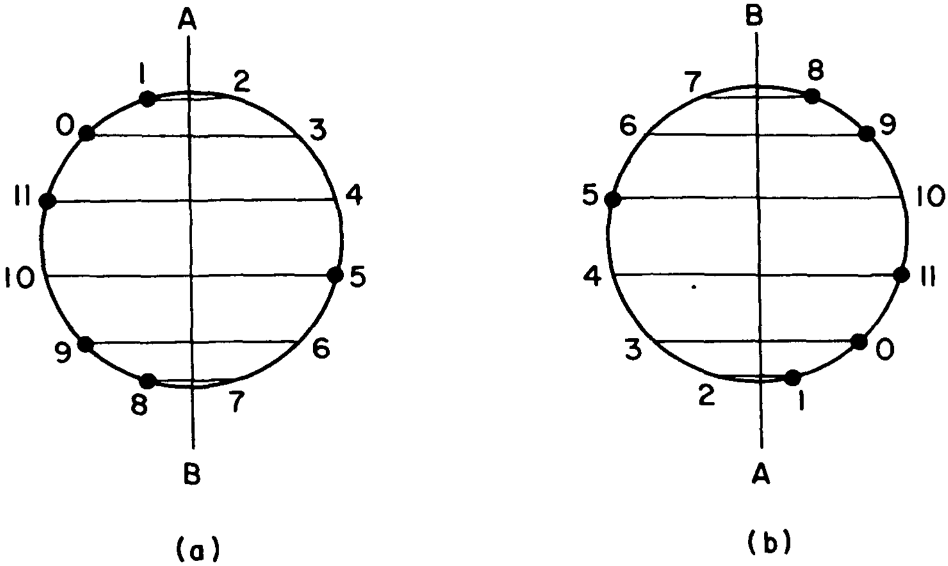

Fig. 3. Left-handed (a) and right-handed (b) diagrams for the S of Fig. 2.

This diagram now suggests an alternative way of labelling canonical RI-invariant twelve-tone rows. First rotate the diagram until the diameter is vertical; this can be done in two ways, with either end-point A or B at the top as in Fig. 3. If the first dot (reading from the top) is on the left-hand side of the diameter (as in Fig. 3(a)), the diagram is said to have a *left-handed orientation*, while if the first dot is on the right-hand side of the diameter (as in Fig. 3(b)), the diagram is said to have a *right-handed orientation*. The numbers of adjacent dots are then read off from the diagram (proceeding from top to bottom down through the line-system). Thus, Fig. 3(a) is a left-handed (3, 1, 2)-diagram (that is, starting from the left side of the diameter: three dots at positions 1, 0 and 11; one dot at position 5; and two dots at positions 9 and 8), while Fig. 3(b) is a right-handed (2, 1, 3)-diagram (that is, starting from the right side of the diameter: two dots at positions 8 and 9;

one dot at position 5; and three dots at positions 11, 0 and 1). Both diagrams clearly correspond to the same S given above. We shall write L for a left-handed diagram and R for a right-handed diagram. Thus, (a) above will be written as ((3, 1, 2), L), and (b) will be written as ((2, 1, 3), R). This notation will be made more precise in Section 4 below.

Suppose now we consider all such diagrams *without labels for coordinates*. One such diagram is given by Fig. 4. As we have already seen, this diagram corresponds to the canonical RI-invariant twelve-tone row

$$S=(0, 1, 5, 8, 9, 11, 4, 6, 7, 10, 2, 3).$$

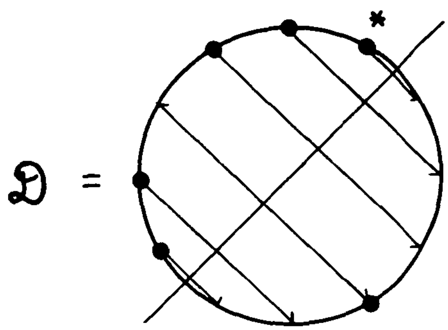

Fig. 4. Canonical diagram, $\mathscr{D}$, corresponding to the S of Fig. 2.

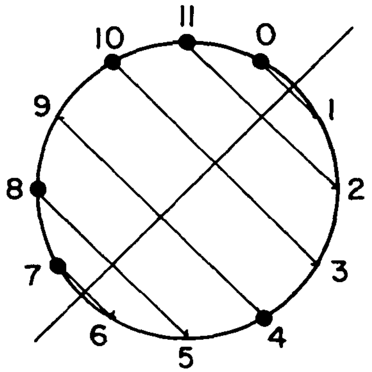

Fig. 5. Diagram, obtained from $\mathscr{D}$, for the canonical RI-invariant twelve-tone row $S=(0, 4, 7, 8, 10, 11, 2, 3, 5, 6, 9, 1)$.

If we now attach coordinate labels to $\mathscr{D}$ beginning with 0 at any one of the six positions marked with dots, say *, and continue to label in a clockwise fashion from 1 through 11, we arrive at Fig. 5. From this diagram we can read off a second canonical RI-invariant twelve-tone row, namely,

$$S=(0, 4, 7, 8, 10, 11, 2, 3, 5, 6, 9, 1).$$

Thus $\mathscr{D}$, which we shall call a *canonical diagram*, corresponds to *more than one* canonical RI-invariant twelve-tone row. This correspondence between canonical diagrams and canonical RI-invariant twelve-tone rows leads to an extremely efficient method for generating all such canonical RI-invariant twelve-tone rows, and also to a method for counting all such rows (without generating them all). An extension of these arguments leads in turn to the generation and counting of all RI-invariant twelve-tone rows. The next section develops these ideas in formal fashion, followed by examples of the technique.

4. The exact number of canonical RI-invariant tone rows

In this section we shall first find it useful to introduce the following notation and terminology. A *composition* of the positive integer k is defined to be an ordered sequence $C=(c_1,c_2,\ldots,c_l)$ of positive integers such that $\sum_{i=1}^{l} c_i = k$. Since a composition of k corresponds uniquely to a choice of a subset of the points (for example, .|..|.|.. shows the composition of 6 into (1, 2, 1, 2)), it follows that the number of compositions of k is 2^{k-1}. An *oriented-composition* of k is an ordered pair $\mathscr{C}=(C,d)$, where C is a composition of k and d is either L or R. $\mathscr{C}$ is said to be a *left-handed* (*right-handed*) *composition* of k if d is L (d is R). A composition C is *even* or *odd* according as l is even or odd. An oriented-composition $\mathscr{C}=(C,d)$ is even or odd according as C is even or odd. The *mate* of a composition $C=(c_1,c_2,\ldots,c_l)$ is the reverse composition $C^{\text{rev}}=(c_l,\ldots,c_2,c_1)$. The mate of an oriented-composition is an oriented-composition $\mathscr{C}'=(C^{\text{rev}},d')$, where d' is d or the opposite of d according as C is even or odd. A composition C is *symmetric* if $C=C^{\text{rev}}$. An oriented-composition $\mathscr{C}=(C,d)$ is symmetric if C is symmetric.

From these definitions we make the following observations:

(1) The number, e_k, of even symmetric (unoriented) compositions of k is zero if k is odd, and $2^{k/2-1}$ if k is even:

(2) if $\mathscr{C}$ and $\mathscr{C}'$ are mates, then they are equal if and only if one (and hence both) of them is even and symmetric; and

(3) if k is odd and $\mathscr{C}$ is an oriented-composition of k, then $\mathscr{C}$ and $\mathscr{C}'$ are distinct. (This is a simple corollary of (1) and (2) above.)

We can now relate these notions to the previous section of this paper. Let S be a canonical RI-invariant $2k$-tone row. To illustrate the main principles we take S to be the twelve-tone row given above, namely

$$S=(0,\ 1,\ 5,\ 8,\ 9,\ 11,\ 4,\ 6,\ 7,\ 10,\ 2,\ 3).$$

Each element of the generating sequence $GS(S)=(g_1,g_2,\ldots,g_{k-1})$ is a positive integer and $0<\sum_{i=1}^{k-1} g_i = s_{k-1} < 2k$; that is, $GS(S)$ is a composition of s_{k-1}. In this example, $GS(S)=(1, 4, 3, 1, 2)$ is a composition of 11. Now denote by $\overline{GS}(S)$ the *extended generating sequence* for S, defined by

$$\overline{GS}(S)=(GS(S),\ 2k-s_{k-1})=(g_1,g_2,\ldots,g_{k-1},2k-s_{k-1}).$$

Thus, $\overline{GS}(S)$ is designed to be a composition of $2k$. For example,

$$\overline{GS}(S)=(1,\ 4,\ 3,\ 1,\ 2,\ 1)$$

is a composition of 12.

Next, denote by $\{\mathscr{C}(S),\mathscr{C}'(S)\}$ the *mate-pair* of oriented-compositions of k corresponding to S, where correspondence is defined using a canonical diagram. Note that $\mathscr{C}(S)$ and $\mathscr{C}'(S)$ may not be distinct. For the above example,

$$\mathscr{C}(S)=((3,\ 1,\ 2),\ \text{L})\quad\text{and}\quad\mathscr{C}'(S)=((2,\ 1,\ 3),\ \text{R}).$$

https://doi.org/10.1017/S0334270000002101 Published online by Cambridge University Press

Let $\mathscr{D}(\mathscr{C})$ denote the canonical diagram corresponding to the mate-pair

$$\{\mathscr{C}(S), \mathscr{C}'(S)\}.$$

Then, if k is odd, it follows that $\mathscr{C}(S)$ and $\mathscr{C}'(S)$ are distinct, and there are k different canonical RI-invariant $2k$-tone rows having the same canonical diagram $\mathscr{D}(\mathscr{C})$. On the other hand, if k is even, then the number of different canonical RI-invariant $2k$-tone rows having the same canonical diagram $\mathscr{D}(\mathscr{C})$ is $k/2$ if $\mathscr{C}(S)=\mathscr{C}'(S)$, and is k otherwise. Thus, in our example, there are six different canonical RI-invariant twelve-tone rows having the same canonical diagram derived from S above. Furthermore, since the number of oriented-compositions of k is 2^k (of which $2e_k$ are even and symmetric), it follows that the number of mate-pairs $\{\mathscr{C}(S), \mathscr{C}'(S)\}$ is $2e_k+\frac{1}{2}(2^k-2e_k)=2^{k-1}+e_k$, and for $2e_k$ of these, $\mathscr{C}$ is even and symmetric. Collecting together these remarks, we can state the following result.

THEOREM 1. *There are $k2^{k-1}$ canonical RI-invariant $2k$-tone rows.*

This follows from the computation that the number, b_k, of such rows is

$$b_k=(k/2)(2e_k)+k(2^{k-1}+e_k-2e_k)=k2^{k-1}.$$

Thus, for twelve-tone rows, $b_6=192$. Furthermore, the generating function of $\{b_k\}$ is $x/(1-2x)^2$.

5. The case $k=2$

There are $2^{2-1}=2$ compositions of $k=2$. They are: $C_1=(2)$ and $C_2=(1, 1)$. Each composition is its own mate (that is, is symmetric). The $2^2=4$ oriented-compositions of 2 are:

$$\mathscr{C}_1=((2), \mathrm{L}), \quad \mathscr{C}_2=((2), \mathrm{R}), \quad \mathscr{C}_3=((1, 1), \mathrm{L}) \quad \text{and} \quad \mathscr{C}_4=((1, 1), \mathrm{R}).$$

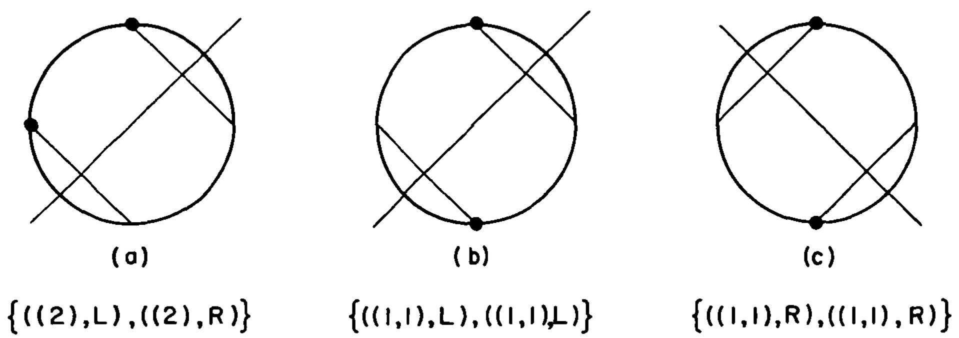

Fig. 6. Canonical diagrams for four-tone rows.

https://doi.org/10.1017/S0334270000002101 Published online by Cambridge University Press

Both $\mathscr{C}_1$ and $\mathscr{C}_2$ are odd and form a mate-pair, while $\mathscr{C}_3$ and $\mathscr{C}_4$ are even, and are each their own mates. Each of the four oriented-compositions is symmetric. The canonical RI-invariant four-tone rows are obtained from the canonical diagrams in Fig. 6. The canonical RI-invariant four-tone rows are, therefore: from Fig. 6(a) (0, 1, 2, 3) and (0, 3, 2, 1); from Fig. 6(b) (0, 2, 3, 1); and from Fig. 6(c) (0, 2, 1, 3). Note that the two four-tone rows (0, 1, 3, 2) and (0, 3, 1, 2) are *not* canonical RI-invariant. The row (0, $\underline{1, 2}$, 3) has the generating sequence $GS=(1)$ and extended generating sequence $\overline{GS}=(1, 3)$, which is a composition of 4. In this case, $e_2=2^{2/2-1}=1$. There are $(2)(2^{2-1})=4$ canonical RI-invariant four-tone rows; that is, $b_2=4$, which is the coefficient of x^2 in $x/(1-2x)^2$.

6. The case $k=6$

The oriented-compositions of $k=6$ which need to be considered are the following:

$\mathscr{C}_1=((6), R)$	$\mathscr{C}_{13}=((2, 3, 1), R)$	$\mathscr{C}_{25}=((2, 1, 2, 1), L)$
$\mathscr{C}_2=((5, 1), R)$	$\mathscr{C}_{14}=((2, 1, 3), R)$	$\mathscr{C}_{26}=((2, 1, 1, 2), R)$
$\mathscr{C}_3=((5, 1), L)$	$\mathscr{C}_{15}=((1, 3, 2), R)$	$\mathscr{C}_{27}=((2, 1, 1, 2), L)$
$\mathscr{C}_4=((4, 2), R)$	$\mathscr{C}_{16}=((1, 2, 3), R)$	$\mathscr{C}_{28}=((1, 2, 2, 1), R)$
$\mathscr{C}_5=((4, 2), L)$	$\mathscr{C}_{17}=((3, 1, 1, 1), R)$	$\mathscr{C}_{29}=((1, 2, 2, 1), L)$
$\mathscr{C}_6=((4, 1, 1), R)$	$\mathscr{C}_{18}=((3, 1, 1, 1), L)$	$\mathscr{C}_{30}=((2, 1, 1, 1, 1), R)$
$\mathscr{C}_7=((1, 4, 1), R)$	$\mathscr{C}_{19}=((1, 3, 1, 1), R)$	$\mathscr{C}_{31}=((1, 2, 1, 1, 1), R)$
$\mathscr{C}_8=((1, 1, 4), R)$	$\mathscr{C}_{20}=((1, 3, 1, 1), L)$	$\mathscr{C}_{32}=((1, 1, 2, 1, 1), R)$
$\mathscr{C}_9=((3, 3), R)$	$\mathscr{C}_{21}=((2, 2, 2), R)$	$\mathscr{C}_{33}=((1, 1, 1, 2, 1), R)$
$\mathscr{C}_{10}=((3, 3), L)$	$\mathscr{C}_{22}=((2, 2, 1, 1), R)$	$\mathscr{C}_{34}=((1, 1, 1, 1, 2), R)$
$\mathscr{C}_{11}=((3, 2, 1), R)$	$\mathscr{C}_{23}=((2, 2, 1, 1), L)$	$\mathscr{C}_{35}=((1, 1, 1, 1, 1, 1), R)$
$\mathscr{C}_{12}=((3, 1, 2), R)$	$\mathscr{C}_{24}=((2, 1, 2, 1), R)$	$\mathscr{C}_{36}=((1, 1, 1, 1, 1, 1), L)$

A complete list of all 192 canonical RI-invariant twelve-tone rows is given in the Appendix of this paper, tabulated by generating sequence and rotation number.

7. The generation of RI-invariant tone rows

In this section we present two results; the first shows how to generate RI-invariant tone rows from a given canonical RI-invariant tone row, while the second gives the exact number of such RI-invariant tone rows. Both results are proved using the correspondence previously set up with the canonical diagrams.

THEOREM 2. *If* Π *is an RI-invariant 2k-tone row, then there exists a canonical RI-invariant 2k-tone row S, an integer* $a \in \bar{Z}_{2k}$, *and a permutation* $\sigma: \bar{Z}_k \to \bar{Z}_k$, *such that*

$$\Pi = \rho_\sigma S \alpha^a,$$

where ρ_σ *is the permutation*

$$\rho_\sigma = \begin{pmatrix} 0 & 1 & 2 & \dots & k-1 & k & k+1 & \dots & 2k-2 & 2k-1 \\ \sigma_0 & \sigma_1 & \sigma_2 & \dots & \sigma_{k-1} & 2k-1-\sigma_{k-1} & 2k-1-\sigma_{k-2} & \dots & 2k-1-\sigma_1 & 2k-1-\sigma_0 \end{pmatrix}.$$

In such a case, S is said to generate Π. *Moreover, if S and S′ both generate* Π, *then S also generates S′, and S and S′ have the same canonical diagram.*

Note that the mapping $\sigma \mapsto \rho_\sigma$ induces a group of transformations on $\bar{Z}_{2k}$ naturally isomorphic to the group of all permutations on $\bar{Z}_k$. This observation gives the formulas $\rho_{\sigma_1\sigma_2} = \rho_{\sigma_1}\rho_{\sigma_2}$ and $\rho_{\sigma^{-1}} = (\rho_\sigma)^{-1}$. The essential relationship between Π and S as given in Theorem 2 can also be written element-wise in the following way:

$$\pi_i = \begin{cases} s_{\sigma_i} + a & \text{if } 0 \leqslant i \leqslant k-1, \\ s_{2k-1-\sigma_{2k-1-i}} + a & \text{if } k \leqslant i \leqslant 2k-1. \end{cases}$$

Before we proceed to prove this result, let us see how it works with an example. From the table in the Appendix to this paper, we first choose a generating sequence, say $GS = (2, 1, 4, 1, 1)$ with rotation factor $X = 7$. This now identifies the canonical RI-invariant twelve-tone row

$$S = (0, 2, 3, 7, 8, 9, 4, 5, 6, 10, 11, 1).$$

Next, we decide on a suitable permutation of $\bar{Z}_6$, say,

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 0 & 1 & 4 & 3 \end{pmatrix}.$$

The RI-invariant twelve-tone row Π can now be obtained from S and σ as follows (we take $a = 7$):

$$\pi_0 = s_{\sigma_0} + a = s_5 + 7 = 4; \qquad \pi_1 = s_{\sigma_1} + a = s_2 + 7 = 10;$$
$$\pi_2 = s_{\sigma_2} + a = s_0 + 7 = 7; \qquad \pi_3 = s_{\sigma_3} + a = s_1 + 7 = 9;$$
$$\pi_4 = s_{\sigma_4} + a = s_4 + 7 = 3; \qquad \pi_5 = s_{\sigma_5} + a = s_3 + 7 = 2;$$
$$\pi_6 = s_{11-\sigma_5} + a = s_8 + 7 = 1; \qquad \pi_7 = s_{11-\sigma_4} + a = s_7 + 7 = 0;$$
$$\pi_8 = s_{11-\sigma_3} + a = s_{10} + 7 = 6; \qquad \pi_9 = s_{11-\sigma_2} + a = s_{11} + 7 = 8;$$
$$\pi_{10} = s_{11-\sigma_1} + a = s_9 + 7 = 5; \qquad \pi_{11} = s_{11-\sigma_0} + a = s_6 + 7 = 11,$$

so that the row is

$$\Pi = (4, 10, 7, 9, 3, 2, 1, 0, 6, 8, 5, 11).$$

https://doi.org/10.1017/S0334270000002101 Published online by Cambridge University Press

Clearly, different values of the integer a, different permutations σ, or both, will always result in different RI-invariant twelve-tone rows.

To prove Theorem 2, let

$$S=\rho_\Sigma\Pi\alpha^{-\pi_0}=(\rho_\sigma)^{-1}\Pi\alpha^{-\pi_0},$$

where Σ is the (unique) permutation of $\bar{Z}_k$ for which the first k elements of $S=\rho_\Sigma\Pi\alpha^{-\pi_0}$ are in increasing order, and where $\sigma=\Sigma^{-1}$. Since

$$\begin{aligned}\beta S\gamma\alpha^c &= \beta\rho_\Sigma\Pi\alpha^{-\pi_0}\gamma\alpha^c\\ &= \beta\rho_\Sigma\Pi\gamma\alpha^{c+\pi_0}\\ &= (\beta\rho_\Sigma\beta)(\beta\Pi\gamma\alpha^{\pi_{k-1}+\pi_k})\alpha^{c+\pi_0-(\pi_{k-1}+\pi_k)} \quad (\text{since } \beta^2=1)\\ &= \rho_\Sigma\Pi\alpha^{c+\pi_0-(\pi_{k-1}+\pi_k)} \quad (\text{since } \beta \text{ commutes with } \rho_\Sigma \text{ and } \Pi \text{ is RI-invariant})\\ &= \rho_\Sigma\Pi\alpha^{-\pi_0}\\ &= S,\end{aligned}$$

where $c=\pi_{k-1}+\pi_k-2\pi_0$, then S is a canonical RI-invariant $2k$-tone row, and

$$\Pi=\rho_\sigma S\alpha^{\pi_0}.$$

Suppose now that S and S' both generate Π; that is, that there exist permutations σ,σ' and integers $a,a'\in\bar{Z}_{2k}$ such that

$$\Pi=\rho_\sigma S\alpha^a=\rho_{\sigma'}S'\alpha^{a'}.$$

Then,

$$\begin{aligned}S' &= (\rho_{\sigma'})^{-1}\rho_\sigma S\alpha^{a-a'}\\ &= \rho_{(\sigma')^{-1}\sigma}S\alpha^{a-a'}\\ &= \rho_{\hat\sigma}S\alpha^{\hat a}, \quad (\hat\sigma=(\sigma')^{-1}\sigma,\ \hat a=a-a')\end{aligned}$$

so that S also generates S'. Let $\mathscr{D}$ (respectively $\mathscr{D}'$) be the canonical diagram for S (respectively S'). Since the first k elements of the tone row S' are merely the first k elements of S rearranged and transposed by $\hat a$, then $\mathscr{D}$ and $\mathscr{D}'$ can differ only in their line-systems. But the line-system for $\mathscr{D}'$ is given by the correspondences

$$s'_i=s_{\hat\sigma_i}+\hat a\leftrightarrow s_{2k-1-\hat\sigma_i}+\hat a=s'_{2k-1-i},$$

so $\mathscr{D}$ and $\mathscr{D}'$ have the same line-systems and hence $\mathscr{D}=\mathscr{D}'$. This completes the proof of the theorem.

THEOREM 3. *There are $k2^k k!$ RI-invariant $2k$-tone rows.*

To prove this theorem, note that, from Theorem 2, each RI-invariant $2k$-tone row arises from exactly one canonical diagram. Conversely, let $\mathscr{D}$ be a canonical diagram and let $\{\mathscr{C},\mathscr{C}'\}$ be the corresponding mate-pair of oriented compositions. If $\mathscr{C}\neq\mathscr{C}'$, then we may select any of the $2k$ points as the origin and choose the k

dots on the circle in any order (the remaining positional order being thus determined). In this manner, $(2k)(k!)$ distinct RI-invariant $2k$-tone rows are generated from $\mathscr{D}$. On the other hand, if $\mathscr{C}=\mathscr{C}'$, then by selecting any of the k points *on one fixed side of the axis of symmetry* as the origin and choosing the k dots on the circle in any order, one of $(k)(k!)$ distinct RI-invariant $2k$-tone rows is generated from $\mathscr{D}$. Thus, the total number, r_k, of RI-invariant $2k$-tone rows is given by

$$r_k=(2^{k-1}-e_k)(2k)(k!)+(2e_k)(k)(k!)=k2^k k!,$$

which completes the proof.

Thus, for example, $r_6=276{,}480$. Furthermore, the exponential generating function for $\{r_k\}$ is $2x/(1-2x)^2$.

8. The case $k=2$ (continued)

The RI-invariant four-tone rows are obtained from the canonical diagrams in Section 5 above. The first pair {((2), L), ((2), R)} generates the $(2)(2)(2!)=8$ RI-invariant four-tone rows (using the method of Theorem 3):

$$(0, 3, 2, 1),\quad (3, 0, 1, 2),\quad (2, 3, 0, 1),\quad (3, 2, 1, 0),$$
$$(1, 2, 3, 0),\quad (2, 1, 0, 3),\quad (0, 1, 2, 3),\quad (1, 0, 3, 2).$$

The second pair {((1, 1), L), ((1, 1), L)} generates the $(1)(2)(2!)=4$ RI-invariant four-tone rows:

$$(0, 2, 3, 1),\quad (2, 0, 1, 3),\quad (1, 3, 0, 2),\quad (3, 1, 2, 0).$$

The third pair {((1, 1), R), ((1, 1), R)} generates the $(1)(2)(2!)=4$ RI-invariant four-tone rows:

$$(0, 2, 1, 3),\quad (2, 0, 3, 1),\quad (1, 3, 2, 0),\quad (3, 1, 0, 2).$$

There are $(2)(2^2)(2!)=16$ RI-invariant four-tone rows, which is also the coefficient of $x^2/2!$ in the expansion of $2x/(1-2x)^2$. Note that the four-tone rows:

$$(0, 1, 3, 2),\quad (0, 3, 1, 2),\quad (1, 0, 2, 3),\quad (1, 2, 0, 3),$$
$$(2, 1, 0, 3),\quad (2, 3, 1, 0),\quad (3, 0, 2, 1),\quad (3, 2, 0, 1),$$

are *not* RI-invariant.

Acknowledgements

We should like to thank Professor Easley Blackwood of the Department of Music at the University of Chicago for introducing us to this subject and for his constant encouragement. As noted in the introduction to this paper, it was his enquiry that led to the research contained herein. Discussions with Allen Moy and Nicholas Kuhn were found to be extremely helpful, especially concerning the use of canonical diagrams.

Work on this paper was carried out while Alan J. Izenman was a Visiting Assistant Professor in the Department of Statistics, University of Chicago, Chicago, Illinois. Support for this research was provided in part by National Science Foundation Grants No. SOC72–05228 A04 and Nos. MCS72–04364 A04. James Fill is a National Science Foundation Fellow and Robert R. McCormick Fellow at the University of Chicago.

Appendix

A COMPLETE LIST OF GENERATING SEQUENCES FOR CANONICAL RI-INVARIANT TWELVE-TONE ROWS, TABULATED BY ROTATION FACTOR X.

A typical entry in the table is 11213(4), for which $X=7$. The first five numbers represent the generating sequence, $GS(S)=(1, 1, 2, 1, 3)$, of some canonical RI-invariant twelve-tone row S. The generating sequence together with the accompanying rotation factor X will yield the interval sequence

$$IS(S)=(1, 1, 2, 1, 3, 7, 3, 1, 2, 1, 1),$$

and the corresponding canonical RI-invariant twelve-tone row

$$S=(0, 1, 2, 4, 5, 8, 3, 6, 7, 9, 10, 11).$$

The extended generating sequence is given by all six numbers of the tabulated entry, namely $\overline{GS}(S)=(1, 1, 2, 1, 3, 4)$.

	$X=1$	$X=3$	$X=5$	$X=7$	$X=9$	$X=11$
1	11111(7)	71111(1)	17111(1)	11711(1)	11171(1)	11117(1)
2	21111(6) 11116(2)	62111(1)	16211(1)	11621(1)	11162(1)	
3		61111(2)	26111(1)	12611(1)	11261(1)	11112(6) 11126(1)
4	13111(5) 11151(3)	51311(1) 31115(1)	15131(1)	11513(1)		
5			15111(3)	31511(1)	11131(5) 13151(1)	11315(1) 51113(1)

https://doi.org/10.1017/S0334270000002101 Published online by Cambridge University Press

	$X=1$	$X=3$	$X=5$	$X=7$	$X=9$	$X=11$
6	51112(2)		25111(2)	22511(1)	11122(5) 12251(1)	11225(1)
7	11125(2)	52111(2)	25211(1)	12521(1)	11252(1)	21112(5)
8	22111(5)	52211(1) 21115(2)	15221(1)	11522(1)		11152(2)
9	11411(4)	41141(1)	14114(1)	11411(4)	41141(1)	14114(1)
10	14113(2)			11321(4) 21411(3)	32141(1) 41132(1)	13214(1)
11	41121(3)	34112(1)		11213(4) 13411(2)	21341(1)	12134(4)
12	11314(2)		14211(3)	31421(1)	21131(4) 13142(1)	42113(1)
13	31211(4)	43121(1)	12114(3) 14312(1)		11431(2)	21143(1)
14	11241(3)	41311(2) 31124(1)	24131(1)	12413(1)		13112(4)
15	12311(4)	31141(2) 41231(1)	14123(1) 23114(1)			11412(3)
16	22211(4) 21142(2)	42221(1)	22114(2) 14222(1)		11422(2)	
17		42211(2) 21124(2)	24221(1)	12422(1)		22112(4) 11242(2)
18	11224(2) 42112(2)		24211(2)	22421(1)	21122(4) 12242(1)	
19		24112(2)		11222(4) 22411(2)	22241(1)	12224(1) 41122(2)
20	13131(3)	31313(1)	13131(3)	31313(1)	13131(3)	31313(1)
21	12231(3) 31312(2)	31223(1)	23131(2)	22313(1)	13122(3)	
22	32121(3) 12133(2)	33212(1)		21213(3) 13321(2)	21332(1)	
23	21321(3) 13213(2)	32132(1)		13213(2) 21321(3)	32132(1)	
24		21313(2)	13221(3)	31322(1)	22131(3)	13132(2) 32213(1)
25		33121(2)	12123(3) 23312(1)		12331(2)	21233(1) 31212(3)
26		31231(2)	12312(3) 23123(1)		31231(2)	23123(1) 12312(3)
27	22221(3)	22132(2) 32222(1)		13222(2) 22213(2)		21322(2)

	$X=1$	$X=3$	$X=5$	$X=7$	$X=9$	$X=11$
28	21232(2)	32221(2)	22123(2) 23222(1)		12322(2)	22212(3)
29	32212(2)	21223(2)	23221(2)	22322(1)	22122(3)	12232(2)
30	12223(2)	23212(2)		21222(3) 22321(2)	22232(1)	32122(2)
31	23122(2)		12222(3) 22312(2)		22231(2) 31222(2)	22223(1)
32	22222(2)	22222(2)	22222(2)	22222(2)	22222(2)	22222(2)

References

[1] Milton Babbitt, "Twelve-tone invariants as compositional determinants", *The Musical Quart.* 46 (1960), 246–259.

[2] James A. Fill and Alan J. Izenman, "Invariance properties of Schoenberg's tone row system", *J. Austral. Math. Soc. B* 21 (1979), 268–282.

[3] George Perle, *Serial composition and atonality* (University of California Press, second edition, 1968).

Department of Statistics
University of Chicago
5734 South University Avenue
Chicago
Illinois 60637
United States of America

and

Department of Applied Statistics
School of Statistics
University of Minnesota
Saint Paul
Minnesota 55108
United States of America