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MEASURE OF NON-COMPACTNESS AND INTERPOLATION METHODS ASSOCIATED TO POLYGONS

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Abstract. We establish an estimate for the measure of non-compactness of an interpolated operator acting from a J-space into a K-space. Our result refers to general Banach N-tuples. We also derive estimates for entropy numbers if some of the N-tuples reduce to a single Banach space.

0. Introduction. The investigation of the behaviour of compactness under interpolation methods for N-tuples of Banach spaces associated with polygons was started by Cobos and Peetre in [**8**]. There they studied the case when the interpolated operator acts between two K-spaces or two J-spaces. Later, Cobos, Kühn and Schonbek [**7**] continued this research by considering operators acting from a J-space into a K-space. Optimality of all these results was analyzed in [**3**].

It is natural to investigate now how far from being compact an interpolated operator can be, a question that was already considered by Edmunds and Teixeira [**12**] and by the present authors [**5**] in the case of the real method for couples, and by Nikolova [**10**] in the present context of N-tuples of Banach spaces. Nikolova derived estimates for the measure of non-compactness of an interpolated operator provided that one of the N-tuples degenerates into a single Banach space or that the image N-tuple satisfies a certain approximation condition.

We deal here with general N-tuples, without requiring any approximation hypothesis, and we establish an estimate for the measure of non-compactness when the interpolated operator T acts from a J-space into a K-space. In the special situation where one of the restrictions of T is compact, we recover the compactness result of Cobos, Kühn and Schonbek [**7**].

Our techniques are based on some ideas introduced in [**7**] that allow us to use efficiently the information known for the real interpolation method for couples. The relevant estimate in this last case was derived by the authors in [**5**].

The organization of the paper is as follows. In Section 1 we recall some basic facts on measure of non-compactness and on methods associated with polygons. Section 2 contains the estimate for the measure of non-compactness. Finally, in Section 3, we study degenerate cases when one of the N-tuples reduces to a single

Banach space. In these special cases we show estimates for entropy numbers that improve Nikolova's results mentioned before.

1. Preliminaries Let A and B be Banach spaces and $T\in\mathcal{L}(A,B)$ be a bounded linear operator acting from A into B. The n-th entropy number $e_n(T)$ of T is defined as the infimum for all $r>0$ such that there are $b_1,\dots,b_m\in B$ with $m\le 2^{n-1}$ and

$$T(\mathcal{U}_A) \subseteq \bigcup_{j=1}^{m}\{b_j + r\mathcal{U}_B\}.$$

Here $\mathcal{U}_A$ and $\mathcal{U}_B$ are the closed unit balls of A and B respectively. The measure of non-compactness $\beta(T)$ of T is defined as the infimum of all $r>0$ such that there exists a finite number of elements $b_1,\dots,b_s\in B$ so that

$$T(\mathcal{U}_A) \subseteq \bigcup_{j=1}^{s}\{b_j + r\mathcal{U}_B\}.$$

Clearly $\| T \| = e_1(T) \geq e_2(T) \geq \cdots \geq 0$, and $e_n(T)\to\beta(T)$ as $n\to\infty$. Also $\beta(T)=0$ if and only if T is compact. We refer to [**2**], [**9**] and [**11**] for others properties of these notions.

Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon in the affine plane R^2 with vertices $P_j=(x_j, y_j)$, $(j=1,\dots, N)$. Let $\overline{A} = \{A_1, \dots, A_N\}$ be a Banach N-tuple, that is to say, a family of N Banach spaces A_j all of them continuously embedded in a common linear Hausdorff space. In what follows, it will be useful to imagine each space of the N-tuple $\overline{A}$ as sitting on the vertex P_j.

By means of the polygon Π, we define the following family of norms in the sum $\Sigma(\overline{A}) = A_1 + \cdots + A_N$:

$$K(t, s; a) = K(t, s; a; \overline{A}) = \inf\left\{ \sum_{j=1}^{N} t^{x_j} s^{y_j} \| a_j \|_{A_j} : a = \sum_{j=1}^{N} a_j,\, a_j \in A_j \right\} \quad t, s > 0.$$

Similarly, in $\Delta(\overline{A})=A_1\cap\cdots\cap A_N$, we consider the family of norms

$$J(t, s; a) = J(t, s; a; \overline{A}) = \max_{1\le j\le N}\left\{ t^{x_j} s^{y_j} \| a \|_{A_j} \right\}.$$

Given (α,β) in the interior of Π, $(\alpha,\beta)\in \operatorname{Int}\Pi$, and $1\le q\le\infty$, the K-interpolation space $\overline{A}_{(\alpha,\beta),q;K}$ is formed by all elements $a\in\Sigma(\overline{A})$ for which the norm

$$\| a \|_{(\alpha,\beta),q;K} = \left(\sum_{(m,n)\in Z^2} \left(2^{-\alpha m-\beta n} K(2^m, 2^n; a)\right)^q \right)^{\frac{1}{q}}$$

is finite (the sum should be replaced by the supremum if $q=\infty$).

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The J-interpolation space is formed by all elements $a\in\Sigma(\overline{A})$ which can be represented as

$$a = \sum_{(m,n)\in Z^2} u_{m,n} \quad (\text{convergence in } \Sigma(\overline{A}))$$

with $(u_{m,n})\subset\Delta(\overline{A})$ and

$$\left(\sum_{(m,n)\in Z^2} (2^{-\alpha m-\beta n} J(2^m, 2^n; u_{m,n}))^q\right)^{\frac{1}{q}} < \infty.$$

The norm in $\overline{A}_{(\alpha,\beta),q;J}$ is

$$\| a \|_{(\alpha,\beta),q;J} = \inf\left\{\left(\sum_{(m,n)\in Z^2} (2^{-\alpha m-\beta n} J(2^m, 2^n; u_{m,n}))^q\right)^{\frac{1}{q}}\right\}$$

where the infimum is taken over all representations $(u_{m,n})$ as above. It is possible to give continuous characterizations for the spaces $\overline{A}_{(\alpha,\beta),q;K}$ and $\overline{A}_{(\alpha,\beta),q;J}$ using integrals instead of sums, but they will not be required here (see [**8**] for more details).

Note that the real interpolation space $(A_0, A_1)_{\theta,q}$ can be described by a similar scheme, but replacing the polygon Π by the segment [0,1], the N-tuple by a couple (A_0, A_1) and (α, β) by a point $\theta\in(0,1)$. In the case of couples, it is well known that J- and K-spaces coincide with equivalence of norms, i.e.

$$(A_0, A_1)_{\theta,q;K} = (A_0, A_1)_{\theta,q;J} = (A_0, A_1)_{\theta,q}$$

(see [**1**] or [**13**]). However, working with N-tuples ($N\geq 3$), K- and J-spaces do not agree in general. We only have the continuous inclusion $\overline{A}_{(\alpha,\beta),q;J}\hookrightarrow\overline{A}_{(\alpha,\beta),q;K}$ (see [**8**, Theorem 1.3]).

Let $\overline{B}=\{B_1,\dots,B_n\}$ be another Banach N-tuple, which we also think of as sitting on the vertices of another copy of Π. By $T : \overline{A}\to\overline{B}$ we mean a bounded linear operator from $\Sigma(\overline{A})$ into $\Sigma(\overline{B})$ whose restriction to each A_j defines a bounded operator from A_j into B_j, $j=1,\dots,N$. We denote the norm of $T : A_j\to B_j$ by $\|T\|_j$.

It is not hard to check that if $T : \overline{A}\to\overline{B}$, then the restriction of T to $\overline{A}_{(\alpha,\beta),q;K}$ gives a bounded operator

$$T : \overline{A}_{(\alpha,\beta),q;K} \to \overline{B}_{(\alpha,\beta),q;K}.$$

According to [**6**, Theorem 1.9], its norm can be estimated by

$$\| T\|_{(\alpha,\beta),q;K} = \| T : \overline{A}_{(\alpha,\beta),q;K} \to \overline{B}_{(\alpha,\beta),q;K} \| \leq C \max_{\{i,j,k\}\in\mathcal{P}_{(\alpha,\beta)}} \{\| T\|_i^{c_i} \| T\|_j^{c_j} \| T\|_k^{c_k}\}. \quad (1)$$

Here C is a constant that depends only on (α,β) and Π, $\mathcal{P}_{(\alpha,\beta)}$ stands for the collection of all triples $\{i,j,k\}$ such that the point (α,β) belongs to the interior of the triangle $\overline{P_iP_jP_k}$ and (c_i,c_j,c_k) are the barycentric coordinates of (α,β) with respect to the vertices $\{P_i, P_j, P_k\}$. A similar estimate holds for the restriction of T to the J-spaces. If we consider instead the operator T acting from a J-space into a K-space, then it was shown in [**6**, Theorem 3.2] that

$$\| T : \overline{A}_{(\alpha,\beta),q;J} \to \overline{B}_{(\alpha,\beta),q;K} \| \leq C \prod_{j=1}^{N} \| T \|_j^{\theta_j} \tag{2}$$

where $\overline{\theta} = (\theta_1, \ldots, \theta_N)$ are some barycentric coordinates of (α, β) with respect to the vertices $P_1, \ldots, P_N$ of Π (i.e. $0 < \theta_1, \ldots, \theta_N < 1$, $\sum_{j=1}^{N} \theta_j = 1$ and $\sum_{j=1}^{N} \theta_j P_j = (\alpha, \beta)$), and C is a constant depending only on $\overline{\theta}$.

Inequality (1) for J-spaces yields

$$\| a \|_{(\alpha,\beta),q;J} \leq C_1 \max_{\{i,j,k\} \in \mathcal{P}_{(\alpha,\beta)}} \{ \| a \|_{A_i}^{c_i} \| a \|_{A_j}^{c_j} \| a \|_{A_k}^{c_k} \}, \quad a \in \Delta(\overline{A}), \tag{3}$$

while for the K-norm it follows from (2) that

$$\| a \|_{(\alpha,\beta),q;K} \leq C_2 \prod_{j=1}^{N} \| a \|_{A_j}^{\theta_j}, \quad a \in \Delta(\overline{A}). \tag{4}$$

Given any double sequence of Banach spaces $(W_{m,n})_{(m,n)\in Z^2}$ and any sequence of non-negative numbers $(\lambda_{m,n})_{(m,n)\in Z^2}$ we write $\ell_q(\lambda_{m,n} W_{m,n})$ to designate the vector-valued ℓ_q space modelled on the $W_{m,n}$, that is to say,

$$\ell_q(\lambda_{m,n} W_{m,n}) = \Big\{ w = (w_m) : w_{m,n} \in W_{m,n} \text{ and}$$

$$\| w \|_{\ell_q(\lambda_{m,n} W_{m,n})} = \Big(\sum_{(m,n)\in Z^2} (\lambda_{m,n} \| w_{m,n} \|_{W_{m,n}})^q \Big)^{\frac{1}{q}} < \infty \Big\}.$$

2. Estimates for the measure of non-compactness.

THEOREM 2.1. *Let $\Pi = \overline{P_1 \ldots P_N}$ be a convex polygon with vertices $P_j = (x_j, y_j)$, let $(\alpha,\beta) \in \operatorname{Int} \Pi$ and $1 \leq q \leq \infty$. There exist constants $\gamma > 0$ and $0 < \tau < 1$, depending only on Π and (α,β), such that for any N-tuples, $\overline{A} = \{A_1, \ldots, A_N\}$ and $\overline{B} = \{B_1, \ldots, B_N\}$, and any operator $T : \overline{A} \to \overline{B}$ the measure of non-compactness of the interpolated operator can be estimated by*

$$\beta(T : \overline{A}_{(\alpha,\beta),q;J} \to \overline{B}_{(\alpha,\beta),q;K}) \leq \gamma \min_{1\leq i\leq N} \{\beta(T : A_i \to B_i)\}^{\tau} \max_{1\leq i\leq N} \{\| T \|_i\}^{1-\tau}.$$

Proof. As we pointed out in the Introduction, we shall use in the proof some ideas developed in [**7**] in order to use efficiently the estimate established in [**5**] for the real method.

First of all, by [**7**, Remark 4.1], we can assume without loss of generality that Π

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is such that $P_1=(0,0)$, $P_2=(1,0)$ and $P_N=(0,1)$. We can also suppose that $\beta_1 = \min_{1\leq j\leq N}\{\beta_i\}$, where $\beta_i=\beta(T: A_i\to B_i)$.

Since $(\alpha, \beta)\in \mathrm{Int}\Pi$, there exists $0<\theta<1$ with $(\alpha', \beta')=(\alpha/\theta, \beta/\theta)\in \mathrm{Int}\,\Pi$. Write $A_{i_\theta}^\theta=(A_1, A_i)_{\theta,1}$, $B_i^\theta=(B_1, B_i)_{\theta,1}$ and consider the N-tuples $\overline{A}^\theta=\{A_1^\theta, \ldots, A_N^\theta\}$, $\overline{B}^\theta=\{B_1^\theta, \ldots, B_N^\theta\}$. According to the formula we established in [**5**, Theorem 1.2], the measure of non-compactness $\tilde{\beta}_i$ of $T: A_i^\theta \to B_i^\theta$ can be estimated by

$$\tilde{\beta}_i = \beta(T: A_i^\theta \to B_i^\theta) \leq C_\theta \beta_1^{1-\theta}\beta_i^\theta.$$

On the other hand, by [**7**, Theorem 4.7], we can compare spaces generated by $\overline{A}$, $\overline{B}$ and (α, β) with those defined by $\overline{A}^\theta$, $\overline{B}^\theta$ and (α', β'). Namely, the following continuous inclusions hold:

$$\overline{A}_{(\alpha,\beta),q;J}\hookrightarrow\overline{A}^\theta_{(\alpha',\beta'),q;J}, \quad \overline{A}^\theta_{(\alpha',\beta'),q;K}\hookrightarrow\overline{A}_{(\alpha,\beta),q;K}.$$

Hence, there is a constant C depending only on (α,β) and Π such that

$$\beta(T: \overline{A}_{(\alpha,\beta),q;J} \to \overline{B}_{(\alpha,\beta),q;K}) \leq C\beta(T: \overline{A}^\theta_{(\alpha',\beta'),q;J} \to \overline{B}^\theta_{(\alpha',\beta'),q;K}). \tag{5}$$

This shows that in order to establish the theorem it suffices to work with $\beta(T: \overline{A}^\theta_{(\alpha',\beta'),q;J} \to \overline{B}^\theta_{(\alpha',\beta'),q;K})$. With this aim, we put

$$G^\theta_{m,n} = \left(\Delta(\overline{A}^\theta), J(2^m, 2^n, .\overline{A}^\theta)\right), \quad F^\theta_{m,n} = \left(\sum(\overline{A}^\theta), K(2^m, 2^n, .; \overline{A}^\theta)\right), \ (m,n)\in Z^2$$

and we shall work with vector-valued sequence spaces modelled on these Banach spaces.

Let π be the operator defined by $\pi(u_{m,n}) = \sum_{(m,n)\in Z^2} u_{m,n}$. Clearly $\pi: \ell_1(2^{-mx_i-ny_i}G^\theta_{m,n}) \to A_i^\theta$ is bounded with norm ≤ 1 for $i=1,\ldots,N$. Moreover $\pi: \ell_q(2^{-\alpha' m-\beta' n}G^\theta_{m,n}) \to \overline{A}^\theta_{(\alpha',\beta'),q;J}$ is a metric surjection.

Consider next the operator j that associates to each $b\in\Sigma(\overline{B}^\theta)$ the constant sequence $j(b)=(\ldots,b,b,b,\ldots)$. This time $j: B_i^\theta \to \ell_\infty(2^{-mx_i-ny_i}F^\theta_{m,n})$ is bounded with norm ≤ 1 for $i=1,\ldots,N$, and $j: \overline{B}^\theta_{(\alpha',\beta'),q;K} \to \ell_q(2^{-\alpha' m-\beta' n}F_{m,n})$ is a metric injection.

So we have the following diagram of bounded operators.

$$\begin{array}{ccccccc}
\ell_1(2^{-mx_1-ny_1}G^\theta_{m,n}) & \overset{\pi}{\to} & A_1^\theta & \overset{T}{\to} & B_1^\theta & \overset{j}{\to} & \ell_\infty(2^{-mx_1-ny_1}F^\theta_{m,n}) \\
\cdots & & \cdots & & \cdots & & \cdots \\
\cdots & & \cdots & & \cdots & & \cdots \\
\ell_1(2^{-mx_N-ny_N}G^\theta_{m,n}) & \overset{\pi}{\to} & A_N^\theta & \overset{T}{\to} & B_N^\theta & \overset{j}{\to} & \ell_\infty(2^{-mx_N-ny_N}F^\theta_{m,n}) \\
\hline
\ell_q(2^{-\alpha' m-\beta' n}G^\theta_{m,n}) & \overset{\pi}{\to} & \overline{A}^\theta_{(\alpha',\beta'),q;J} & \overset{T}{\to} & \overline{B}^\theta_{(\alpha',\beta'),q;K} & \overset{j}{\to} & \ell_q(2^{-\alpha' m-\beta' n}F^\theta_{m,n})
\end{array}$$

Write $\hat{\ell}_1 = \{\ell_1(2^{-mx_1-ny_1}G^\theta_{m,n}),\ldots, \ell_1(2^{-mx_N-ny_N}G^\theta_{m,n})\}$, $\hat{\ell}_\infty = \{\ell_\infty(2^{-mx_1-ny_1}G^\theta_{m,n}), \ldots, \ell_\infty(2^{-mx_N-ny_N}G^\theta_{m,n})\}$ and put $\hat{T}=jT\pi$. Using the properties mentioned above of j and

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π, we get

$$\beta(T:\overline{A}^{\theta}_{(\alpha',\beta'),q;J}\to\overline{B}^{\theta}_{(\alpha',\beta'),q;K})\leq 2\beta(\hat{T}:\ell_q(2^{-\alpha' m-\beta' n}G^{\theta}_{m,n})\to\ell_q(2^{-\alpha' m-\beta' n}F^{\theta}_{m,n})). \quad (6)$$

We write, for simplicity, $\beta(\hat{T})=\beta(\hat{T}:\ell_q(2^{-\alpha' m-\beta' n}G^{\theta}_{m,n})\to\ell_q(2^{-\alpha' m-\beta' n}F^{\theta}_{m,n}))$. In order to estimate this value let us introduce on $\hat{\ell}_1$ families of operators $\{P^{(r)}_k\}_{k=1}^{\infty}$, $r=0,1,2,3,4$ defined by $P_k{}^{(r)}((\xi_{m,n}))=(u_{m,n})$ where

$$u_{m,n}=\begin{cases}\xi_{m,n} & \text{if } (m,n)\in\Omega^{(r)}_k\\ 0 & \text{otherwise}\end{cases}$$

and where the sets $\{\Omega_k{}^{(r)}\}$ are given by

$$\begin{aligned}
\Omega^{(0)}_k&=\{(m,n)\in Z^2:|m|<k,|n|<k\},\\
\Omega^{(1)}_k&=\{(m,n)\in Z^2:m\leq -k,|n|<k\},\\
\Omega^{(2)}_k&=\{(m,n)\in Z^2:m\geq k,|n|<k\},\\
\Omega^{(3)}_k&=\{(m,n)\in Z^2:n\leq -k\},\\
\Omega^{(4)}_k&=\{(m,n)\in Z^2:n\geq k\}.
\end{aligned}$$

It is not hard to check that the following properties hold.

(I) The identity operator on $\Sigma(\hat{\ell}_1)$ can be decomposed as

$$I=\sum_{r=0}^{4}P^{(r)}_k,\quad k=1,2,\ldots$$

(II) They are uniformly bounded, i.e.

$$\| P^{(r)}_k:\ell_1(2^{-mx_i-ny_i}G^{\theta}_{m,n})\to\ell_1(2^{-mx_i-ny_i}G^{\theta}_{m,n})\|=1$$

for any $k\in$N, $0\leq r\leq 4$, $1\leq i\leq N$.

(III) For each $k\in$N, we have that

$$\begin{aligned}
P^{(1)}_k&:\ell_1(2^{-m}G^{\theta}_{m,n})\to\ell_1(G^{\theta}_{m,n}),\\
P^{(2)}_k&:\ell_1(G^{\theta}_{m,n})\to\ell_1(2^{-m}G^{\theta}_{m,n}),\\
P^{(3)}_k&:\ell_1(2^{-n}G^{\theta}_{m,n})\to\ell_1(G^{\theta}_{m,n}),\\
P^{(4)}_k&:\ell_1(G^{\theta}_{m,n})\to\ell_1(2^{-n}G^{\theta}_{m,n}),
\end{aligned}$$

and their norms are equal to 2^{-k}.

(IV) For each $k\in$N, $P_k{}^{(0)}:\Sigma(\hat{\ell}_1)\to\Delta(\hat{\ell}_1)$ is bounded.

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Since $\hat{T} = \hat{T}P_k^{(0)} + \hat{T}P_k^{(1)} + \hat{T}P_k^{(2)} + \hat{T}P_k^{(3)} + \hat{T}P_k^{(4)}$, we get

$$\beta(\hat{T}) \leq \beta(\hat{T}P_k^{(0)}) + \sum_{r=1}^{4} \| \hat{T}P_k^{(r)} \|$$

where all the operators are considered from $\ell_q(2^{-\alpha' m-\beta' n}G_{m,n}^{\theta})$ into $\ell_q(2^{-\alpha' m-\beta' n}F_{m,n}^{\theta})$. Let us estimate each of these terms. We start with $\beta(\hat{T}P_k^0)$.

Let $\ell_q^{(2k-1)^2}$ be $R^{(2k-1)^2}$ with the ℓ_q-norm. Since $\ell_q^{(2k-1)^2}$ is finite dimensional, given any $\varepsilon > 0$, there exists a finite set $\{\mu^r\}_{r=1}^{l} \subseteq \mathcal{U}_{\ell_q^{(2k-1)^2}}$ such that for any $\lambda \in \mathcal{U}_{\ell_q^{(2k-1)^2}}$

$$\min_{1\leq r\leq l}\{\| \lambda - \mu^r \|_{\ell_q^{(2k-1)^2}}\} \leq \varepsilon.$$

Given any $u = (u_{m,n}) \in \mathcal{U}_{\ell_q(2^{-\alpha' m-\beta' n}G_{m,n}^{\theta})}$, since

$$\| (2^{-\alpha' m-\beta' n}J(2^m, 2^n; u_{m,n}))_{|m|,|n|<k} \|_{\ell_q^{(2k-1)^2}} \leq \left(\sum_{(m,n)\in Z^2} (2^{-\alpha' m-\beta' n}J(2^m, 2^n; u_{m,n}))^q \right)^{1/q} \leq 1$$

we can find $r \in [1, l]$ satisfying that

$$2^{-\alpha' m-\beta' n}J(2^m, 2^n; u_{m,n}) \leq \mu_{m,n}^r + \varepsilon$$

for any m, n with $|m|, |n| < k$, where $\mu^r = (\mu_{m,n}^r)_{|m|,|n|<k}$. Hence

$$\| u_{m,n} \|_{A_i^{\theta}} \leq (\mu_{m,n}^r + \varepsilon)2^{(\alpha'-x_i)m+(\beta'-y_i)n}, \quad 1 \leq i \leq N, |m|, |n| < k.$$

According to the definition of $\tilde{\beta}_i$, if $\tilde{k}_i > \tilde{\beta}_i$, we can find a finite set of vectors $\{b^{i,\upsilon}\} \subseteq B_i^{\theta}$, $\upsilon = 1, \ldots, h_i$, $1 \leq i \leq N$, such that

$$\min_{1\leq \upsilon\leq h_i}\left\{\| T(u_{m,n}) - (\mu_{m,n}^r + \varepsilon)2^{(\alpha'-x_i)m+(\beta'-y_i)n}b^{i,\upsilon} \|_{B_i^{\theta}}\right\} \leq \tilde{k}_i(\mu_{m,n}^r + \varepsilon)2^{(\alpha'-x_i)m+(\beta'-y_i)n}, \; 1 \leq i \leq N.$$

So, for each $|m|, |n| < k$, there is a finite set $\{d_{m,n}^p\} \subseteq B_1^{\theta} \cap \cdots \cap B_N^{\theta}$ of, say, $w = w(m,n)$ vectors such that for some p

$$\| T(u_{m,n}) - d_{m,n}^p \|_{B_i^{\theta}} \leq 2\tilde{k}_i(\mu_{m,n}^r + \varepsilon)2^{(\alpha'-x_i)m+(\beta'-y_i)n}, \; 1 \leq i \leq N.$$

Let

$$\mathcal{D} = \left\{ \sum_{|m|,|n|<k} d_{m,n}^p : p = p(m, n) \in [1, w(m, n)] \right\}.$$

Then $\mathcal{D}$ is a finite subset of $\overline{B}_{(\alpha',\beta'),q;K}^{\theta}$ and is such that for each $u = \sum_{(m,n)\in Z^2} u_{m,n} \in \mathcal{U}_{\ell_q(2^{-\alpha' m-\beta' n}G_{m,n}^{\theta})}$ there exists some $\sum_{|m|,|n|<k} d_{m,n}^p \in \mathcal{D}$ with

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$$\begin{aligned}
&K\Big(2^s, 2^t; \sum_{|m|,|n|<k} (T(u_{m,n}) - d^p_{m,n})\Big) \\
&\leq \sum_{|m|,|n|<k} K(2^s, 2^t; T(u_{m,n}) - d^p_{m,n}) \\
&\leq \sum_{|m|,|n|<k} \min_{1\leq i\leq N}\{2^{sx_i}2^{ty_i} \parallel T(u_{m,n}) - d^p_{m,n} \parallel_{B_i^\theta}\} \\
&\leq \sum_{|m|,|n|<k} 2 \min_{1\leq i\leq N}\{2^{sx_i}2^{ty_i}\tilde{k}_i(\mu^r_{m,n}+\varepsilon)2^{(\alpha'-x_i)m+(\beta'-y_i)n}\} \\
&= \sum_{(m,n)\in Z^2} 2(\tilde{\mu}^r_{m,n}+\varepsilon) \min_{1\leq i\leq N}\{2^{(s-m)x_i+(t-n)y_i+\alpha' m+\beta' n}\tilde{k}_i\} \\
&= \sum_{(m',n')\in Z^2} 2(\tilde{\mu}^r_{s-m',t-n'}+\varepsilon) \min_{1\leq i\leq N}\{2^{m'x_i+n'y_i+\alpha'(s-m')+\beta'(t-n')}\tilde{k}_i\}
\end{aligned}$$

where

$$\mu^r_{m,n} = \begin{cases} \mu^r_{m,n} & \text{if } |m|, |n| < k \\ -\varepsilon & \text{otherwise} \end{cases}.$$

Thus

$$\begin{aligned}
&\parallel T\pi P_k^{(0)}(u) - \sum_{|m|,|n|<k} d^p_{m,n} \parallel_{(\alpha',\beta'),q;K} \\
&= \left[\sum_{(s,t)\in Z^2} (2^{-\alpha' s-\beta' t}K(2^s, 2^t; \sum_{|m|,|n|<k}(T(u_{m,n}) - d^p_{m,n})))^q\right]^{1/q} \\
&\leq \left[\sum_{(s,t)\in Z^2} (2^{-\alpha' s-\beta' t} \sum_{(m',n')\in Z^2} 2(\tilde{\mu}^r_{s-m',t-n'}+\varepsilon) \min_{1\leq i\leq N}\{2^{m'x_i+n'y_i+\alpha'(s-m')+\beta'(t-n')}\tilde{k}_i\})^q\right]^{1/q} \\
&= \left[\sum_{(s,t)\in Z^2} (\sum_{(m',n')\in Z^2} 2(\tilde{\mu}^r_{s-m',t-n'}+\varepsilon) \min_{1\leq i\leq N}\{2^{m'x_i+n'y_i-\alpha' m'-\beta' n'}\tilde{k}_i\})^q\right]^{1/q} \\
&\leq 2 \sum_{(m',n')\in Z^2} \left(\sum_{(s,t)\in Z^2} (\tilde{\mu}^r_{s-m',t-n'}+\varepsilon)^q \min_{1\leq i\leq N}\{2^{m'x_i+n'y_i-\alpha' m'-\beta' n'}\tilde{k}_i\}^q\right)^{1/q} \\
&= 2 \sum_{(m',n')\in Z^2} \left[\min_{1\leq i\leq N}\{2^{m'x_i+n'y_i-\alpha' m'-\beta' n'}\tilde{k}_i\}(\sum_{\substack{-k+m'<s<k+m' \\ -k+n'<t<k+n'}} (\tilde{\mu}^r_{s-m',t-n'}+\varepsilon)^q)^{1/q}\right] \\
&\leq 2(1+\varepsilon(2k-1)^{2/q}) \sum_{(m',n')\in Z^2} 2^{-\alpha' m'-\beta' n'} \min_{1\leq i\leq N}\{2^{m'x_i+n'y_i}\tilde{k}_i\}
\end{aligned}$$

To evaluate the last series observe that since $(\alpha',\beta') \in \operatorname{Int}\Pi$, we can choose $\varepsilon_1 > 0$ such that $(\alpha',\beta') + \varepsilon_1 h \in \operatorname{Int}\Pi$ for all possible vectors $h = (\pm 1, \pm 1)$. By [**7**, Lemma 4.2], there exist positive real numbers $\{\alpha_i(h)\}_{i=1}^N$ such that

$$\sum_{i=1}^{N}\alpha_i(h)=1 \text{ and } (\alpha',\beta')+\varepsilon_1 h=\sum_{i=1}^{N}\alpha_i(h)P_i.$$

Taking into account that $\min_{1\le i\le N}\delta_i \le \prod_{i=1}^{N}\delta_i^{\nu_i}$ for $\delta_i,\nu_i>0$ with $\sum_{i=1}^{N}\nu_i=1$, we obtain

$$2^{-\alpha'm'-\beta'n'}\min_{1\le i\le N}\{2^{m'x_i+n'y_i}\tilde{k}_i\}\le 2^{-\alpha'm'-\beta'n'}\prod_{i=1}^{N}(2^{m'x_i+n'y_i}\tilde{k}_i)^{\alpha_i(h)}=2^{\varepsilon_1<(m,n),h>}\prod_{i=1}^{N}\tilde{k}_i^{\alpha_i(h)}$$

where $<\ ,\ >$ stands for the inner product of R^2.
Put $\tau_1=\min\{\alpha_i(h):1\le i\le N,\ h=(\pm1,\pm1)\}$. Then we have

$$\begin{aligned}\prod_{i=1}^{N}\tilde{k}_i^{\alpha_i(h)}&=\max_{1\le i\le N}\{\tilde{k}_i\}\prod_{i=1}^{N}\Big(\frac{\tilde{k}_i}{\max_{1\le i\le N}\{\tilde{k}_i\}}\Big)^{\alpha_i(h)}\\&\le \max_{1\le i\le N}\{\tilde{k}_i\}\left(\frac{\min_{1\le i\le N}\{\tilde{k}_i\}}{\max_{1\le i\le N}\{\tilde{k}_i\}}\right)^{\tau_1}\\&=(\min_{1\le i\le N}\{\tilde{k}_i\})^{\tau_1}(\max_{1\le i\le N}\{\tilde{k}_i\})^{1-\tau_1}.\end{aligned}$$

Taking the minimum over all $h=(\pm1,\pm1)$ we obtain

$$2^{-\alpha'm'-\beta'n'}\min_{1\le i\le N}\{2^{m'x_i+n'y_i}\tilde{k}_i\}\le 2^{-|m'|\varepsilon_1-|n'|\varepsilon_1}(\min_{1\le i\le N}\{\tilde{k}_i\})^{\tau_1}(\max_{1\le i\le N}\{\tilde{k}_i\})^{1-\tau_1}.$$

This implies that

$$\sum_{(m',n')\in Z^2}2^{-\alpha'm'-\beta'n'}\min_{1\le i\le N}\{2^{m'x_i+n'y_i}\tilde{k}_i\}\le\left(\min_{1\le i\le N}\{\tilde{k}_i\}\right)^{\tau_1}\left(\max_{1\le i\le N}\{\tilde{k}_i\}\right)^{1-\tau_1}\sum_{(m',n')\in Z^2}2^{-|m'|\varepsilon_1-|n'|\varepsilon_1},$$

and therefore,

$$\beta(\hat{T}P_k^{(0)})\le\beta(T\pi_k^{(0)})\le 2\Big(\sum_{(m',n')\in Z^2}2^{-|m'|\varepsilon_1-|n'|\varepsilon_1}\Big)\left(\min_{1\le i\le N}\{\tilde{\beta}_i\}\right)^{\tau_1}\left(\max_{1\le i\le N}\{\tilde{\beta}_i\}\right)^{1-\tau_1}.$$

Put $\gamma_1=2\Big(\sum_{(m',n')\in Z^2}2^{-|m'|\varepsilon_1-|n'|\varepsilon_1}\Big)$. Recalling that $\tilde{\beta}_i\le C_\theta\beta_1^{1-\theta}\beta_i^{\theta}$ with $\beta_1=\min_{1\le i\le N}\{\beta_i\}$, we conclude

$$\beta(\hat{T}P_k^{(0)})\le\gamma_1C_\theta\beta_1^{\tau_1}\beta_1^{(1-\theta)(1-\tau_1)}\left(\max_{1\le i\le N}\{\beta_i\}\right)^{\theta(1-\tau_1)}=\gamma_1C_\theta\beta_1^{1-\theta+\theta\tau_1}\left(\max_{1\le i\le N}\{\beta_i\}\right)^{\theta(1-\tau_1)}.$$

Next we estimate the norm of the operator

$$\hat{T}P_k^{(1)} : \ell_q(2^{-\alpha' m-\beta' n}G^\theta_{m,n}) \to \ell_q(2^{-\alpha' m-\beta' n}F^\theta_{m,n}).$$

The arguments given in [**8**, Theorem 3.1] show that

$$\ell_q(2^{-\alpha' m-\beta' n}G^\theta_{m,n}) \to (\hat{\ell}_1)_{(\alpha',\beta'),q;J}$$

$$(\hat{\ell}_\infty)_{(\alpha',\beta'),q;K} \to \ell_q(2^{-\alpha' m-\beta' n}F^\theta_{m,n})$$

with norms ≤ 1. If $\overline{\theta}=(\theta_1,\ldots,\theta_N)$ are some barycentric coordinates of (α',β') with respect to $P_1,\ldots,P_N$, it follows from (2) and (I) that

$$\left\|\hat{T}P_k^{(1)}\right\| \leq \left\|\hat{T}P_k^{(1)}\right\|_{(\hat{\ell}_1)_{(\alpha',\beta'),q;J},(\hat{\ell}_\infty)_{(\alpha',\beta'),q,K}} \leq C\left\|\hat{T}P_k^{(1)}\right\|_2^{\theta_2} \max_{1\leq i\leq N}\left\{\left\|\hat{T}P_k^{(1)}\right\|_i\right\}^{1-\theta_2}$$
$$\leq C\left\|\hat{T}P_k^{(1)}\right\|_2^{\theta_2} \max_{1\leq i\leq N}\{\|T\|_i\}^{1-\theta_2}.$$

Further since

$$\left\|\hat{T}P_1^{(1)}\right\|_2 \geqslant \left\|\hat{T}P_2^{(1)}\right\|_2 \geqslant \cdots \geqslant 0$$

there exists $\lambda\geq 0$ such that $\| \hat{T}P_k^{(1)} \|_2 \to \lambda$ as $k \to \infty$. Choose vectors $(u^k)_{k\in N} \subset \mathcal{U}_{\ell_1(2^{-m}G^\theta_{m,n})}$ such that

$$\left\|\hat{T}P_k^{(1)}(u^k)\right\|_{\ell_\infty(2^{-m}F^\theta_{m,n})} \to \lambda \text{ as } k \to \infty.$$

By the definition of $\tilde{\beta}_2$, given any $\varepsilon > 0$, there exists a finite set $\{b_1{}^2,b_2{}^2,\ldots,b_s{}^2\}$ in B_2^θ such that

$$T\pi(\mathcal{U}_{\ell_1(2^{-m}G^\theta_{m,n})}) \subseteq \bigcup_{r=1}^{s}\{b_r^2 + (\tilde{\beta}_2 + \varepsilon)\mathcal{U}_{B_2^\theta}\}.$$

For some subsequence $(k')\subset$N and some r, say $r=1$, it follows that

$$T\pi P_{k'}^{(1)}(u^{k'}) \in \{b_1^2 + (\tilde{\beta}_2 + \varepsilon)\mathcal{U}_{B_2^\theta}\} \text{ for all } k'.$$

Using property (III), we have that for any $m, n\in$Z

$$2^{-m}K(2^m, 2^n; b_1^2) \leq 2^{-m}\left(2^m\left\|b_1^2 - T\pi P_{k'}^{(1)}(u^{k'})\right\|_{B_2^\theta} + \left\|T\pi P_{k'}^{(1)}(u^{k'})\right\|_{B_1^\theta}\right)$$
$$\leq (\tilde{\beta}_2 + \varepsilon) + 2^{-m-k'}\|T\|_1 \to \tilde{\beta}_2 + \varepsilon \text{ as } k' \to \infty.$$

This implies

$$\|j(b_1^2)\|_{\ell_\infty(2^{-m}F^\theta_{m,n})} = \sup_{(m,n)\in Z^2}\{2^{-m}K(2^m, 2^n; b_1^2)\} \leq \tilde{\beta}_2 + \varepsilon,$$

whence

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$$\lambda = \lim_{k'\to\infty} \left|\left| \hat{T}P_{k'}^{(1)}(u^{k'}) \right|\right|_{\ell_\infty(2^{-m}F_{m,n}^{\theta})}$$
$$\leq \sup_{k'} \left[\left|\left| \hat{T}P_{k'}^{(1)}(u^{k'}) - j(b_1^2) \right|\right|_{\ell_\infty(2^{-m}F_{m,n}^{\theta})} + \left|\left| j(b_1^2) \right|\right|_{\ell_\infty(2^{-m}F_{m,n}^{\theta})} \right] \leq 2(\tilde{\beta}_2 + \varepsilon).$$

Given any $\varepsilon > 0$, there then exists $k_1 \in \mathrm{N}$ such that for all $k \geq k_1$,

$$\left|\left| \hat{T}P_k^{(1)} \right|\right|_2^{\theta_2} \leq (2\tilde{\beta}_2)^{\theta_2} + \varepsilon$$

and so

$$\left|\left| \hat{T}P_k^{(1)} \right|\right| \leq C(2\tilde{\beta}_2)^{\theta_2} \max_{1\leq i\leq N} \{||T||_i\}^{1-\theta_2} + \varepsilon \leq C2^{\theta_2}\beta_1^{(1-\theta)\theta_2}\beta_2^{\theta\theta_2} \max_{1\leq i\leq N}\{||T||_i\}^{1-\theta_2} + \varepsilon.$$

Similar arguments show that

$$\left|\left| \hat{T}P_k^{(2)} \right|\right| \leq C2^{\theta_1}\beta_1^{\theta_1} \max_{1\leq i\leq N}\{||T||_i\}^{1-\theta_1} + \varepsilon,$$
$$\left|\left| \hat{T}P_k^{(3)} \right|\right| \leq C2^{\theta_N}\beta_1^{(1-\theta)\theta_N}\beta_N^{\theta\theta_N} \max_{1\leq i\leq N}\{||T||_i\}^{1-\theta_N} + \varepsilon$$
$$\left|\left| \hat{T}P_k^{(4)} \right|\right| \leq C2^{\theta_1}\beta_1^{\theta_1} \max_{1\leq i\leq N}\{||T||_i\}^{1-\theta_1} + \varepsilon.$$

Therefore

$$\beta(\hat{T}) \leq C_\theta\gamma_1\beta_1^{1-\theta+\theta\tau_1} \max_{1\leq i\leq N}\{\beta_i\}^{\theta(1-\tau_1)} + C2^{\theta_2}\beta_1^{(1-\theta)\theta_2}\beta_2^{\theta\theta_2} \max_{1\leq i\leq N}\{\| T \|_i\}^{1-\theta_2}$$
$$+ 2C2^{\theta_1}\beta_1^{\theta_1} \max_{1\leq i\leq N}\{\| T \|_i\}^{1-\theta_1} + C2^{\theta_N}\beta_1^{(1-\theta)\theta_N}\beta_N^{\theta\theta_N} \max_{1\leq i\leq N}\{\| T \|_i\}^{1-\theta_N} + 4\varepsilon.$$

Writing $\gamma_2 = \gamma_1 C_\theta + C2^{\theta_2} + C2^{\theta_1+1} + C2^{\theta_N}$ and $\tau = \min\{1-\theta+\theta\tau_1, (1-\theta)\theta_2, \theta_1, (1-\theta)\theta_N\}$, we get

$$\beta(\hat{T}) \leq \gamma_2(\min\{\beta_i\})^\tau(\max\{\| T \|_i\})^{1-\tau}.$$

Combining this inequality with (5) and (6) we finally obtain the desired estimate

$$\beta(T : \overline{A}_{(\alpha,\beta),q;J} \to \overline{B}_{(\alpha,\beta),q;K}) \leq \gamma(\min\{\beta_i\})^\tau(\max\{\| T \|_i\})^{1-\tau}.$$

If one of the restrictions $T : A_i \to B_i$ is compact, so $\beta_i = 0$, we recover the compactness theorem of Cobos, Kühn and Schonbek (see [**7**, Theorem 4.8]).

3. Estimates for entropy numbers. When one of the N-tuples degenerates to a single Banach space, i.e. $A_1 = \cdots = A_N = A$ or $B_1 = \cdots = B_N = B$, we can improve Theorem 2.1 by estimating entropy numbers of the interpolated operator.

Proposition 3.1. *Let* $\Pi = \overline{P_1 \dots P_N}$ *be a convex polygon with vertices* P_j, *let* $(\alpha,\beta) \in \operatorname{Int} \Pi$ *and* $1 \le q \le \infty$. *For any Banach N-tuple* $\overline{A} = \{A_1, \dots, A_N\}$, *any Banach space B and any operator* $T : \overline{A} \to \overline{B}$, *we have*

(i) $e_{n_1+\cdots+n_N-N+1}(T : \overline{A}_{(\alpha,\beta),q;J} \to B) \le C_1 N e_{n_1}(T_1)^{\theta_1} \cdots e_{n_N}(T_N)^{\theta_N}$,
(ii) $e_{n_1+\cdots+n_N-N+1}(T : \overline{A}_{(\alpha,\beta),q;K} \to B) \le C_2 N \max_{\{i,j,k\} \in \mathcal{P}_{(\alpha,\beta)}} \{e_{n_i}(T_i)^{c_i} e_{n_j}(T_j)^{c_j} e_{n_k}(T_k)^{c_k}\}$.

Here $T_i = T_{|A_i}$, $i = 1,\dots,N$, $\overline{\theta} = (\theta_1,\dots,\theta_N)$ *are barycentric coordinates of* (α,β), C_1 *is a constant depending only on* $\overline{\theta}$, *and* C_2 *is another constant that depends only on* Π *and* (α,β).

Proof. For $i = 1,\dots,N$, take any $k_i > e_{n_i}(T_i)$ and consider the following norm on $\Sigma(\overline{A})$:

$$|||a||| = \inf\left\{ k_1 \| a_1 \| + \dots + k_N \| a_N \| : a = \sum_{i=1}^{N} a_i;\ a_i \in A_i \right\}.$$

Given any $a \in \overline{A}_{(\alpha,\beta),q;J}$ with $\|a\|_{(\alpha,\beta),q;J} < 1$, by the Hahn-Banach theorem, we can find a bounded functional $f \in (\Sigma(\overline{A}))^*$ such that $f(a) = |||a|||$ and $\|f\|_{A_i} \le k_i$ for $i = 1,\dots,N$. According to (2), the norm of the restriction of f to $\overline{A}_{(\alpha,\beta),q;J}$ satisfies

$$\| f \|_{(\overline{A}_{(\alpha,\beta),q;J})^*} \le C_1 k_1^{\theta_1} \cdots k_N^{\theta_N}.$$

Hence

$$|||a||| = |f(a)| \le C_1 k_1^{\theta_1} \cdots k_N^{\theta_N} \| a \|_{(\alpha,\beta),q;J} < C_1 k_1^{\theta_1} \cdots k_N^{\theta_N}.$$

It follows that there is a representation $a = \sum_{i=1}^{N} a_i$ of a with $\| a_i \|_{A_i} \le C_1 k_1^{\theta_1} \cdots k_i^{\theta_i - 1} \cdots k_N^{\theta_N}$, $1 \le i \le N$. Thus

$$\frac{a_i}{C_1 k_1^{\theta_1} \cdots k_i^{\theta_i - 1} \cdots k_N^{\theta_N}} \in \mathcal{U}_{A_i}.$$

By definition of entropy numbers, there exists $b_1^i, \dots, b_{s_i}^i$ with $s_i \le 2^{n_i - 1}$ so that

$$T(\mathcal{U}_{A_i}) \subset \bigcup_{j=1}^{s_i} \{b_j^i + k_i \mathcal{U}_B\},\ 1 \le i \le N.$$

We can then choose j_i in such a way that

$$\| T(a_i) - C k_1^{\theta_1} \cdots k_i^{\theta_i - 1} \cdots k_N^{\theta_N} b_{j_i}^i \|_B \le C_1 k_1^{\theta_1} \cdots k_N^{\theta_N},$$

and so

$$\| T(a) - (C_1 k_1^{\theta_1 - 1} \cdots k_N^{\theta_N} b_{j_1}^1 + \cdots + C_1 k_1^{\theta_1} \cdots k_N^{\theta_N - 1} b_{j_N}^N) \|_B \leqslant C_1 N k_1^{\theta_1} \cdots k_N^{\theta_N}.$$

This yields the result

$$e_{n_1+\cdots+n_N-N+1}(T : \overline{A}_{(\alpha,\beta),q;J} \to B) \le C_1 N e_{n_1}(T_1)^{\theta_1} \cdots e_{n_N}(T_N)^{\theta_N}.$$

Inequality (ii) follows from similar arguments but now using (1) to estimate the norm of the restriction of f to $\overline{A}_{(\alpha,\beta),q;K}$.

REMARK 3.2. Inequality (i) does not hold for K-spaces, as we show next by means of an example.

Let $\Pi=\{(0,0), (1,0), (0,1), (1,1)\}$ be the unit square, let $\overline{A}=\{\ell_\infty^n,\ell_\infty,\ell_\infty,\ell_\infty\}$, $B=l_\infty$ and let T be the identity operator.

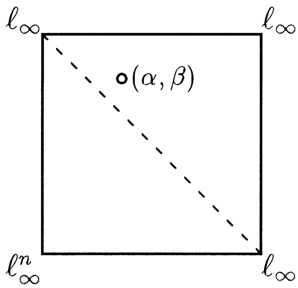

Figure 2.1

Choose (α,β) as in Fig. 2.1, i.e. in the interior of the triangle $\overline{(1,0),(0,1),(1,1)}$. Then, since ℓ_∞^n is n-dimensional, $T : \ell_\infty^n\to\ell_\infty$ is compact. But $T:(\ell_\infty^n,\ell_\infty,\ell_\infty,\ell_\infty)_{(\alpha,\beta),q;K}\to\ell_\infty$ fails to be compact, because, according to [**4** Theorem 1.5], $(\ell_\infty^n,\ell_\infty,\ell_\infty,\ell_\infty)_{(\alpha,\beta),q;K}=\ell_\infty$. In other words, $\lim_{n\to\infty} e_n(T:\overline{A}_{(\alpha,\beta),q;K}\to B)\neq 0$ although $\lim_{n\to\infty} e_n(T:A_1\to B)=0$.

Next we turn our attention to the case when the operator starts from a degenerate N-tuple. This time the stronger result corresponds to K-spaces.

PROPOSITION 3.3. *Let $\Pi=\overline{P_1\ldots P_N}$ be a convex polygon with vertices P_j, let $(\alpha,\beta)\in\mathrm{Int}\Pi$ and $1\le q\le\infty$. For any Banach N-tuple $\overline{B}=\{B_1,\ldots,B_N\}$, any Banach space A and any operator $T:A\to\overline{B}$, we have*

(i) $e_{n_1+\cdots+n_N-N+1}(T:A\to\overline{B}_{(\alpha,\beta),q;K})\le 2C_1Ne_{n_1}(T_1)^{\theta_1}\cdots e_{n_N}(T_N)^{\theta_N}$,
(ii) $e_{n_1+\cdots+n_N-N+1}(T:A\to\overline{B}_{(\alpha,\beta),q;J})\le 2C_2N\max_{\{i,j,k\}\in\mathcal{P}_{(\alpha,\beta)}}\{e_{n_i}(T_i)^{c_i}e_{n_j}(T_j)^{c_j}e_{n_k}(T_k)^{c_k}\}$.

Here $T_i=T:A\to B_i$, $i=1,\ldots,N$, $\overline{\theta}=(\theta_1,\ldots,\theta_N)$ are barycentric coordinates of (α,β), C_1 is a constant depending only on $\overline{\theta}$, and C_2 is another constant that depends only on Π and (α,β).

Proof. Given any $k_i>e_{n_i}(T_i)$, there are $\{y_{j_i}^i\}_{1\le j_i\le s_i}\subseteq B_i$ with $s_i\le 2^{n_i-1}$ and

$$T(\mathcal{U}_A)\subset\bigcup_{j_i=1}^{s_i}\{y_{j_i}^i+k_i\mathcal{U}_{B_i}\},\ 1\le i\le N.$$

Hence

$$T(\mathcal{U}_A)\subset\bigcup_{\substack{1\le j_1\le s_1\\ \cdots\cdots\\ 1\le j_N\le s_N}}\left(\bigcap_{i=1}^{N}\{y_{j_i}^i+k_i\mathcal{U}_{B_i}\}\right).$$

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Take $w_{(j_1,\ldots,j_N)} \in \bigcap_{i=1}^{N}\{y^i_{j_i} + k_i\mathcal{U}_{B_i}\}$ if the last set is non-empty. Then the number of the $w_{(j_1,\ldots,j_N)}$ is at most $2^{n_1+\cdots+n_N-N}$, and given any $a\in\mathcal{U}_A$ we can find $(j_1,\ldots,j_N)$ such that

$$\| Ta - w_{(j_1,\ldots,j_N)} \|_{(\alpha,\beta),q;K} \leq C_1 \prod_{i=1}^{N} \| Ta - w_{(j_1,\ldots,j_N)} \|^{\theta_i}_{B_i} \leq 2C_1 \prod_{i=1}^{N} k_i^{\theta_i}$$

where we have used (4) in the first inequality. This implies (i). Part (ii) follows by using (3) instead of (4).

Remark 3.4. Let $\Pi = \{(0,0),(1,0),(0,1),(1,1)\}$ be the unit square, let $A = \ell_1(n) = \{\xi = (\xi_n) : \| \xi \|_{\ell_1(n)} = \sum_{n=1}^{\infty} n|\xi_n| < \infty\}$, $\overline{B} = \{\ell_1, \ell_1(n), \ell_1(n), \ell_1(n)\}$ and let T be the identity operator. Taking (α,β) as in Remark 3.2, it follows from [**4**, Theorem 1.5], that $\overline{B}_{(\alpha,\beta),q;J} = \ell_1(n)$. Therefore $\lim_{n\to\infty} e_n(T : A \to \overline{B}_{(\alpha,\beta),q;J}) \neq 0$ although $\lim_{n\to\infty} e_n(T : A \to B_1) = 0$. Consequently, estimate (i) does not hold in general for J-spaces.

Remark 3.5. Proposition 3.1(ii) and Proposition 3.3(ii) yield Nikolova's results [**10**] mentioned in the Introduction, because $\lim_{n\to\infty} e_n(T) = \beta(T)$.

Compactness results in degenerate cases established by Cobos and Peetre in [**8**, Section 4], and Cobos, Kühn and Schonbek [**7**, Proposition 4.5 and 4.6], follow also from Propositions 3.1 and 3.3.

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