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A Note on Quaternionic Hyperbolic Ideal Triangle Groups

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Abstract. In this paper, the quaternionic hyperbolic ideal triangle groups are parametrized by a real one-parameter family $\{\phi_s : s \in \mathbb{R}\}$. The indexing parameter s is the tangent of the quaternionic angular invariant of a triple of points in $\partial\mathbf{H}^2_{\mathbb{H}}$ forming this ideal triangle. We show that if $s > \sqrt{125/3}$, then ϕ_s is not a discrete embedding, and if $s \le \sqrt{35}$, then ϕ_s is a discrete embedding.

1 Introduction

A basic problem in geometry and representation theory is the deformation problem. Suppose that $\phi_0 \colon \Gamma \to G_1$ is a discrete embedding of a finitely generated group Γ into a Lie group G_1. Suppose also that $G_1 \subset G_2$, where G_2 is a larger Lie group. The deformation problem amounts to finding and studying discrete embeddings $\phi_s \colon \Gamma \to G_2$ that extend ϕ_0.

When Γ is the fundamental group of a surface, $G_1 = \mathrm{Isom}(\mathbf{H}^2_{\mathbb{R}})$, the isometry group of the hyperbolic plane, and $G_2 = \mathrm{Isom}(\mathbf{H}^3_{\mathbb{R}})$, the isometry group of hyperbolic three space, one is dealing with the theory of quasifuchsian groups, which is quite well developed (see [16] and the references therein).

A complex hyperbolic ideal triangle group is a representation of the form $\phi_s \colon \Gamma \to \mathrm{PU}(2,1)$. Here Γ is the free product $\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$. The indexing parameter, s, is the tangent of the angular invariant of the ideal triangle formed by the three complex lines fixed by the generators. The representation ϕ_s maps the standard generators of Γ to distinct order-two complex inversions, such that any product of two distinct generators is parabolic. Modulo conjugation, there is a one-parameter family

$$\{\phi_s : s \in \mathbb{R}\}$$

of such representations.

Goldman and Parker [6] took one of the first steps on the road to a theory of complex hyperbolic quasifuchsian groups. They defined and partially classified which complex hyperbolic ideal triangle groups are discrete and faithful.

Theorem GP ([6]) *If $|s| > \sqrt{125/3}$, then ϕ_s is not a discrete embedding. If $|s| \le \sqrt{35}$, then ϕ_s is a discrete embedding.*

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Let g_s be the product of all three generators of $\phi_s(\Gamma)$ taken in any order. In [6], it was shown that g_s is loxodromic for $|s| \in [0, \sqrt{125/3})$, parabolic for $|s| = \sqrt{125/3}$, and elliptic for $|s| > \sqrt{125/3}$. If g_s is elliptic with finite order, then ϕ_s is not an embedding. If g_s is elliptic with infinite order, then ϕ_s is not discrete. Goldman and Parker conjectured that ϕ_s remains a discrete embedding for $|s| \in (35, 125/3]$.

Schwartz proved a sharp version of the Goldman–Parker conjecture.

Theorem S ([13,15]) *ϕ_s is a discrete embedding if and only if g_s is not elliptic. Also, ϕ_s is indiscrete if g_s is elliptic.*

The significance of the above two theorems was that it proposed the first complete description of a complex hyperbolic deformation problem. The results in [6,13] are seminal. Since then much research have been done into discreteness of non-ideal cases as well as fundamental domain construction and exploration of the structure of limit sets and domain of discontinuity (see [7,11,12,14] and the references therein).

Real hyperbolic geometry is extensively studied and complex hyperbolic geometry is still a central subject of recent research. The quaternionic hyperbolic space is less well understood. As interest in quaternionic hyperbolic space has grown, many of real and complex hyperbolic problems have been translated into the quaternionic arena [1–3, 8]. This paper is concerned with the deformation problem mentioned above in quaternionic hyperbolic geometry.

Let $\mathrm{PSp}(2,1)$ be the isometry group of quaternionic hyperbolic space $\mathbf{H}^2_{\mathbb{H}}$ and $\Gamma = \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ be freely generated by involutions $\varepsilon_0, \varepsilon_1, \varepsilon_2$. Let $\mathrm{Hom}(\Gamma, \mathrm{PSp}(2,1))$ be the space of homomorphism $\Gamma \to PSp(2,1)$. We consider a homomorphism in $\mathrm{Hom}(\Gamma, \mathrm{PSp}(2,1))$ that geometrically arises from a triple $u = (u_0, u_1, u_2)$ of points in $\partial\mathbf{H}^2_{\mathbb{H}}$. We mention that such a triple u is parametrized up to $\mathrm{PSp}(2,1)$-equivalence by the quaternionic Cartan angular invariant [1] $\mathbb{A}_{\mathbb{H}}(u) \in [0, \pi/2]$.

Let Q_0 be the quaternionic line $L_{u_1u_2}$ spanned by u_1 and u_2. Similarly let $Q_1 = L_{u_0u_2}$ and $Q_2 = L_{u_0u_1}$. We denote inversion in Q_j by $\tau_j \in \mathrm{PSp}(2,1)$ and define the representation ϕ_u above by

$$\phi_u\colon \varepsilon_j \to \tau_j. \tag{1.1}$$

When u_0, u_1, u_2 lie in the boundary of a quaternionic line Q, that is $\mathbb{A}_{\mathbb{H}}(u) = \pi/2$, then each $Q_j = Q$ and the representation ϕ_u takes Γ onto the cyclic group of order two generated by inversion in Q. On the other hand when u_0, u_1, u_2 lie on an $\mathbb{R}$-circle bounding an $\mathbb{R}$-plane R, that is $\mathbb{A}_{\mathbb{H}}(u) = 0$, then by a change of co-ordinates we may take R to be the subspace $\mathbf{H}^2_{\mathbb{R}} \subset \mathbf{H}^2_{\mathbb{H}}$. Then ϕ_u embeds Γ as a lattice in the subgroup $\mathrm{PO}(2,1)$ stabilizing $\mathbf{H}^2_{\mathbb{R}}$. As the triple u is determined up to $\mathrm{PSp}(2,1)$-equivalence by the quaternionic Cartan angular invariant $\mathbb{A}_{\mathbb{H}}(u)$. The resulting map

$$[0, \pi/2] \to \mathrm{Hom}(\Gamma, \mathrm{PSp}(2,1))/\,\mathrm{PSp}(2,1)$$

yields a one-parameter family of representations interpolating between these two cases.

Let

$$s = \tan(\mathbb{A}_h(u)), \quad \phi_s := \phi_u. \tag{1.2}$$

Our main result is the following theorem.

Theorem 1.1 *If $s > \sqrt{125/3}$, then ϕ_s is not a discrete embedding. If $s \le \sqrt{35}$, then ϕ_s is a discrete embedding.*

As should be apparent, our exposition and results are based on the paper [6] of Goldman and Parker. As is suggested by complex hyperbolic geometry [13, 15], we also propose the following conjecture in quaternionic hyperbolic geometry.

Conjecture 1.1 *ϕ_s is a discrete embedding if and only if g_s is not elliptic; ϕ_s is indiscrete if g_s is elliptic.*

The paper is organized as follows. Section 2 contains some necessary background material on quaternionic hyperbolic geometry. Section 3 contains some properties of bisections and Dirichlet polyhedra, which is crucial in constructing the Dirichlet polyhedra of some involved subgroups. Section 4 contains a type criterion of $\phi_s(\varepsilon_0\varepsilon_1\varepsilon_2)$ by the parameter s. Section 5 contains the proof of Theorem 1.1. Section 6 contains some remark about Conjecture 1.1.

2 Background

2.1 Quaternionic Hyperbolic Space

We briefly recall some necessary material on quaternionic hyperbolic geometry here and we refer to [1, 4, 8] for further details.

We recall that a real quaternion is of the form $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}$ where $q_i \in \mathbb{R}$ and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Let $\overline{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$ and $|q| = \sqrt{\overline{q}q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ be the conjugate and modulus of q, respectively. We define $\mathfrak{R}(q) = (q+\overline{q})/2$ and $\mathfrak{I}(q) = (q-\overline{q})/2$. Let $\mathbb{S} = \{v = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} : v_1^2 + v_2^2 + v_3^2 = 1\}$. Every unit quaternion v can be written as

$$v = \exp(\theta\mathbf{I}) := \cos\theta + \mathbf{I}\sin\theta = \cos(-\theta) + (-\mathbf{I})\sin(-\theta)$$

for some $\theta \in [0, \pi]$ and $\mathbf{I} \in \mathbb{S}$.

Let $\mathbb{H}^{2,1}$ be a copy of the vector space $\mathbb{H}^3$ equipped with the Hermitian form

$$\langle z, w\rangle = \mathbf{w}^* J\mathbf{z} = \bar{w_1}z_1 + \bar{w_2}z_2 - \bar{w_3}z_3,$$

where

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We define $\mathrm{Sp}(n,1) = \{g \in \mathrm{GL}(n+1,\mathbb{H}) : A^*JA = J\}$. Let

$$\begin{aligned} V_0 &= \left\{\mathbf{z} \in \mathbb{H}^{2,1} \setminus \{0\} : \langle \mathbf{z}, \mathbf{z}\rangle = 0\right\}, \\ V_- &= \left\{\mathbf{z} \in \mathbb{H}^{2,1} : \langle \mathbf{z}, \mathbf{z}\rangle < 0\right\}, \\ V_+ &= \left\{\mathbf{z} \in \mathbb{H}^{2,1} : \langle \mathbf{z}, \mathbf{z}\rangle > 0\right\}. \end{aligned}$$

Let $\mathbb{P}\colon \mathbb{H}^{2,1} \smallsetminus \{0\} \to \mathbb{HP}^2$ be the right projection onto $\mathbb{H}$-projective space given by

$$\mathbb{P}(z_1, z_2, z_3)^T = (z_1 z_3^{-1}, z_2 z_3^{-1})^T \in \mathbb{H}^2.$$

The ball model of the quaternionic hyperbolic 2-space is defined to be $\mathbf{H}^2_{\mathbb{H}} = \mathbb{P}(V_-)$ with the boundary $\partial\mathbf{H}^2_{\mathbb{H}} = \mathbb{P}(V_0)$. We mention that $g \in \mathrm{Sp}(2,1)$ acts on $\mathbf{H}^2_{\mathbb{H}} \cup \partial\mathbf{H}^2_{\mathbb{H}}$ as $g(z) = \mathbb{P}g\mathbb{P}^{-1}(z)$. The Bergman metric on $\mathbf{H}^2_{\mathbb{H}}$ is given by the distance formula

$$\cosh^2 \frac{\rho(z,w)}{2} = \frac{\langle \mathbf{z}, \mathbf{w}\rangle\langle \mathbf{w}, \mathbf{z}\rangle}{\langle \mathbf{z}, \mathbf{z}\rangle\langle \mathbf{w}, \mathbf{w}\rangle}, \tag{2.1}$$

where $z, w \in \mathbf{H}^2_{\mathbb{H}}$, $\mathbf{z} \in \mathbb{P}^{-1}(z)$, $\mathbf{w} \in \mathbb{P}^{-1}(w)$. The holomorphic isometry group of $\mathbf{H}^2_{\mathbb{H}}$ is $\mathrm{PSp}(2,1) = \mathrm{Sp}(2,1)/\pm I_3$.

2.2 Quaternionic Inversion

Definition 2.1 For distinct $z, w \in \overline{\mathbf{H}^2_{\mathbb{H}}}$ with lifts $\mathbf{z}$ and $\mathbf{w}$, respectively, we define the *quaternionic line* spanned by z, w as the set

$$L_{zw} = \mathbb{P}\big(\{\mathbf{x} : \mathbf{x} = \mathbf{z}\lambda + \mathbf{w}\mu,\ \lambda, \mu \in \mathbb{H}\}\big) \cap \overline{\mathbf{H}^2_{\mathbb{H}}}.$$

We need the following proposition to define the polar vector of a quaternionic line.

Proposition 2.1 *If $\mathbf{z}, \mathbf{w} \in V_- \cup V_0 \smallsetminus \{0\}$ and $\mathbb{P}(\mathbf{z}) \neq \mathbb{P}(\mathbf{w})$, then there exists a unique $\mathbf{c} \in V_+$ under projection such that $\langle \mathbf{c}, \mathbf{z}\rangle = \langle \mathbf{c}, \mathbf{w}\rangle = 0$.*

Proof Without loss of generality, let $\mathbf{z} = (z_1, z_2, 1)^T, \mathbf{w} = (w_1, w_2, 1)^T$, and $\mathbf{c} = (c_1, c_2, c_3)^T$. By $\langle \mathbf{c}, \mathbf{z}\rangle = \langle \mathbf{c}, \mathbf{w}\rangle = 0$, we have that $\langle \mathbf{c}, \mathbf{z}\rangle = \overline{z_1}c_1 + \overline{z_2}c_2 - c_3 = 0$ and $\langle \mathbf{c}, \mathbf{w}\rangle = \overline{w_1}c_1 + \overline{w_2}c_2 - c_3 = 0$. Since $\mathbf{z} \neq \mathbf{w}$, we have that $z_1 \neq w_1$ or $z_2 \neq w_2$. We obtain under projection a unique $\mathbf{c}$ given by

$$\mathbf{c} = \begin{cases} \Big((\overline{z_1} - \overline{w_1})^{-1}(\overline{w_2} - \overline{z_2}), 1, \overline{z_1}(\overline{z_1} - \overline{w_1})^{-1}(\overline{w_2} - \overline{z_2}) + \overline{z_2}\Big)^T, & \text{provided } z_1 \neq w_1, \\ \Big(1, (\overline{w_2} - \overline{z_2})^{-1}(\overline{z_1} - \overline{w_1}), \overline{w_2}(\overline{w_2} - \overline{z_2})^{-1}(\overline{z_1} - \overline{w_1}) + \overline{w_1}\Big)^T, & \text{provided } z_2 \neq w_2. \end{cases}$$

It is easy to verify that $\mathbf{c} \in V_+$. ∎

By Proposition 2.1, any vector $\mathbf{c} \in V_+$ determines the two-dimensional quaternionic subspace $\{\mathbf{z} \in \mathbb{H}^{2,1} | \langle \mathbf{c}, \mathbf{z}\rangle = 0\}$. The projection of this subspace is a quaternionic line L determined by $\mathbf{c}$. We call the vector $\mathbf{c} \in V_+$ the polar vector of quaternionic line L.

Given a quaternionic line L, there is a unique isometry $\tau_L \in \mathrm{PSp}(2,1)$ of order 2 whose fixed point set equals L. We call this isometry τ_L the quaternionic inversion in L, which is given by $\tau_L(z) = \mathbb{P}\big(-\mathbf{z} + 2\frac{\langle \mathbf{z},\mathbf{c}\rangle}{\langle \mathbf{c},\mathbf{c}\rangle}\mathbf{c}\big)$, where $\mathbf{c}$ is the polar vector of L and $\mathbf{z} \in \mathbb{P}^{-1}(z)$.

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2.3 Quaternionic Cartan Angular Invariant

Let $u = (u_0, u_1, u_2)$ be any triple of distinct points in $\overline{\mathbf{H}^2_{\mathbb{H}}}$ and $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$ be arbitrary lifts of u_0, u_1, u_2, respectively. It was shown in [2] that the number

$$\langle \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_0 \rangle \langle \mathbf{u}_2, \mathbf{u}_1 \rangle \langle \mathbf{u}_0, \mathbf{u}_2 \rangle \in \mathbb{H}, \quad \text{and} \quad \Re(\langle \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2 \rangle) \le 0.$$

Hence we can reformulate the definition of quaternionic Cartan angular invariant given by Apanasov and Kim [1] as follows.

Definition 2.2 The quaternionic Cartan angular invariant of a triple $u = (u_0, u_1, u_2)$ of distinct points in $\overline{\mathbf{H}^2_{\mathbb{H}}}$ is the angular $\mathbb{A}_{\mathbb{H}}(u)$, $0 \le \mathbb{A}_{\mathbb{H}}(u) \le \frac{\pi}{2}$, given by

$$\mathbb{A}_{\mathbb{H}}(u) = \arccos \frac{\Re\big(-\langle \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2 \rangle\big)}{|\langle \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2 \rangle|}.$$

where $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2$ are lifts of u_0, u_1, u_2, respectively.

The quaternionic Cartan angular invariant $\mathbb{A}_{\mathbb{H}}(u)$ was first used by Apanasov and Kim [1] to study the deformation of quaternionic hyperbolic manifolds. Similar to the complex case, the quaternionic Cartan angular invariant has the following properties [1].

Proposition 2.2 *Let $u = (u_0, u_1, u_2)$ and $v = (v_0, v_1, v_2)$ be two triples of distinct points in $\partial \mathbf{H}^2_{\mathbb{H}}$.*

(i) *Three points u_0, u_1, u_2 lie in the same $\mathbb{R}$-circle if and only if $\mathbb{A}_{\mathbb{H}}(u) = 0$.*
(ii) *Three points u_0, u_1, u_2 lie in the boundary of an $\mathbb{H}$-line if and only if $\mathbb{A}_{\mathbb{H}}(u) = \pi/2$.*
(iii) *Then $\mathbb{A}_{\mathbb{H}}(u) = \mathbb{A}_{\mathbb{H}}(v)$ if and only if there exists an isometry $f \in \mathrm{PSp}(2,1)$ such that $f(u_i) = v_i$, $i = 0, 1, 2$.*

For more details of quaternionic Cartan angular invariant, see [1, 2].

2.4 The Classification of Elements in $\mathrm{PSp}(2,1)$

Following Chen and Greenberg [4], a non-trivial element $g \in \mathrm{Sp}(2,1)$ is called *elliptic* if it has a fixed point in $\mathbf{H}^2_{\mathbb{H}}$, *parabolic* if it has exactly one fixed point which lies in $\partial \mathbf{H}^2_{\mathbb{H}}$, and *loxodromic* if it has exactly two fixed points which lie in $\partial \mathbf{H}^2_{\mathbb{H}}$. This classification is exhaustive and exclusive. We refine this classification further as follows.

(i) Let g be elliptic: If g has mutually distinct eigenvalues, then g is called a *regular elliptic*. If two of the eigenvalues of g are equal to each other, then it is called a *complex elliptic*. If all the eigenvalues of g are equal, then we call it a *simple elliptic*.

(ii) Suppose g is hyperbolic: The isometry g is called a *regular hyperbolic* if it has a non-real eigenvalue of norm different from 1. If all the eigenvalues of g are real numbers, then it is called *strictly hyperbolic*. If g has two and only two real eigenvalues, then it is called a *screw hyperbolic*.

(iii) Suppose g is parabolic: Let g be unipotent, *i.e.*, all eigenvalues of g are 1. If the minimal polynomial of g is $(x-1)^2$, then it is called a *vertical translation*. It is

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a *non-vertical translation* if the minimal polynomial is $(x-1)^3$. Suppose g is non-unipotent, *i.e.*, it has a non-real eigenvalue. Suppose the multiplicity of the non-real eigenvalue is 3. Then g is an *ellipto-translation* or *ellipto-parabolic* according as the minimal polynomial of g has degree 2 or 3. If g has two distinct eigenvalues, it is called a *screw parabolic*.

Write $\mathbb{H} = \mathbb{C} \oplus \mathbf{j}\mathbb{C}$. For $A \in \mathrm{Sp}(2,1)$, express $A = A_1 + \mathbf{j}A_2$, where $A_1, A_2 \in M_3(\mathbb{C})$. This gives an embedding $A \mapsto A_{\mathbb{C}}$ of $\mathrm{Sp}(2,1)$ into $\mathrm{GL}(6,\mathbb{C})$, where

$$A_{\mathbb{C}} = \begin{pmatrix} A_1 & -\overline{A_2} \\ A_2 & \overline{A_1} \end{pmatrix}.$$

Cao and Gongopadhyay obtained a classification of isometries of $\mathbf{H}^2_{\mathbb{H}}$ in [3, Theorem 1.1]. This is an analogue of [5, Theorem 6.2.4].

Proposition 2.3 *Define an embedding $\chi: A \mapsto A_{\mathbb{C}}$, where $A \in \mathrm{Sp}(2,1)$ and $A_{\mathbb{C}} \in \mathrm{GL}(6,\mathbb{C})$. The characteristic polynomial of $A_{\mathbb{C}}$ is of the form*

$$\chi_A(x) = x^6 - ax^5 + bx^4 - cx^3 + bx^2 - x + 1,$$

where a, b, c are real numbers. Define $G = 27(a-c) + 9ab - 2a^3$, $H = 3(b-3) - a^2$, and $\Delta = G^2 + 4H^3$. Then we have the following.

(i) *A acts as a regular hyperbolic if and only if $\Delta > 0$;*
(ii) *A acts as a regular elliptic if and only if $\Delta < 0$;*
(iii) *A acts as a strictly hyperbolic, or a screw hyperbolic, or a complex elliptic, or a screw parabolic if and only if $\Delta = 0$ and $G \neq 0$.*

3 Bisectors and Dirichlet Polyhedra

It is well known that there exist no totally geodesic hypersurfaces, that is, surfaces of real codimension 1, in rank 1 symmetric spaces distinct from real hyperbolic spaces. A reasonable substitute are the bisectors introduced by Mostow [10]. The bisector equidistant from two points $x, y \in \mathbf{H}^2_{\mathbb{H}}$ is defined as follows:

$$\mathcal{B}(x,y) = \left\{ z \in \mathbf{H}^2_{\mathbb{H}} : \rho(x,z) = \rho(y,z) \right\}.$$

The half-space containing the center x and bounded by the bisector $\mathcal{B}(x,y)$ is denoted by $\mathcal{B}^+(x,y) = \left\{ z \in \mathbf{H}^2_{\mathbb{H}} : \rho(x,z) < \rho(y,z) \right\}$. By use of the bisections, the usual construction of Dirichlet fundamental polyhedra for real hyperbolic space straightforwardly extends to $\mathbf{H}^2_{\mathbb{H}}$ giving rise to a locally finite fundamental domain $D_G(x)$ based at $x \in \mathbf{H}^2_{\mathbb{H}}$ for the discrete group $G \subset \mathrm{PSp}(2,1)$:

$$D_G(x) = \left\{ z \in \mathbf{H}^2_{\mathbb{H}} : \rho(x,z) < \rho(g(x),z), \forall g \in G \setminus \mathrm{id} \right\} = \bigcap_{g \in G \setminus \mathrm{id}} \mathcal{B}^+\big(x, g(x)\big).$$

The following lemma is a simple case of Klein's combination theorem (see [9] for the real hyperbolic case and [6] for the quaternionic hyperbolic case), whose extension to quaternionic hyperbolic space is straightforward.

Lemma 3.1 *Let G_1, G_2 be discrete subgroups of $\mathrm{PSp}(2,1)$ with connected fundamental domains D_1 and D_2. Let E_1 and E_2 be the interior of the complement of D_1 and D_2, respectively. Suppose that $E_1 \cap E_2 = \varnothing$ and $D_1 \cap D_2 \neq \varnothing$. Then $G = \langle G_1, G_2 \rangle$ is discrete* *and $D = D_1 \cap D_2$ is a fundamental domain for G. In particular if D_1, D_2 are Dirichlet polyhedra based at $x \in \mathbf{H}^2_{\mathbb{H}}$ for G_1 and G_2, then D is the Dirichlet polyhedron based at x for G.*

Repeating word-for-word the arguments for the complex hyperbolic space in [5,6], we obtain the following result.

Lemma 3.2 *Suppose Q is an $\mathbb{H}$-linear subspace of $\mathbf{H}^2_{\mathbb{H}}$ and let $\Pi_Q \colon \mathbf{H}^2_{\mathbb{H}} \to Q$ be an orthogonal projection onto Q endowed with hyperbolic distance ρ. Suppose that $x \in Q$ and G is a discrete group of automorphisms of $\mathbf{H}^2_{\mathbb{H}}$ leaving Q invariant. Then*

$$D_G(x) = \Pi_Q^{-1}\big(D_G(x) \cap Q\big).$$

We use the following lemma about Dirichlet polyhedra in real hyperbolic 4-space $\mathbf{H}^1_{\mathbb{H}}$ (see [6] for the case of the hyperbolic plane).

Lemma 3.3 *Let $Q = \mathbf{H}^1_{\mathbb{H}}$ and let $p_1, p_2 \in Q$ be distinct points. Let σ_i denote inversion in p_i, for $i = l, 2$ and $\Sigma = \langle \sigma_1, \sigma_2 \rangle$. Let $x \in \mathcal{B}(p_1, p_2) \subset Q$ and let $\gamma_i = \mathcal{B}\big(x, \sigma_i(x)\big)$ denote the totally geodesic hypersurface equidistant from x and $\sigma_i(x)$.*

(i) *If $\gamma_1 \cap \gamma_2 = \varnothing$, then the Dirichlet polyhedron $D_\Sigma(x)$ equals*

$$\mathcal{B}^+\big(x, \sigma_1(x)\big) \cap \mathcal{B}^+\big(x, \sigma_2(x)\big)$$

and is bounded by γ_1 and γ_2.

(ii) *Otherwise*

$$D_\Sigma(x) = \mathcal{B}^+\big(x, \sigma_1(x)\big) \cap \mathcal{B}^+\big(x, \sigma_2(x)\big) \cap \mathcal{B}^+\big(x, \sigma_1\sigma_2(x)\big) \cap \mathcal{B}^+\big(x, \sigma_1\sigma_2(x)\big)$$

and has four faces which are totally geodesic hypersurfaces

$$\gamma_1, \gamma_2, \gamma_{12} = \mathcal{B}\big(x, \sigma_1\sigma_2(x)\big) \quad \textit{and} \quad \gamma_{21} = \mathcal{B}\big(x, \sigma_2\sigma_1(x)\big).$$

Proof We remark that Figure 2 in [6, p. 77] can still be used as a schematic diagram to indicate the relative positions of the involving geometric objects.

We first observe that Σ only contains hyperbolic elements (which are integer powers of $\sigma_1\sigma_2$) and inversions (which are conjugates of σ_i by hyperbolic elements).

We denote the geodesic connecting p_1 and p_2 by α. This is the axis of $\sigma_1\sigma_2$ and so all our fixed points of inversions in Σ lie on the geodesic α. Moreover, as x is equidistant from both p_1 and p_2, the orthogonal projection of x onto α is the midpoint of p_1 and p_2. If $x \in \alpha$, then γ_1 and γ_2 are orthogonal to α and $\gamma_1 \cap \gamma_2 = \varnothing$. This is case (i).

So we suppose that $x \notin \alpha$ and we let $\beta = \mathcal{B}(p_1, p_2)$ be the totally geodesic hypersurface through x perpendicular to α. It is easy to see that $\gamma_{12} = \sigma_1(\beta)$ and $\gamma_{21} = \sigma_2(\beta)$.

Consider the inversion $\sigma \in \Sigma$. Without loss of generality suppose that p, the fixed point of σ, is closer to p_1 than to p_2 and let $\gamma = \mathcal{B}\big(x, \sigma(x)\big)$, the totally geodesic hypersurface equidistant from x and $\sigma(x)$. Since the translate length of hyperbolic elements in Σ is $k\rho(p_1, p_2)$, where k is a positive integer, the point p must lie in α satisfying $\rho(p, p_1) \geq \rho(p_1, p_2)$. This implies that the point p is in the complement of both $\overline{\mathcal{B}^+}\big(x, \sigma_1(x)\big)$ and $\overline{\mathcal{B}^+}\big(x, \sigma_1\sigma_2(x)\big)$. The angle between α and γ_1 at p_1 is less

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than the angle between α and γ at p. Since the angle between α and γ at p is acute, by convexity γ lies in the complement of $\overline{\mathcal{B}^+}\big(x, \sigma_1(x)\big) \cap \overline{\mathcal{B}^+}\big(x, \sigma_1\sigma_2(x)\big)$.

Thus the only candidates for the faces of $D_\Sigma(x)$ are $\gamma_1, \gamma_2, \gamma_{12}$ and γ_{21}. If γ_1 and γ_2 are disjoint, then β lies completely in $\mathcal{B}\big(x, \sigma_1(x)\big) \cap \mathcal{B}\big(x, \sigma_2(x)\big)$, therefore γ_{12} and γ_{21} lie outside this set. This is case (i) also. If $\gamma_1 \cap \gamma_2 = l \neq \varnothing$, then $\sigma_1(l) \subset \gamma_{12}$ and $\sigma_2(l) \subset \gamma_{21}$ so these totally geodesic hypersurfaces are in the boundary of $D_\Sigma(x)$. This is case (ii). ∎

As in the complex case [5, 6], there exist quaternionic lines playing the role of bisectors of asymptotic pairs of quaternionic lines. We record this fact in the following lemma.

Lemma 3.4 *Let Q_1, Q_2 be two asymptotic quaternionic lines. Then there exists a unique quaternionic lines Q_{12} so that inversion in Q_{12} interchanges Q_1 and Q_2. Moreover, if Q_0 is a quaternionic line asymptotic to Q_1 and Q_2, and such that $\partial Q_1 \cap \partial Q_2 \notin \partial Q_0$, then Q_0 meets Q_{12}.*

4 The Type of g_s

In this section, we will obtain a type criterion of $\phi_s(\varepsilon_0\varepsilon_1\varepsilon_2)$ by the parameter s.

Proposition 4.1 *Let $g_s = \phi_s(\varepsilon_0\varepsilon_1\varepsilon_2)$, where ϕ_s is given by (1.2). Then g_s is loxodromic for $s \in [0, \sqrt{125/3})$, parabolic for $s = \sqrt{125/3}$, and elliptic for $s > \sqrt{125/3}$.*

Proof We recall that $\phi_s = \phi_u$, which are given by (1.1) and (1.2). We lift ϕ_u to a representation $\Gamma \to \mathrm{PSp}(2,1)$, which we also denote by ϕ_u. Choose co-ordinates so that u_0, u_1, u_2 lift to the following null vectors in $\mathbb{H}^{2,1}$:

$$\widetilde{u_0} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \widetilde{u_1} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \widetilde{u_2} = \begin{pmatrix} \exp(2\mathbb{A}\mathbf{j}) \\ 0 \\ 1 \end{pmatrix}.$$

Then $\mathbb{A}_{\mathbb{H}}(u) = \mathbb{A}$ and the three quaternionic lines $Q_0 = L_{u_1u_2}$, $Q_1 = L_{u_0u_2}$, and $Q_2 = L_{u_0u_1}$ are represented by the polar vectors:

$$\mathbf{c}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{c}_1 = \begin{pmatrix} 1 \\ \exp(-2\mathbb{A}\mathbf{j}) \\ \exp(-2\mathbb{A}\mathbf{j}) \end{pmatrix}, \quad \mathbf{c}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

The corresponding quaternionic inversions are given by

$$\tau_j(z) = \mathbb{P}\Big(-\mathbf{z} + 2\frac{\langle \mathbf{z}, \mathbf{c}_j\rangle}{\langle \mathbf{c}_j, \mathbf{c}_j\rangle}\mathbf{c}_j\Big), \quad j = 0, 1, 2,$$

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where $\mathbf{z}$ is an arbitrary lift of z. These inversions are represented in $\mathrm{Sp}(2,1)$ by the following isometries:

$$\tau_0=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix},$$
$$\tau_1=\begin{pmatrix}1&2\exp(2\mathbb{A}\mathbf{j})&-2\exp(2\mathbb{A}\mathbf{j})\\2\exp(-2\mathbb{A}\mathbf{j})&1&-2\\2\exp(-2\mathbb{A}\mathbf{j})&2&-3\end{pmatrix},$$
$$\tau_2=\begin{pmatrix}1&-2&2\\-2&1&-2\\-2&2&-3\end{pmatrix}.$$

Therefore

$$\begin{aligned}\tau_0\tau_1\tau_2&=\begin{pmatrix}-1&2+2\exp(2\mathbb{A}\mathbf{j})&-2-2\exp(2\mathbb{A}\mathbf{j})\\2+2\exp(-2\mathbb{A}\mathbf{j})&-3-4\exp(-2\mathbb{A}\mathbf{j})&4+4\exp(-2\mathbb{A}\mathbf{j})\\-2-2\exp(-2\mathbb{A}\mathbf{j})&4+4\exp(-2\mathbb{A}\mathbf{j})&-5-4\exp(-2\mathbb{A}\mathbf{j})\end{pmatrix}\\&=\begin{pmatrix}-1&2+2\cos(2\mathbb{A})&-2-2\cos(2\mathbb{A})\\2+2\cos(2\mathbb{A})&-3-4\cos(2\mathbb{A})&4+4\cos(2\mathbb{A})\\-2-2\cos(2\mathbb{A})&4+4\cos(2\mathbb{A})&-5-4\cos(2\mathbb{A})\end{pmatrix}\\&\quad+\mathbf{j}\begin{pmatrix}0&2\sin(2\mathbb{A})&-2\sin(2\mathbb{A})\\-2\sin(2\mathbb{A})&4\sin(2\mathbb{A})&-4\sin(2\mathbb{A})\\2\sin(2\mathbb{A})&-4\sin(2\mathbb{A})&4\sin(2\mathbb{A})\end{pmatrix}.\end{aligned}$$

By the embedding $\chi\colon A\mapsto A_{\mathbb{C}}$ in Proposition 2.3, the corresponding matrix of $\tau_0\tau_1\tau_2$ in $\mathrm{GL}(6,\mathbb{C})$ is the following matrix:

$$\begin{pmatrix}-1&2+2\cos(2\mathbb{A})&-2-2\cos(2\mathbb{A})&0&-2\sin(2\mathbb{A})&2\sin(2\mathbb{A})\\2+2\cos(2\mathbb{A})&-3-4\cos(2\mathbb{A})&4+4\cos(2\mathbb{A})&2\sin(2\mathbb{A})&-4\sin(2\mathbb{A})&4\sin(2\mathbb{A})\\-2-2\cos(2\mathbb{A})&4+4\cos(2\mathbb{A})&-5-4\cos(2\mathbb{A})&-2\sin(2\mathbb{A})&4\sin(2\mathbb{A})&-4\sin(2\mathbb{A})\\0&2\sin(2\mathbb{A})&-2\sin(2\mathbb{A})&-1&2+2\cos(2\mathbb{A})&-2-2\cos(2\mathbb{A})\\-2\sin(2\mathbb{A})&4\sin(2\mathbb{A})&-4\sin(2\mathbb{A})&2+2\cos(2\mathbb{A})&-3-4\cos(2\mathbb{A})&4+4\cos(2\mathbb{A})\\2\sin(2\mathbb{A})&-4\sin(2\mathbb{A})&4\sin(2\mathbb{A})&-2-2\cos(2\mathbb{A})&4+4\cos(2\mathbb{A})&-5-4\cos(2\mathbb{A})\end{pmatrix}.$$

The characteristic polynomial of the above matrix is

$$\begin{aligned}\chi_{\tau_0\tau_1\tau_2}(x)=x^6&+\big(16\cos(2\mathbb{A})+18\big)x^5+\big(128\cos(2\mathbb{A})+127\big)x^4-4\big(64\cos^2(2\mathbb{A})\\&+72\cos(2\mathbb{A})+9\big)x^3+\big(128\cos(2\mathbb{A})+127\big)x^2\\&+\big(16\cos(2\mathbb{A})+18\big)x+1.\end{aligned}$$

By Proposition 2.3 we have

$$\begin{gathered}a=-16\cos(2\mathbb{A})-18,\ b=128\cos(2\mathbb{A})+127,\ c=4\big(64\cos^2(2\mathbb{A})+72\cos(2\mathbb{A})+9\big),\\G=8192\cos(2\mathbb{A})^3+2304\cos(2\mathbb{A})^2-16128\cos(2\mathbb{A})-10368,\\H=-256\cos^2(2\mathbb{A})-192\cos(2\mathbb{A})+48\end{gathered}$$

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and

$$\begin{aligned}\Delta &= -113246208\cos^5(2\mathbb{A}) - 334430208\cos^4(2\mathbb{A}) - 215875584\cos^3(2\mathbb{A})\\ &\qquad + 226492416\cos^2(2\mathbb{A}) + 329121792\cos(2\mathbb{A}) + 107937792\\ &= 113246208\big(\cos(2\mathbb{A})+1\big)^3\big(1-\cos(2\mathbb{A})\big)\Big(\cos(2\mathbb{A})+\frac{61}{64}\Big).\end{aligned}$$

It is obvious that if $\mathbb{A} = 0$, then $\tau_0\tau_1\tau_2$ is loxodromic, and if $\mathbb{A} = \pi/2$, then $\tau_0\tau_1\tau_2$ is elliptic. Henceforth we assume that $\mathbb{A} \neq 0, \mathbb{A} \neq \pi/2$.

Therefore $\Delta > 0$ if and only if $\cos(2\mathbb{A}) > -\frac{61}{64}$. That is, $\frac{\sqrt{6}}{16} < \cos\mathbb{A} < 1$ and

$$0 < s = \tan(\mathbb{A}) < \sqrt{125/3}.$$

Similarly, $\Delta < 0$ if and only if $s > \sqrt{125/3}$.

If $\Delta = 0$, then $\cos(2\mathbb{A}) = -\frac{61}{64}$, $s = \sqrt{125/3}$, and $G = \frac{125}{32}$. In this case

$$\tau_0\tau_1\tau_2 = \begin{pmatrix} -1 & \frac{3+5\sqrt{15}\mathbf{j}}{32} & \frac{-3-5\sqrt{15}\mathbf{j}}{32}\\ \frac{3-5\sqrt{15}\mathbf{j}}{32} & \frac{13+5\sqrt{15}\mathbf{j}}{16} & \frac{3-5\sqrt{15}\mathbf{j}}{16}\\ \frac{-3+5\sqrt{15}\mathbf{j}}{32} & \frac{3-5\sqrt{15}\mathbf{j}}{16} & \frac{-19+5\sqrt{15}\mathbf{j}}{16}\end{pmatrix},$$

which is a screw parabolic.

By Proposition 2.3, g_s is loxodromic for $s \in [0, \sqrt{125/3})$, parabolic for $s = \sqrt{125/3}$, and elliptic for $s > \sqrt{125/3}$. ∎

5 The Proof of Theorem 1.1

We are now ready to prove our main theorem. We only need to prove the following theorem, which is a reformulation of Theorem 1.1.

Theorem 5.1 *Let $u = (u_0, u_1, u_2)$ be a triple of points in $\partial\mathbf{H}^2_{\mathbb{H}}$ and ϕ_u be given by (1.1).*

(i) *If ϕ_u is a discrete embedding, then $0 \leq \mathbb{A}_{\mathbb{H}}(u) \leq \arccos\frac{\sqrt{6}}{16} \approx 81.1938°$.*

(ii) *Conversely ϕ_u is a discrete embedding when $0 \leq \mathbb{A}_{\mathbb{H}}(u) \leq \arccos\frac{1}{6} \approx 80.4059°$.*

Proof We will use the same symbols as those of Section 4 for our proof. Suppose that ϕ_u is a discrete embedding. As we know $\tau_0\tau_1\tau_2$ has infinite order in Γ so that $\tau_0\tau_1\tau_2$ can not be a regular elliptic element. By Proposition 4.1, we have $\frac{\sqrt{6}}{16} \leq \cos\mathbb{A} \leq 1$. Hence $0 \leq \mathbb{A}_{\mathbb{H}}(u) \leq \arccos\frac{\sqrt{6}}{16} \approx 81.1938°$. This proves assertion (i).

In order to prove assertion (ii), we will show that the homomorphism $\phi_u : \Gamma \to \mathrm{Sp}(2,1)$ given by (1.1) is faithful and discrete provided that $\cos(\mathbb{A}_{\mathbb{H}}(u)) \leq \frac{1}{6}$.

Consider the quaternionic lines $Q_3 = \tau_0 Q_1$ and $Q_4 = \tau_0 Q_2$ with corresponding inversions $\tau_3 = \tau_0\tau_1\tau_0$, and $\tau_4 = \tau_0\tau_2\tau_0$. Since $\langle\tau_1, \tau_2, \tau_3, \tau_4\rangle$ is a subgroup of index two in $\langle\tau_0, \tau_1, \tau_2\rangle$, we only need to prove that $\langle\tau_1, \tau_2, \tau_3, \tau_4\rangle$ is discrete. We prove this by constructing a fundamental polyhedron for the group $\langle\tau_1, \tau_2, \tau_3, \tau_4\rangle$.

Let Q_{12} (resp. Q_{34}) be the bisector (in the sense of Lemma 3.4) of Q_1 and Q_2 (resp. Q_3 and Q_4). Note that $\langle \mathbf{c}_1, \mathbf{c}_1 \rangle = \langle \mathbf{c}_2, \mathbf{c}_2 \rangle = 1$. The polar vector of Q_{12} is

$$\mathbf{c}_{12} = \mathbf{c}_1 + \mathbf{c}_2 = \begin{pmatrix} 2 \\ -1 + \exp(-2\mathbb{A}\mathbf{j}) \\ -1 + \exp(-2\mathbb{A}\mathbf{j}) \end{pmatrix}$$

We shall take the point $x = Q_0 \cap Q_{12}$ as the center of our Dirichlet polyhedron. Now by symmetry $x = \tau_0(x) = Q_0 \cap Q_{34}$ and so $x = Q_0 \cap Q_{12} \cap Q_{34}$. By computation, a vector representing x is

$$\widetilde{x'} = \begin{pmatrix} \frac{-1+\exp(2\mathbb{A}\mathbf{j})}{2} \\ 0 \\ 1 \end{pmatrix}$$

Let $x_j = \tau_j(x)$, $j = 1, 2, 3, 4$. We claim that the three point x, x_1, and x_4 lie on a common quaternionic line, which we call Q_{14}. Similarly x, $x_2 = \tau_0(x_4)$, $x_3 = \tau_0(x_1)$ lie on a common quaternionic line $Q_{23} = \tau_0(Q_{14})$.

In fact, by applying the following isometry

$$h = \begin{pmatrix} \frac{-\exp(-\mathbb{A}\mathbf{j})}{\sqrt{2}\cos\mathbb{A}} & -\frac{1}{\sqrt{2}} & \frac{\tan\mathbb{A}\mathbf{j}}{\sqrt{2}} \\ \frac{\exp(-\mathbb{A}\mathbf{j})}{\sqrt{2}\cos\mathbb{A}} & -\frac{1}{\sqrt{2}} & -\frac{\tan\mathbb{A}\mathbf{j}}{\sqrt{2}} \\ \frac{1-\exp(-2\mathbb{A}\mathbf{j})}{2\cos\mathbb{A}} & 0 & \frac{1}{\cos\mathbb{A}} \end{pmatrix} \in \mathrm{Sp}(2,1)$$

to $\widetilde{x'}$ and its images under inversions $\tau_1, \tau_2, \tau_3, \tau_4$, we obtain vectors representing the points x, x_1, x_2, x_3, and x_4

$$\widetilde{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \widetilde{x_1} = \begin{pmatrix} -2\sqrt{2}\exp(-\mathbb{A}\mathbf{j}) \\ 0 \\ 3 \end{pmatrix}, \quad \widetilde{x_2} = \begin{pmatrix} 0 \\ -2\sqrt{2}\exp(\mathbb{A}\mathbf{j}) \\ 3 \end{pmatrix},$$
$$\widetilde{x_3} = \begin{pmatrix} 0 \\ 2\sqrt{2}\exp(-\mathbb{A}\mathbf{j}) \\ 3 \end{pmatrix}, \quad \widetilde{x_4} = \begin{pmatrix} 2\sqrt{2}\exp(\mathbb{A}\mathbf{j}) \\ 0 \\ 3 \end{pmatrix}.$$

Hence Q_{14} is the quaternionic line $\left\{ (z_1, 0)^T : |z_1| \le 1 \right\} = \mathbf{H}^1_{\mathbb{H}} \times \{0\}$ and Q_{23} is the quaternionic line $\left\{ (0, w_2)^T : |w_2| \le 1 \right\} = \{0\} \times \mathbf{H}^1_{\mathbb{H}}$.

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The corresponding inversions $h\tau_i h^{-1}$, $i = 0,1,2,3,4$, which we still denote them by the same symbols τ_i, are the following isometries:

$$\tau_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 3 & 0 & 2\sqrt{2}\exp(-\mathbb{A}\mathbf{j}) \\ 0 & -1 & 0 \\ -2\sqrt{2}\exp(\mathbb{A}\mathbf{j}) & 0 & -3 \end{pmatrix},$$
$$\tau_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 2\sqrt{2}\exp(\mathbb{A}\mathbf{j}) \\ 0 & -2\sqrt{2}\exp(-\mathbb{A}\mathbf{j}) & -3 \end{pmatrix},$$
$$\tau_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & -2\sqrt{2}\exp(-\mathbb{A}\mathbf{j}) \\ 0 & 2\sqrt{2}\exp(\mathbb{A}\mathbf{j}) & -3 \end{pmatrix},$$
$$\tau_4 = \begin{pmatrix} 3 & 0 & -2\sqrt{2}\exp(\mathbb{A}\mathbf{j}) \\ 0 & -1 & 0 \\ 2\sqrt{2}\exp(-\mathbb{A}\mathbf{j}) & 0 & -3 \end{pmatrix}.$$

Let $x_{14} = \tau_1\tau_4(x)$, $x_{41} = \tau_4\tau_1(x)$, $x_{23} = \tau_2\tau_3(x)$ and $x_{32} = \tau_3\tau_2(x)$. Then we obtain vectors representing the points x_{14}, x_{41}, x_{23} and x_{32}

$$\widetilde{x_{14}} = \begin{pmatrix} -12\sqrt{2}\cos\mathbb{A} \\ 0 \\ 9+8\exp(2\mathbb{A}\mathbf{j}) \end{pmatrix}, \quad \widetilde{x_{41}} = \begin{pmatrix} 12\sqrt{2}\cos\mathbb{A} \\ 0 \\ 9+8\exp(-2\mathbb{A}\mathbf{j}) \end{pmatrix},$$

and

$$\widetilde{x_{23}} = \begin{pmatrix} 0 \\ -12\sqrt{2}\cos\mathbb{A} \\ 9+8\exp(-2\mathbb{A}\mathbf{j}) \end{pmatrix}, \quad \widetilde{x_{32}} = \begin{pmatrix} 0 \\ 12\sqrt{2}\cos\mathbb{A} \\ 9+8\exp(2\mathbb{A}\mathbf{j}) \end{pmatrix}.$$

Since $x, x_1, x_4 \in Q_{14}$, the group $G_{14} = \langle\tau_1,\tau_4\rangle$ leaves invariant the quaternionic line Q_{14}. Similarly the group $G_{23} = \tau_0 G_{14}\tau_0 = \langle\tau_2,\tau_3\rangle$ leaves the quaternionic line Q_{23} invariant. By Lemma 3.2, the Dirichlet polyhedron D_{14} based at x for the action of G_{14} on $\mathbf{H}^2_{\mathbb{H}}$ is $(\mathbb{H}\times D_1)\cap\mathbf{H}^2_{\mathbb{H}}$, where D_1 is the Dirichlet polyhedron based at x for the action of G_{14} on $Q_{14} = \mathbf{H}^1_{\mathbb{H}}\times\{0\}$. Also $D_{23} = (D_2\times\mathbb{H})\cap\mathbf{H}^2_{\mathbb{H}}$, where D_2 is the polyhedron now regarded as a subset of $Q_{23} = \{0\}\times\mathbf{H}^1_{\mathbb{H}}$.

Let E_{14} (resp. E_{23}) be the interior of the complement of D_{14} in $\mathbf{H}^2_{\mathbb{H}}$ (resp. D_{23}). It follows from Lemma 3.1 that to show $\langle\tau_1,\tau_2,\tau_3,\tau_4\rangle$ is discrete, it suffices to show that $E_{14}\cap E_{23} = \varnothing$.

Let $(z_1,z_2)^T \in E_{14}$ and $(w_1,w_2)^T\in E_{23}$. Then the following inequalities

$$|z_1| \geq 1/\sqrt{2} \quad\text{and}\quad |w_2|\geq 1/\sqrt{2}$$

imply that $E_{14}\cap E_{23} = \varnothing$. Hence we only need to show that D_1 and D_2 contain a ball with center 0 and radius $1/\sqrt{2}$ in $\mathbf{H}^1_{\mathbb{H}}$. By Lemma 3.3, it suffices to check that the totally geodesic hypersurfaces

$$\gamma_1 = \mathcal{B}\big(x,\tau_1(x)\big),\ \gamma_4 = \mathcal{B}\big(x,\tau_4(x)\big),\ \gamma_{14} = \mathcal{B}\big(x,\tau_1\tau_4(x)\big),\ \gamma_{41} = \mathcal{B}\big(x,\tau_4\tau_1(x)\big)$$

and

$$\gamma_2 = \mathcal{B}\big(x,\tau_2(x)\big),\ \gamma_3 = \mathcal{B}\big(x,\tau_3(x)\big),\ \gamma_{23} = \mathcal{B}\big(x,\tau_2\tau_3(x)\big),\ \gamma_{32} = \mathcal{B}\big(x,\tau_3\tau_2(x)\big)$$

all lie outside the interior of this ball.

Let $z_1 = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in D_1$. By the distance formula (2.1), we obtain the expressions for $\gamma_1, \gamma_4, \gamma_{14}$ and γ_{41} as follows:

$$\gamma_1 = \Big\{ z = (z_1, 0)^T \in Q_{14} : \big(q_0 + \tfrac{3\sqrt{2}\cos\mathbb{A}}{4} \big)^2 + q_1^2 + \big(q_2 - \tfrac{3\sqrt{2}\sin\mathbb{A}}{4} \big)^2 + q_3^2 = \big(\tfrac{\sqrt{2}}{4} \big)^2 \Big\},$$

$$\gamma_4 = \Big\{ z = (z_1, 0)^T \in Q_{14} : \big(q_0 - \tfrac{3\sqrt{2}\cos\mathbb{A}}{4} \big)^2 + q_1^2 + \big(q_2 - \tfrac{3\sqrt{2}\sin\mathbb{A}}{4} \big)^2 + q_3^2 = \big(\tfrac{\sqrt{2}}{4} \big)^2 \Big\},$$

$$\gamma_{14} = \Big\{ z = (z_1, 0)^T \in Q_{14} : \big(q_0 + \tfrac{9+8\cos 2\mathbb{A}}{12\sqrt{2}\cos\mathbb{A}} \big)^2 + q_1^2 + \big(q_2 + \tfrac{9+8\sin 2\mathbb{A}}{12\sqrt{2}\cos\mathbb{A}} \big)^2 + q_3^2 = \big(\tfrac{1}{12\sqrt{2}\cos\mathbb{A}} \big)^2 \Big\},$$

$$\gamma_{41} = \Big\{ z = (z_1, 0)^T \in Q_{14} : \big(q_0 - \tfrac{9+8\cos 2\mathbb{A}}{12\sqrt{2}\cos\mathbb{A}} \big)^2 + q_1^2 + \big(q_2 - \tfrac{9+8\sin 2\mathbb{A}}{12\sqrt{2}\cos\mathbb{A}} \big)^2 + q_3^2 = \big(\tfrac{1}{12\sqrt{2}\cos\mathbb{A}} \big)^2 \Big\}.$$

It is obvious that $|z_1| \geq 1/\sqrt{2}$ for all $z \in \gamma_1, \gamma_4$.

For all $z \in \gamma_{14}$ we have $|z_1| \geq 1/\sqrt{2}$, provided that

$$\sqrt{\Big(\frac{9+8\cos 2\mathbb{A}}{12\sqrt{2}\cos\mathbb{A}} \Big)^2 + \Big(\frac{9+8\sin 2\mathbb{A}}{12\sqrt{2}\cos\mathbb{A}} \Big)^2} \geq \frac{1}{\sqrt{2}} + \frac{1}{12\sqrt{2}\cos\mathbb{A}},$$

that is, $\cos\mathbb{A} \geq \frac{1}{6}$.

Similarly we can obtain $|w_2| \geq 1/\sqrt{2}$ providing that $\cos\mathbb{A} \geq \frac{1}{6}$. Note that

$$0 \leq \mathbb{A}_{\mathbb{H}}(u) \leq \arccos\frac{1}{6} \approx 80.4059^\circ.$$

This proves Theorem 5.1 (ii). ∎

6 Some Remarks About Conjecture 1.1

By normalization we can see in Section 4 that the null vectors or the polar vectors all lie in the subspace $(\mathbb{R} \oplus \mathbf{j}\mathbb{R})^{2,1} \subset \mathbb{H}^{2,1}$. This means that the matrices of quaternionic inversions τ_0, τ_1, τ_2 all lie in a copy of $U(2,1) < \mathrm{Sp}(2,1)$. Geometrically, this means that every quaternionic hyperbolic triangle group preserves a totally geodesically embedded copy of complex hyperbolic space. This is to say that every $\mathrm{Sp}(2,1)$ representation of a triangle group factors through a $U(2,1)$ representation. Based on this observation we believe that the results of [13, 15] will also follow in the setting of quaternionic hyperbolic geometry and Conjecture 1.1 is therefore true.

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