RANDOMLY PACKED AND SOLIDLY PACKED SPHERES

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1. Introduction. In the classical packing problem unit spheres are placed without overlapping in D-dimensional space. When $D = 2$, the densest packing is a familiar regular arrangement of circles, each circle touching six others. In this packing, circles cover a fraction $\pi/\sqrt{12} = 0.9069\ldots$ of the area of the plane. The densest packing is not known for $D \geqslant 3$.

Most of the packings to be considered here use spheres of many different sizes. In this way greater densities are obtainable; small spheres can fill up some of the space left over after large spheres have been packed. This phenomenon occurs when cement is mixed with various grades of sand to make concrete. In one of the experiments of Westman and Hugill **(14)** a coarse grade and a fine grade of sand, each of which occupied about 60 per cent of its container by volume, were combined to get a mixture containing 82 per cent sand by volume.

Let $f(R)$ denote the volume fraction occupied by spheres of radius R or greater. A related function is the number $m(R)$ of spheres per unit volume which have radii R or greater. If all radii are less than B, then

$$f(R) = -S \int_R^B x^D dm(x)$$

and

$$m(R) = -S^{-1} \int_R^B x^{-D} df(x),$$

where $S = \pi^{D/2}/\Gamma(1 + \frac{1}{2}D)$ is the volume of a D-dimensional unit sphere.

A problem of main interest here will be to find what kinds of functions $f(R)$ or $m(R)$ can be achieved by actual packings. Section 2 gives a bound on $f(R)$ for 2-dimensional packings and shows that $f(R) < 1 - \text{const.} \times R^2$ for small R.

The remaining sections are concerned with two random processes which pack spheres. These processes depend on an arbitrarily chosen function; thus they provide a variety of functions $f(R)$ and $m(R)$. Random packings of equal spheres have application in the theory of liquid structure (see Bernal **1**). Section 7 considers this application in more detail.

The volume fraction $f = f(0)$ occupied by all spheres combined will be called the *density* of a packing. If $f = 1$, as it may if the packing contains

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arbitrarily small spheres, then the packing will be called a *solid* packing. Section 6 gives a simple condition on one of the random packings to decide whether or not it is solid. The other random packing is too wasteful to be solid but it can be made to have densities arbitrarily close to 1.

2. Two-dimensional packings. To fill a container solidly one may start packing large spheres and later add smaller and smaller spheres to fill up the remaining empty space. Figure 1 illustrates an extreme case of this procedure in the plane. Portions of three large circles are shown. Imagine

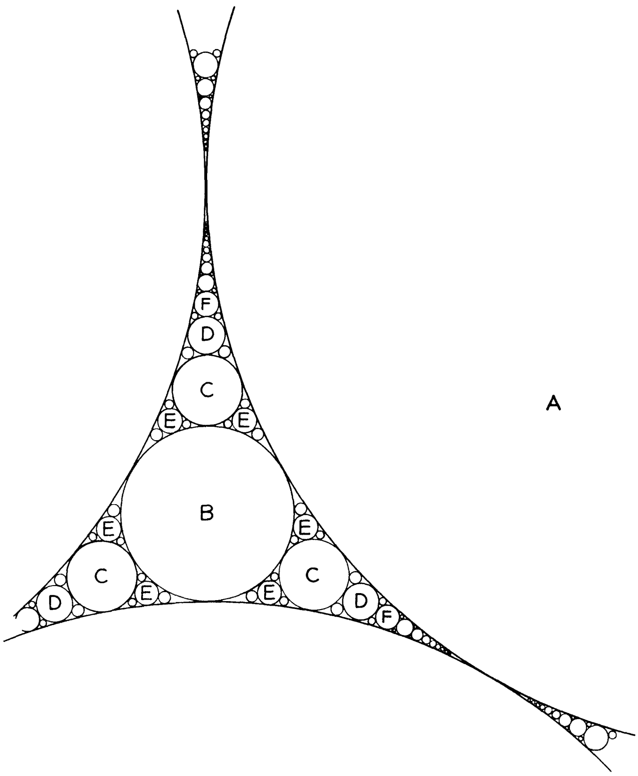

FIGURE 1. A solid plane packing.

circles of this size packed throughout the plane as densely as possible. Circle B is the largest circle which can be added next. After adding circles of this size wherever possible the next largest size (circle C) is added, etc.

Table I gives the radii of the circles in Figure 1 and the number per unit area of each kind of circle. To extend this table an IBM 7090 computer quickly found the radii of several hundred circles (Coxeter (**3**, p. 15), gives Soddy's formula for the radius of a circle tangent to three mutually tangent circles).

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TABLE I

Circle	Number in unit area	Radius, R	$f(R)$	$m(R)$
A	1	0.5373	0.9069	1
B	2	0.0831	0.9503	3
C	6	0.0337	0.9718	9
D	6	0.0183	0.9781	15
E	12	0.0117	0.9832	27
F	6	0.0114	0.9857	33
G	6	0.00786	0.9869	39
H	12	0.00593	0.9882	51
I	12	0.00586	0.9895	63

Figure 2 shows the data plotted as two staircase curves. Empirical approximation formulas

$$(1) \qquad m(R) \approx 0.07R^{-1.3}, \qquad f(R) \approx 1 - 0.41R^{0.7},$$

fit the data reasonably well; these approximations appear as straight lines in Figure 2. The dashed curve labelled FTMF comes from a bound which Fejes Tóth and Molnár conjectured in **(5)**; the proof was completed by Florian **(6)**. Their bound, which applies to all packings of non-overlapping circles of radius $\leqslant B$, is

$$(2) \qquad f(R) \leqslant \{\pi p^2 + 2(1 - p^2)\arcsin p(1 + p)^{-1}\}/\{2p(1 + 2p)^{\frac{1}{2}}\},$$

where $p = R/B$. In Figure 2 and Table I, $B = (12)^{-\frac{1}{4}}$.

As $R \to 0$ the bound (2) becomes

$$f(R) \leqslant 1 - (2 - \tfrac{1}{2}\pi)(R/B) + O(R^2).$$

Thus, for the packing shown in Figure 1, $f(R)$ seems to approach 1 more slowly than the bound (2) ($1 - 0(R^{0.7})$ as compared with $1 - O(R)$). Since Figure 1 uses circles as large as possible at every step it is hard to imagine another packing for which $f(R)$ approaches 1 much faster. It would be interesting to know the largest exponent

$$\limsup_{R\to\infty}\{\log(1 - f(R))/\log R\}$$

achievable in a solid packing.

THEOREM. *When non-overlapping circles are packed in the plane, the circles of radius R or greater leave uncovered a fraction*

$$1 - f(R) \geqslant (2\sqrt{3} - \pi)m(R)R^2 = 0.32251\, m(R)R^2$$

of the plane area.

Since $m(R) > (\pi B^2)^{-1}$, the theorem shows immediately that $f(R) \leqslant 1 - 0.10266(R/B)^2$, but this result is weaker than (2). If a better estimate of

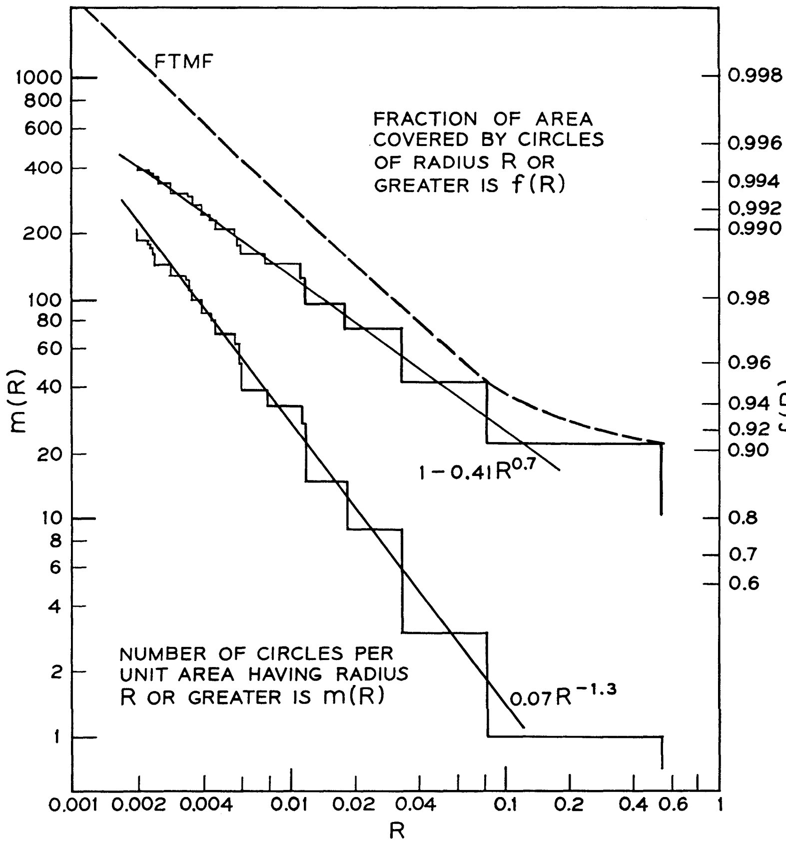

FIGURE 2. Distribution of radii.

$m(R)$ is available, the theorem can give a stronger bound. For example, with $m(R)$ given by (1) the theorem becomes $f(R) \leqslant 1 - 0.023R^{0.7}$. The exponent 0.7 agrees with (1). The factor $2\sqrt{3} - \pi$ in the theorem is as large as possible; equality holds when circles of equal radius R are close-packed.

To prove the theorem it suffices to consider packings of circles into an arbitrary convex hexagon. The result for the plane will be the limiting case for large hexagons. If M circles of radius $\geqslant R$ pack inside a hexagon H, the area in H left uncovered will be shown to ecxeed 0.32251 MR^2. An equivalent result is that the uncovered area exceeds 0.10266 times the combined area of M circles of radius R. The proof will use the method by which Fejes Tóth (**4**, p. 60) treats the packing problem with equal circles.

Given circles $C_1, C_2, \ldots, C_M$ packed into a convex hexagon H, enclose each C_i in a convex polygon P_i. The P_i must be non-overlapping and must fit together exactly to cover all of H. Fejes Tóth constructs such polygons P_i as follows. Imagine the plane of H and the C_i as a huge sphere and construct planes π_i which intersect the sphere in the C_i. In this way one obtains a polyhedron, the faces of which are convex polygonal sections of the π_i containing the C_i. In the limit as the sphere becomes a plane, these faces intersect H in the desired polygons P_i.

The collection of sides of polygons P_i forms a planar linear graph. With the possible exception of the six corners of H, each vertex of the graph is a point at which at least three edges meet. It follows from Euler's theorem that the average number of sides of a polygon P_i is at most 6; i.e., $p_1 + p_2 + \ldots + p_M \leqslant 6M$, where p_i is the number of sides of P_i **(4**, p. 16**)**.

Of all possible p-gons which contain a given circle C, the ones which have least area are regular p-gons with sides tangent to C. Thus the (uncovered) area in P_i but not in C_i is

$$\text{area}(P_i) - \text{area}(C_i) \geqslant \text{area}(C_i)\{(p_i/\pi)\tan(\pi/p_i) - 1\}$$

and the total area left uncovered is

$$\geqslant R^2 \sum_{i=1}^{M} \{p_i \tan(\pi/p_i) - \pi\} .$$

To get a lower bound one may find $p_1, p_2, \ldots, p_M$ which minimize this sum subject to $p_1 + p_2 + \ldots + p_M \leqslant 6M$. Since $p \tan(\pi/p)$ is a decreasing convex function of p, the minimum is achieved by $p_1 = p_2 = \ldots = 6$. Then the area left uncovered is at least $(2\sqrt{3} - \pi)MR^2$ and the theorem follows.

Three-dimensional packings analogous to Figure 1 have more practical interest. However, the geometry is more involved and just a few sphere sizes are known. Horsfield **(8)** added smaller spheres to the interstices between spheres in the closest lattice packing (of density 0.74048 . . .) and found, with four sizes of smaller spheres, that the density could be increased to 0.85. White and Walton **(15)** performed similar calculations starting from a different packing, the tetragonal sphenoidal system. Hudson **(9)** added spheres of only a single extra size but allowed the new spheres to be small so that they filled up fractions approaching 0.74 of the interstices. Rankin **(11)** showed, for all D-dimensional packings of spheres of radius $\leqslant B$, that

$$f(R) \leqslant \frac{(D+2)(B+R)^{D+2}}{(B^2 + 6BR + R^2)^{\frac{1}{2}D+1}} .$$

However Rankin's bound exceeds 1 when R/B is less than some value (depending on D). The bound gives no improvement on $f(R) \leqslant 1$ if $D = 2$; when $D = 3$, there is an improvement if $0.71590 < R/B \leqslant 1$.

3. Random packing. This section will give two random processes for packing unequal spheres. Both packings evolve as the final result of processes depending on a time parameter t. Both processes use an arbitrary prescribed function $R(t)$ to control the distribution of radii in the ultimate packing. The processes, in a sense, try to fill space by adding spheres one at a time.

To describe these processes exactly, consider a $(D + 1)$-dimensional space with Cartesian co-ordinates $(x_1, \ldots, x_D, t)$ satisfying

$$|x_i| < \infty, \qquad i = 1, \ldots, D, 0 \leqslant t < \infty.$$

Select a random pattern of points from this space by a Poisson process of density α points per unit $(D + 1)$-dimensional volume. Each selected point represents a sphere which one tries to add to the packing. The co-ordinates $(x_1, \ldots, x_D)$ are the spatial co-ordinates of the centre of the trial sphere and t is the time at which the trial is made. The radius of a sphere to be tried at time t is the function $R(t)$.

The two packing processes differ in the criteria by which they decide which trial spheres remain in the packing. The first, and most natural, criterion rejects a trial sphere only if it overlaps a sphere which was added to the packing at an earlier time. The second criterion is more stringent but also more convenient analytically. It rejects a sphere if it overlaps another trial sphere with a smaller value of t. In particular a sphere is rejected if it overlaps an earlier trial sphere which itself was rejected. Thus for a given Poisson pattern of points the second packing is only a subset of the first packing.

According to the criterion of the second packing a trial sphere with parameters $(x_1^0, \ldots, x_D^0, t^0)$ remains in the packing if and only if the region

$$0 \leqslant t \leqslant t^0, \qquad \sum_{i=1}^{D} (x_i - x_i^0)^2 \leqslant (R(t) + R(t^0))^2$$

contains no points of the Poisson pattern. The expected number of Poisson points in the region is α times the region's volume, i.e.,

$$\alpha V = \alpha \int_0^{t^0} S(R(t) + R(t^0))^D dt. \tag{3}$$

The probability that the sphere remains in the packing is just $\exp\{-\alpha V\}$. No such simple result holds for the first packing method. In the special case that $R(t)$ is equal to a constant, the first packing scheme is a model of automobile parking. Here the spheres represent automobiles which try parking places at random and remain if other parked cars do not interfere. The one-dimensional case is of interest for parking along a street. Renyi **(12)** showed that cars of equal length park with density 0.748; Ney **(10)** generalized this result to allow unequal car lengths.

4. The second packing. During the time interval t^0 to $t^0 + dt^0$ the expected number of trial spheres per unit volume is $\alpha\, dt^0$. If the second criterion is used, these trials contribute an expected number $\alpha \exp\{-\alpha V\}dt^0$ of packed spheres per unit volume. Here V is given by (3). Thus if $T(R)$ is the set of times t^0 such that $R(t^0) > R$ and $T'(R)$ is the set of times t^0 such that $R(t^0) \geqslant R$, then

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$$m(R) = \alpha \int_{t^0 \epsilon T(R)} \exp\left\{-\alpha \int_0^{t^0} S(R(t) + R(t^0))^D dt\right\} dt^0 \tag{4}$$

and

$$f(R) = \alpha S \int_{t^0 \epsilon T'(R)} \{R(t^0)\}^D \exp\left\{-\alpha S \int_0^{t^0} (R(t) + R(t^0))^D dt\right\} dt^0. \tag{5}$$

In particular

$$f = f(0) = \alpha S \int_0^\infty \{R(t^0)\}^D \exp\left\{-\alpha S \int_0^{t^0} (R(t) + R(t^0))^D dt\right\} dt^0. \tag{6}$$

When $R(t)$ is a constant, (6) becomes $f = 2^{-D}$.

The most natural way to pack unequal spheres is to position the largest spheres first, leaving the smaller spheres to fill in the interstices as in Section 2. This way corresponds to a monotonically decreasing function $R(t)$. Suppose the trial spheres are distributed with a density function $N(R)$; i.e., $N(R)dR$ is the expected number of trial spheres which have centres in a unit volume and have radii between R and $R + dR$. The corresponding monotonically decreasing $R(t)$ satisfies the equation

$$\int_{R(t)}^B N(R)dR = \alpha t. \tag{7}$$

If $R(t)$ is monotonic and has a corresponding function $N(R)$ as in (7), then $m(R)$ and $f(R)$ have convenient expressions in terms of $N(R)$ directly:

$$m(R) = \int_R^B \exp\left\{-S \int_r^B (r+s)^D N(s)ds\right\} N(r)dr, \tag{8}$$

$$f(R) = S \int_R^B \exp\left\{-S \int_r^B (r+s)^D N(s)ds\right\} N(r) r^D dr. \tag{9}$$

These formulas come directly from (4) and (5) when (7) is used to change the variables of integration from times to radii.

In (9)

$$s^D \leqslant (r+s)^D \leqslant 2^D s^D.$$

Simple bounds on $f(R)$ follow:

$$2^{-D}\left(1 - \exp\left\{-S \int_R^B 2^D s^D N(s)ds\right\}\right) \leqslant f(R), \tag{10}$$

$$f(R) \leqslant 1 - \exp\left\{-S \int_R^B s^D N(s)ds\right\}. \tag{11}$$

Bound (11) shows that the integral

$$S \int_0^B s^D N(s)ds = \alpha S \int_0^B \{R(t)\}^D dt \tag{12}$$

must be made large if the density f is to be close to 1. If the integral (12) diverges, then (10) shows that $2^{-D} \leqslant f$, i.e., all such packings are denser than packings of equal spheres.

5. Nearly solid packings. The second packing method does not produce solid packings because trials that fail waste a part of space forever afterward. Packings with large densities require a careful choice of $R(t)$ or $N(R)$. In order to avoid wasting large amounts of space by too many unsuccessful trials $R(t)$ must decrease suitably fast as $t \to \infty$. However, one must simultaneously keep the integral (12) large. This section will construct packings which have densities arbitrarily close to 1.

Consider the case

$$N(R) = \begin{cases} AR^{-(D+1+a)}, & \text{if } 0 < R \leqslant 1, \\ 0, & \text{if } 1 < R < \infty, \end{cases} \tag{13}$$

where a, A are constant parameters. The corresponding function $R(t)$ follows from (7):

$$R(t) = \{1 + a(D + a)t/A\}^{-1/(D+a)}.$$

In what follows, $|a| < 1$.

First suppose that $a \neq 0$. Formula (9) contains the integral

$$I = \int_r^B (r + s)^D N(s) ds = A \int_r^1 (r + s)^D s^{-(D+1+a)} ds$$
$$= A \sum_{k=D}^{D} \binom{D}{k} \frac{r^{-a} - r^k}{a + k} = Ar^{-a}(a^{-1} + q) - A(a^{-1} + q(r)),$$

where

$$q(r) = \sum_{k=1}^{D} \binom{D}{k} r^k \Big/ (a + k)$$

and $q = q(1)$. Since $|a| < 1$, $q(r)$ and q are positive. Also $q(r) \leqslant q$. Then

$$-A(a^{-1} + q) \leqslant I - Ar^{-a}(a^{-1} + q) \leqslant -Aa^{-1} \tag{14}$$

and a lower bound on $f(R)$ is

$$f(R) \geqslant ASe^{AS/a} \int_R^1 e^{-AS(a^{-1}+q)/r^2} r^{-(1+a)} dr$$

or

$$f(R) \geqslant \frac{\{1 - \exp(AS(a^{-1} + q)(1 - R^{-a}))\} \exp(-ASq)}{1 + aq}. \tag{15}$$

A corresponding upper bound follows from the left inequality in (14). It differs from (15) just be a factor $\exp(ASq)$.

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As $R \to 0$ the limiting form of these bounds on $f(R)$ depends on the sign of a:

(16) $$\exp(-ASq) \leqslant (1 + aq)f \leqslant 1, \qquad \text{if } a > 0,$$

(17) $$\exp(-ASq) - \exp(-AS/|a|) \leqslant (1 + aq)f \leqslant 1 - \exp\{AS(a^{-1} + q)\}, \qquad \text{if } -q^{-1} < a < 0.$$

When $a > 0$, f can be made arbitrarily close to 1 by choosing A and a small. When $a < 0$, f will be close to 1 if $|a|$ and A are both small but $A/|a|$ is large.

The special case $a = 0$ of (13) has a similar treatment but with

$$0 \leqslant I + A \log r \leqslant Aq$$

instead of (14). The bounds on $f(R)$ are simpler:

(18) $$(1 - R^{AS})\exp(-ASq) \leqslant f(R) \leqslant 1 - R^{AS}.$$

Again A should be small to get a dense packing. Similar bounds on $m(R)$ when $a = 0$ are

(19) $$A \exp(-ASq)(R^{-(D-AS)} - 1)/(D - AS) \leqslant m(R) \leqslant A(R^{-(D-AS)} - 1)/(D - AS).$$

For small R the bounds (19) on $m(R)$ grow like multiples of $R^{-(D-AS)}$. This suggests a comparison with the plane packing in Section 2. To achieve the same exponent 1.3, set $AS = 0.7$. With this choice of A the factor $\exp(-ASq)$ in (18) is only 0.17 and undoubtedly the random packing is not very dense. Picking smaller values of A to ensure a dense random packing makes $m(R)$ grow with decreasing R more slowly than in Section 2.

6. The first packing. Again consider monotonically decreasing $R(t)$ for which $N(R)$ in (7) exists. When the first packing method is used, solid packings are possible.

THEOREM. *The first packing method produces a solid packing if and only if the limit*

$$J = \lim_{R \to 0} \int_R^B r^D N(r)dr$$

is infinite.

Proof. Consider a point Q in space. The integral

$$S \int_a^b r^D N(r)dr, \qquad 0 \leqslant a \leqslant b,$$

is the expected number of trial spheres which contain Q and have radii between a and b.

If J is finite the expected number of trial spheres (of all radii) containing Q is SJ. Q has positive probability $\exp(-SJ)$ of belonging to no trial sphere.

Then even the trial spheres cover only a fraction $1 - \exp(-SJ)$ of space and $f \leqslant 1 - exp(-SJ) < 1$.

Conversely suppose that $J = \infty$. It remains to show that Q has probability 1 of belonging to a packed sphere. Since there are only countably many open packed spheres, it will suffice to prove that Q has probability 1 of being a limit point of the set of packed spheres. In order for Q not to be a limit point it must lie at the centre of some spherical neighbourhood U which does not intersect any of the packed spheres. Let $E(r)$ be the event that such a neighbourhood U exists and has radius $\geqslant r$. Then the event E that a neighbourhood U exists with some unspecified radius is

$$E = E(2^{-1}) \vee E(2^{-2}) \vee E(2^{-3}) \vee \ldots$$

so that

$$\text{Prob}(E) \leqslant \sum_k \text{Prob}(E(2^{-k})).$$

However, since $J = \infty$ there is probability

$$1 - \exp\left\{-S \int_0^{2^{-k-1}} r^D N(r) dr\right\} = 1 - \exp(-\infty) = 1$$

that some trial sphere S of radius less than 2^{-k-1} contains Q. S lies entirely in the spherical neighbourhood of radius 2^{-k} around Q. Either S is a packed sphere or S overlaps a packed sphere; in either case the neighbourhood of radius 2^{-k} overlaps a packed sphere. Thus

$$\text{Prob}(E(2^{-k})) = 0$$

and $\text{Prob}(E) = 0$.

7. Packing equal spheres. As noted in Section 4, the second packing method achieves density 2^{-D} when all spheres have the same size. This is not very large compared with densities achievable by other means. For example, G. D. Scott **(13)** found that ball bearings poured at random into a 3-dimensional container filled it with density 0.63. The densest regular lattice packings in 2, 3, 4, 5 dimensions have densities 0.907, 0.740, 0.617, 0.465 **(7)**. Some higher-dimensional packings are known with densities close to the upper bound $(D + 2)2^{-\frac{1}{2}(D+2)}$ of Blichfeldt **(2)**. Nevertheless 2^{-D} exceeds the densities of the simplest families of regular packings (e.g., cubic packing) when D is large.

For another comparison, note that 2^{-D} is a lower bound on the densities of all packings to which no extra sphere may be added without overlap. For consider such a packing; let its density be f. Since no extra sphere may be added, each point in space must lie within a distance of two radii from one of the sphere centres. Now replace all the spheres by (overlapping) spheres of twice the original radius. The enlarged spheres have density at most $2^D f$ and also cover all space; thus $2^D f \geqslant 1$.

Table II shows some rough estimates of the density achieved by the first packing method with equal spheres. To obtain these estimates an IBM 7090 computer generated pseudo-random number co-ordinates for centres of trial spheres in a large cube. The trials continued until several hundred consecutive trial spheres failed to be packed. Thus in the final packing one expects that only a fraction of 1 per cent of the volume of the cube consists of points at which the centre of an additional packed sphere could lie. However, this small fraction could consist of many tiny pieces scattered throughout the cube; in that case many additional spheres could be packed and the numbers in Table II would be too small.

TABLE II

FIRST PACKING METHOD WITH EQUAL SPHERES

Dimension	Density
1	0.748; cf. (**12**)
2	0.5
3	0.3
5	0.1
10	0.004

One theoretical model of a liquid uses non-overlapping spheres to represent the liquid molecules. A function of great importance in such studies is the radial distribution function $\rho(a)$. To define $\rho(a)$, consider a sphere, say with centre at O, and a volume element dV at distance a from O. $\rho(a)dV$ is the probability that another sphere centre lies in the element dV. In particular $\rho(\infty)$ is just the number of spheres per unit volume, i.e., $\rho(\infty) = f/(SR^D)$. The probability that two volume elements dV_1, dV_2, separated by distance a, both contain centres is $\rho(\infty)\rho(a)dV_1\,dV_2$.

The radial distribution function for spheres packed by the second method is obtainable analytically. In order to have packings in which the density is a free parameter, let the packing cease at time T. Then the range of integration in (4) becomes $0 \leqslant t^0 \leqslant T$. The density is

$$f = 2^{-D}\{1 - \exp(-\nu)\} \tag{20}$$

where $\nu = \alpha ST(2R)^D$ is the expected total number of trial centres to appear inside a sphere of radius $2R$.

Consider volume elements dV_1, dV_2 separated by distance a. Draw spheres K_1 and K_2 with centres in dV_1 and dV_2 and with radii $2R$. The elements dV_1 and dV_2 both contain centres of packed spheres only if they contain trial centres. Let the arrival times of the trial centres be t_1 and t_2. In order for these trial spheres to remain as packed spheres, K_1 and K_2 must have had no other trial centres arriving at times before t_1 and t_2.

If $a > 4R$, the spheres K_1 and K_2 do not overlap. Then if $W = S2^DR^D$ is the volume of K_1 and of K_2,

$$\rho(\infty)\rho(a)dV_1\,dV_2 = \int_0^T \exp(-\alpha t_1 W)\alpha dt_1 \int_0^T \exp(-\alpha t_2 W)\alpha dt_2 \cdot dV_1\,dV_2$$
$$= [\rho(\infty)]^2 dV_1\,dV_2.$$

Then

(21) $$\rho(a) = \rho(\infty) = (1 - \exp(-\nu))/W, \qquad 4R \leqslant a.$$

If $2R < a < 4R$, the spheres K_1, K_2 overlap. Let W' be the volume of the intersection $K_1 K_2$. Then $\rho(\infty)\rho(a)$ is the integral

$$\alpha^2 \int_0^T \int_0^T \exp\{-\alpha t_1(W - W') - \alpha t_2(W - W') - \alpha\,\mathrm{Max}(t_1, t_2)W'\}dt_1\,dt_2.$$

The final result is

(22) $$\rho(a) = \begin{cases} 0, & \text{if } a \leqslant 2R, \\ \dfrac{2\rho(\infty)}{2^{2D}f^2}\left\{2^D f - \dfrac{1 - (1 - 2^D)f^{1+Q}}{1 + Q}\right\}, & \text{if } 2R < a \leqslant 4R, \\ \rho(\infty), & \text{if } R < a. \end{cases}$$

In (22), f is the density (20) $(0 \leqslant f \leqslant 2^{-D})$, $\rho(\infty)$ is given by (21), and $Q = 1 - W'/W$. If $D = 1$, $Q = a/(2R)$. Otherwise

$$Q = 1 - 2\int_0^\theta \sin^D x\,dx \Big/ \int_0^\pi \sin^D x\,dx,$$

where $\cos\theta = a/(4R)$. In particular

$$Q = 1 - (2\theta - \tfrac{1}{2}\sin 2\theta)/\pi, \qquad \text{if } D = 2,$$
$$Q = (48(a/R) - (a/R)^3)/128, \qquad \text{if } D = 3.$$

When $f = 2^{-D}$, (22) simplifies to $\rho(a) = 2\rho(\infty)/(1 + Q)$ for $2R < a \leqslant 4R$. In this range $\rho(a)$ decreases monotonically down to the value $\rho(\infty)$ at $a = 4R$. When $D = 2$, $\rho(2R) = 1.3595\,\rho(\infty)$. When $D = 3$, $\rho(2R) = (32/27)\rho(\infty) = 1.1852\,\rho(\infty)$.

The radial distribution function for the first random packing method seems hard to find analytically. A computer simulation was tried for the case $D = 3$. A total of 199 spheres of radius 0.10 were packed into three unit cubes to get an average density of 0.278. In each cube opposite faces were identified to minimize boundary effects. The radial distribution function had a peak value $\rho(2R) = 1.9\rho(\infty)$; for $a > 3R$ no statistically significant difference between $\rho(a)$ and $\rho(\infty)$ was detectable.

Radial distribution functions for molecules in a liquid, as measured by X-ray diffraction, typically have higher peak values and oscillate about the value $\rho(\infty)$ as a increases. The random packings considered here are not dense enough to show this behaviour.

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