BULL. AUSTRAL. MATH. SOC. 22D05
VOL. 40 (1989) [417–419]

LOCALLY DYADIC TOPOLOGICAL GROUPS

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A topological space is said to be locally dyadic if every neighbourhood of a point contains a dyadic neighbourhood of that point. It is proved here that every locally compact Hausdorff topological group is locally dyadic.

Referring to a result by Kuz'minov [5] which says that every compact Hausdorff group is dyadic, Comfort [2] remarks " ... there appears to be no valid proof of Kuz'minov's theorem available in the English language." We prove here that every locally compact Hausdorff group is locally dyadic, and deduce Kuz'minov's result.

Recall that a topological space is said to be a *dyadic space* if it is a quotient space of a Cantor cube $\{0, 1\}^{\aleph}$, where $\{0, 1\}$ is the discrete 2-point space and $\aleph$ is any cardinal number. It is well-known that every compact metrizable space is dyadic.

We define a topological space to be *locally dyadic* if every neighbourhood of any point contains a dyadic neighbourhood of that point. Clearly every discrete space and every compact metrizable space is locally dyadic.

THEOREM. *Every locally compact Hausdorff group is locally dyadic.*

PROOF: In [1] it is proved that every locally compact Hausdorff group is homeomorphic to $\mathbf{R}^n \times K \times D$, where $\mathbf{R}$ is the group of real numbers with the usual topology, n is a non-negative integer, K is a compact Hausdorff group, and D is a discrete space. It is clear that any finite product of locally dyadic spaces is locally dyadic. Therefore, as $\mathbf{R}^n$ and D are locally dyadic, and any open continuous image of a locally dyadic space is locally dyadic, it suffices to prove that K is locally dyadic.

It is known [6] that every compact Hausdorff group K is homeomorphic to $K_0 \times K/K_0$, where K_0 is the connected component of the identity of K. The group K/K_0 is a compact totally disconnected topological group, and from [4, Section 9.15] we have that it is homeomorphic to $\{0, 1\}^{\aleph}$, for some cardinal number $\aleph$. Thus K/K_0 is locally

Received 14 December 1988
We thank Peter Nyikos for drawing our attention to the question by Eric van Douwen, [8], which asked if every locally compact Hausdorff topological group has a dyadic neighbourhood of the identity. This question is answered in the affirmative, being a corollary of our theorem. According to the Problem section information in Volume 11 (1986) of Topology Proceedings, van Douwen's question has also been answered by Uspensky, though no proof is given.

dyadic. We are now left with the task of showing that the compact connected group K_0 is locally dyadic.

By [7, Section 6.5.6] every compact connected Hausdorff group is a quotient of a product $\prod_{i\in I} G_i \times A$, where each G_i is a compact (connected simply-connected simple) Lie group and A is a compact connected abelian group. It is clear that any product of compact metrizable spaces is locally dyadic (and dyadic), and so $\prod_{i\in I} G_i$ is locally dyadic. Therefore it is enough to prove that any compact connected abelian Hausdorff group A is locally dyadic.

As A is compact and connected, its dual group $\widehat{A}$ is discrete and torsion-free. Therefore the minimal divisible closure of $\widehat{A}$ is torsion-free (and divisible); that is, it is a vector space over $\mathbf{Q}$. In other words, it is a restricted direct sum of m copies of the (discrete) rational group $\mathbf{Q}$, where m is some cardinal number. Taking the dual of $\widehat{A}$, we see that A is a quotient group of the product group $\left(\widehat{\mathbf{Q}_d}\right)^m$, where $\mathbf{Q}_d$ is the discrete group of rational numbers. Observing that $\mathbf{Q}_d$ is countable, its dual group, $\widehat{\mathbf{Q}_d}$ is compact metrizable, and so the product group $\left(\widehat{\mathbf{Q}_d}\right)^m$ is locally dyadic. Hence the quotient group A is locally dyadic. This completes the proof. $\square$

It is easily verified that every compact locally dyadic space is dyadic. Hence we have the following:

Corollary. [5] *Every compact Hausdorff group is dyadic.*

Of course Kuz'minov's theorem can also be deduced from the proof of the above theorem.

Remark. In a recent communication with Comfort, we suggested that every dyadic Hausdorff space might be locally dyadic. This is indeed true, and we outline Comfort's proof here.

Let X be a dyadic Hausdorff space and U any neighbourhood of $x \in X$. Then there is a continuous surjection $f : \prod_{i\in I} D_i \to X$, where each D_i is the discrete 2-point space $\{0,1\}$ and I is some index set. As X is completely regular Hausdorff, U contains a neighbourhood V of x which is a zero-set. Then $f^{-1}(V)$ is a zero-set in $\prod_{i\in I} D_i$. Using [3, Section 2.7.12(c)], this implies that it is homeomorphic to $B \times \prod_{i\in I\setminus J} D_i$, where each $D_i = \{0,1\}$, J is a countable subset of I and B is a closed subspace (and hence a continuous image) of the Cantor space $\prod_{i\in J} D_i$. This implies that B is dyadic, and hence both $f^{-1}(V)$ and V are dyadic.

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https://doi.org/10.1017/S0004972700017469 Published online by Cambridge University Press