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Endomorphism Rings of Finite Global Dimension

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Abstract. For a commutative local ring R, consider (noncommutative) R-algebras Λ of the form $\Lambda = \mathrm{End}_R(M)$ where M is a reflexive R-module with nonzero free direct summand. Such algebras Λ of finite global dimension can be viewed as potential substitutes for, or analogues of, a resolution of singularities of $\mathrm{Spec}\, R$. For example, Van den Bergh has shown that a three-dimensional Gorenstein normal $\mathbb{C}$-algebra with isolated terminal singularities has a crepant resolution of singularities if and only if it has such an algebra Λ with finite global dimension and which is maximal Cohen–Macaulay over R (a "noncommutative crepant resolution of singularities"). We produce algebras $\Lambda = \mathrm{End}_R(M)$ having finite global dimension in two contexts: when R is a reduced one-dimensional complete local ring, or when R is a Cohen–Macaulay local ring of finite Cohen–Macaulay type. If in the latter case R is Gorenstein, then the construction gives a noncommutative crepant resolution of singularities in the sense of Van den Bergh.

This paper takes for its starting point two results of Auslander.

Theorem A *Let Λ be a left Artinian ring with radical $\mathfrak{r}$ and assume that $\mathfrak{r}^n = 0$, $\mathfrak{r}^{n-1} \neq 0$. Set $M = \bigoplus_{i=1}^{n} \Lambda/\mathfrak{r}^i$. Then $\Gamma := \mathrm{End}_\Lambda(M)^{\mathrm{op}}$ is coherent of global dimension at most $n+1$* [1, §III.3].

Theorem B *Let $S = k[[x, y]]$ be the ring of formal power series in two variables over a field k and let G be a finite subgroup of $\mathrm{GL}_2(k)$ with $|G|$ invertible in k. Set $R = S^G$. Then $A := \mathrm{End}_R(S)^{\mathrm{op}}$ has global dimension at most two* [2].

These theorems both relate to Auslander's notion of *representation dimension*, introduced in [1] as a way to measure homologically the failure of an Artin algebra to have finite representation type. The representation dimension of an Artin algebra Λ can be defined as

$$\mathrm{repdim}\, \Lambda = \inf\{\mathrm{gldim}\, \mathrm{End}_\Lambda(M)\},$$

where the infimum is taken over all finitely generated modules M which are generator-cogenerators for $\mathrm{mod}\, \Lambda$. Note that Theorem A does not prove finiteness of the representation dimension; while M has a nonzero free direct summand, it need not be a cogenerator. Auslander showed in [1] that $\mathrm{repdim}\, \Lambda \leq 2$ if and only if Λ has finite representation type, and in 2003, Rouquier [19] gave the first examples with $\mathrm{repdim}\, \Lambda > 3$. Iyama has recently shown [15] that the representation dimension of an Artin algebra is always finite.

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We extend Theorems A and B in two directions. In each case, we consider commutative Noetherian (semi)local base rings.

First, we fill a gap between Auslander's theorems: the case of dimension one. A reduced complete local ring R of dimension 1 always has a finitely generated module whose endomorphism ring has finite global dimension, the normalization $\widetilde{R}$. However, $\widetilde{R}$ is never a generator in the category of R-modules, unless R is already a discrete valuation ring. Theorem 4 produces a finitely generated generator M such that $\mathrm{End}_R(M)$ has finite global dimension. Specifically, M can be taken to be a direct sum of certain overrings S between R and $\widetilde{R}$, and the global dimension of $\mathrm{End}_R(M)$ is bounded by the multiplicity of R. This completes a coherent picture for rings of dimension at most 2; see [20, 21] for related progress in dimension 3.

We also generalize to dimensions $d > 2$ by exploiting the connection with finite representation type. The two-dimensional quotient singularities $\mathbb{C}[\![x, y]\!]^G$, with $G \subseteq \mathrm{GL}_2(\mathbb{C})$ a finite group, are precisely the two-dimensional complete local rings with residue field $\mathbb{C}$ and having finite Cohen–Macaulay (CM) type [2, 9, 12]. Moreover, $\mathbb{C}[\![x, y]\!]$ contains as R-direct summands all indecomposable maximal Cohen–Macaulay R-modules [12]. Theorem 6 states that if R is a d-dimensional Cohen–Macaulay local ring of finite Cohen–Macaulay type, and if M is the direct sum of all indecomposable maximal Cohen–Macaulay R-modules, then $\mathrm{End}_R(M)$ has global dimension at most $\max\{2, d\}$. It follows that the representation dimension of a complete Cohen–Macaulay local ring of finite Cohen–Macaulay type is finite. (See Definition 7 for the definition of representation dimension in this context.)

The proofs of both theorems are based on projectivization [3, II.2]. In the present contexts, this means that the functor $\mathrm{Hom}_R(M, -)$ induces an equivalence of categories between $\mathrm{add}(M)$, the full subcategory of R-modules which are direct summands of finite direct sums of copies of M, and the full subcategory of finitely generated projective modules over $A := \mathrm{End}_R(M)^{\mathrm{op}}$. In particular, if M is an R-generator, then M has an R-free direct summand, and so M is a projective A-module. Auslander's original proof of Theorem A uses this technique, and the proofs of our two main results are very close in spirit to his method. Iyama has refined Auslander's methods into a theory of *rejective subcategories* [13, 14], which he uses to prove that the representation dimension of an order over a complete discrete valuation ring is finite.

Section 3 discusses the implications of Theorem 6 to the theory of non-commutative crepant resolutions [21]. If R is Gorenstein of dimension 2 or 3 and has finite Cohen–Macaulay type, Theorem 6 does indeed produce a non-commutative crepant resolution of R. If R is not Gorenstein, then non-commutative crepant resolutions are not yet defined, but Theorem 6 still gives an analogue. We discuss advantages and disadvantages of this analogy.

The rings under consideration will be Noetherian, and all modules finitely generated. We abbreviate $\mathrm{Hom}_R(-, -)$ by $(-, -)$.

1 Dimension One

In this section we consider reduced one-dimensional semilocal rings R. We always assume that R is complete with respect to its Jacobson radical, equivalently that R

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is isomorphic to a direct product of complete local rings. Let K be the total quotient ring of R, obtained by inverting all nonzerodivisors of R. Recall that a finitely generated R-module M is *torsion-free* provided the natural map $M \to M \otimes_R K$ is injective.

Our goal requires us to consider the module theory of certain birational extensions of reduced rings, that is, extensions $R \subseteq S$ where S is a finitely generated R-module contained in the total quotient ring K of R. Of course, in this situation every finitely generated torsion-free S-module is a finitely generated torsion-free R-module, but not vice versa. The following lemma, however, follows easily by clearing denominators.

Lemma 1 *Let $R \subseteq S$ be a birational extension of reduced rings as above. Let C and D be finitely generated torsion-free S-modules. Then $\mathrm{Hom}_R(C,D) = \mathrm{Hom}_S(C,D)$. Furthermore, if M is a finitely generated torsion-free R-module, and $f\colon C \to M$ is an R-linear map, then the image of f is an S-module.*

For the remainder of this section, $(R,\mathfrak{m})$ will be a reduced complete local ring of dimension one with total quotient ring K and integral closure $\widetilde{R}$. Note that K is a direct product of finitely many fields, and $\widetilde{R}$ is correspondingly a direct product of discrete valuation rings. Since R is complete and reduced, $\widetilde{R}$ is a finitely generated R-module [18, Theorem 11.7].

Set $R^{(1)} := \mathrm{End}_R(\mathfrak{m})$. Since $\mathfrak{m}$ contains a nonzerodivisor, $R^{(1)}$ embeds naturally into K (by sending f to $f(r)/r$, which is independent of the nonzerodivisor r). It is well known that in fact $R^{(1)} \subseteq \widetilde{R}$. Furthermore, $R \subsetneq R^{(1)}$ unless $R = \widetilde{R}$. Now, $R^{(1)}$ may no longer be local (if, for example, we take $R = k[\![x,y]\!]/(xy)$), but by Hensel's Lemma, $R^{(1)}$ is a direct product of complete local rings, $R^{(1)} = R_1^{(1)} \times \cdots \times R_{n_1}^{(1)}$, each of which is again reduced.

Iterating this procedure by taking the endomorphism ring of the maximal ideal of each of the local rings $R_\ell^{(1)}$, $\ell = 1,\dots,n_1$, gives a family of reduced complete local rings $\{R_j^{(i)}\}$. Since $\widetilde{R}/R$ is an R-module of finite length and each $R_j^{(i)}$ is trapped between some $R_l^{(i-1)}$ and $\widetilde{R}$, this family is finite. It follows that the lengths of the chains

$$(\ddagger) \qquad R \subsetneq R_{j_1}^{(1)} \subsetneq \cdots \subsetneq R_{j_n}^{(n)} = \widetilde{R},$$

are bounded above, where each $R_{j_i}^{(i)}$ is a direct factor of the endomorphism ring of the maximal ideal of $R_{(j_{i-1})}^{(i-1)}$.

Let $\mathcal{E}(R)$ denote the family of rings obtained in this way, including R itself. Put $\mathcal{A}(R) = \mathrm{add}(\mathcal{E}(R))$, the full subcategory of mod-R containing all direct summands of finite direct sums of rings in $\mathcal{E}(R)$, considered as R-modules. If S is a direct product of complete local rings S_i, $j = 1,\dots,m$, let $\mathcal{E}(S)$ be the corresponding union of the $\mathcal{E}(S_j)$, and $\mathcal{A}(S) = \mathrm{add}(\mathcal{E}(S))$ the full subcategory containing all direct summands of finite direct sums of rings in $\mathcal{E}(S)$, again considered as S-modules.

Even though we begin with a local ring, the proof of Theorem 4 requires dealing with semilocal rings that crop up along the way. Lemma 2 allows us to reduce to the local case each time.

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Lemma 2 *Let $S = S_1 \times \cdots \times S_k$ be a direct product of rings. Assume that for each $i = 1, \ldots, k$ and for each torsion-free S_i-module D, there is an exact sequence*

$$0 \longrightarrow C_{i,m_i} \longrightarrow C_{i,m_i-1} \longrightarrow \cdots \longrightarrow C_{i,0} \longrightarrow D \longrightarrow 0 \tag{1.1}$$

with each $C_{ij} \in \mathcal{A}(S_i)$ and such that

$$0 \longrightarrow (X, C_{i,m_i}) \longrightarrow (X, C_{i,m_i-1}) \longrightarrow \cdots \longrightarrow (X, C_{i,0}) \longrightarrow (X, D) \longrightarrow 0$$

is exact for all $X \in \mathcal{A}(S_i)$. Then for each torsion-free S-module E, there exists an exact sequence

$$0 \longrightarrow C_m \longrightarrow C_{m-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow E \longrightarrow 0 \tag{1.2}$$

with each $C_j \in \mathcal{A}(S_i)$ and such that

$$0 \longrightarrow (X, C_m) \longrightarrow (X, C_{m-1}) \longrightarrow \cdots \longrightarrow (X, C_0) \longrightarrow (X, E) \longrightarrow 0 \tag{1.3}$$

is exact for all $X \in \mathcal{A}(S)$.

Proof Let E be a torsion-free S-module. Then $E \cong \prod_{i=1}^{n} e_i E$, where e_i is a complete set of orthogonal idempotents for the decomposition $S = S_1 \times \cdots \times S_n$. The exact sequence (1.2) can be taken to be the direct sum of the sequences (1.1) with $D = e_i E$. It remains to show that (1.3) is exact for all $X \in \mathcal{A}(S)$. Since $\mathrm{Hom}_S(Y, Z) = 0$ whenever Y is an S_i-module and Z is an S_j- module with $i \neq j$, this is clear. ■

We can now state the key result which will imply our main theorem in the reduced case.

Proposition 3 *Let $(R, \mathfrak{m})$ be a reduced complete local ring of dimension one and let N be a torsion-free R-module. Let n be the length of the longest chain (‡). Then there exists an exact sequence*

$$0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow N \longrightarrow 0$$

with each $C_i \in \mathcal{A}(R)$ and such that

$$0 \longrightarrow (X, C_n) \longrightarrow (X, C_{n-1}) \longrightarrow \cdots \longrightarrow (X, C_0) \longrightarrow (X, N) \longrightarrow 0$$

is exact for all $X \in \mathcal{A}(R)$.

Proof We proceed by induction on n. If $n = 0$, then $R = \widetilde{R}$ is a discrete valuation ring, and any torsion-free R-module is free. The set $\mathcal{A}(R)$ consists exactly of the free R-modules, so that $0 \to C_0 \xrightarrow{=} N \to 0$ is the required sequence.

Assume that the statement holds for reduced complete local rings of dimension one having a chain (‡) of length at most $n - 1$, and that R has a chain of length n. In

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particular, then the proposition holds for each direct factor of $R^{(1)} = \mathrm{End}_R(\mathfrak{m})$. Let N be a torsion-free R-module.

First suppose that N is an $R^{(1)}$-module. By Lemma 2 then, there is an exact sequence of $R^{(1)}$-modules

$$0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow N \longrightarrow 0, \tag{1.4}$$

with each $C_i \in \mathcal{A}(R^{(1)})$, which remains exact under $(X, -)$ for any $X \in \mathcal{A}(R^{(1)})$. (We use Lemma 1 here to know that $\mathrm{Hom}_R(X, -) = \mathrm{Hom}_{R^{(1)}}(X, -)$.) The only indecomposable module in $\mathcal{A}(R)$ but not in $\mathcal{A}(R^{(1)})$ is the free module R, so the sequence remains exact under $(X, -)$ for any $X \in \mathcal{A}(R)$, as desired.

Next suppose that N is not an $R^{(1)}$-module. Let $N' = \mathrm{Hom}_R(R^{(1)}, N) \subsetneq N$ be the largest $R^{(1)}$-module contained in N. Observe that for any $R^{(1)}$-module X, and any R-linear homomorphism $X \to N$, the image of X is contained in N'. In particular, $(X, N') = (X, N)$ for any $X \in \mathcal{A}(R^{(1)})$. By induction, there is a surjection $f\colon C' \to N'$, with $C' \in \mathcal{A}(R^{(1)})$, such that $(X, f)\colon (X, C') \to (X, N')$ is surjective for all $X \in \mathcal{A}(R^{(1)})$. Since $(X, N') = (X, N)$, we see that applying $(X, -)$ to the composition $C' \to N' \hookrightarrow N$ yields a surjection for all $X \in \mathcal{A}(R^{(1)})$.

Take a free R-module F mapping minimally onto N/N' and lift to a homomorphism $g\colon F \to N$. Then $g^{-1}(N')$ is an $R^{(1)}$-module. Indeed, $g^{-1}(N') \subseteq \mathfrak{m}F$ as F is a minimal free cover of N/N', and since $\mathfrak{m}F$ is clearly a module over $R^{(1)} = \mathrm{End}_R(\mathfrak{m})$, Lemma 1 implies that $g^{-1}(N') = \mathfrak{m}F$, so in particular is an $R^{(1)}$-module.

Define $\pi\colon F \oplus C' \to N$ by $\pi(p, c) = g(p) - f(c)$. Since (X, f) is surjective for all $X \in \mathcal{A}(R^{(1)})$, and g induces a surjection $F \to N/N'$, we see that (X, π) is surjective for all $X \in \mathcal{A}(R)$. We claim that the $L = \ker \pi$ is an $R^{(1)}$-module. Let $\alpha \in R^{(1)}$ and $(p, c) \in L$, so that $g(p) = f(c)$. Since $f(c) \in N'$ and $g^{-1}(N')$ is an $R^{(1)}$-module, $\alpha p \in g^{-1}(N')$. Then, since $f|_{f^{-1}(N')}$ is $R^{(1)}$-linear by Lemma 1, $g(\alpha p) = \alpha g(p) = \alpha f(c)$. Finally, $f\colon C' \to N'$ is $R^{(1)}$-linear, so $\alpha f(c) = f(\alpha c)$. That is, $(\alpha p, \alpha c) \in L$, as claimed.

By the previous case then, there is an exact sequence

$$0 \longrightarrow C_{n-1} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow L \longrightarrow 0$$

such that

$$0 \longrightarrow (X, C_{n-1}) \longrightarrow (X, C_{n-2}) \longrightarrow \cdots \longrightarrow (X, C_0) \longrightarrow (X, L) \longrightarrow 0$$

is exact for all $X \in \mathcal{A}(R^{(1)})$. Splicing this together with the short exact sequence $0 \to L \to F \oplus C' \to N \to 0$, and using once again that the only indecomposable module in $\mathcal{A}(R)$ but not in $\mathcal{A}(R^{(1)})$ is the free module R, we are done. ∎

Theorem 4 *Let $(R, \mathfrak{m})$ be a one-dimensional reduced complete local ring. Put $M = \bigoplus_{S \in \mathcal{E}(R)} S$, a finitely generated R-module. Then $\Gamma := \mathrm{End}_R(M)^{\mathrm{op}}$ has global dimension at most $n + 1$, where n is the length of the longest chain (‡).*

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Proof Let N be a finitely generated Γ-module. Then by [3, II.2] there exists a homomorphism $M_1 \to M_0$ with $M_i \in \operatorname{add}(M)$ such that $(M, M_1) \to (M, M_0) \to N \to 0$ is exact. Let L be the kernel of $M_1 \to M_0$. Then L is torsion-free, so by the proposition has a resolution of length n by modules in $\operatorname{add}(M)$, which remains exact after applying $(M, -)$. Since each (M, M_i) is a projective Λ-module, N has projective dimension at most $n + 1$. ∎

Corollary 5 *A one-dimensional reduced complete local ring has a finitely generated module whose endomorphism ring has global dimension at most* $\mathrm{e}(R)$*, the multiplicity of* R*.*

Proof It is known that $\widetilde{R}/\mathfrak{m}\widetilde{R}$ has dimension $\mathrm{e}(R)$ as a vector space over $R/\mathfrak{m}$. Thus $\widetilde{R}/R$ has length $\mathrm{e}(R) - 1$, so e is a uniform bound on the length of chains (‡). ∎

It seems plausible that Theorem 4 actually holds for rings R such that the integral closure $\widetilde{R}$ is a finitely-generated R-module and a regular ring, for example, the hypersurface $x^2 + y^3 - y^2z^2 = 0$. The proof given above is reminiscent of the algorithm of de Jong [7] for obtaining the integral closure by taking iterated endomorphism rings.

2 Finite Cohen–Macaulay Type

As mentioned in the introduction, the original motivation for Auslander's representation dimension was to study Artin algebras of finite representation type, that is, Artin algebras with only finitely many isomorphism classes of finitely generated modules. For (commutative Noetherian) rings of higher dimension, this property has been generalized to *finite Cohen–Macaulay type*. A nonzero finitely generated module M over a d-dimensional ring R is called *maximal Cohen–Macaulay* (MCM) if there exists an M-regular sequence $x_1, \ldots, x_d$. We then say that R has finite CM type provided there are, up to isomorphism, only finitely many indecomposable MCM R-modules.

The one-dimensional CM local rings of finite CM type are completely characterized [6, 8, 11, 22, 23]. In dimension two, the complete local rings containing the complex numbers and having finite CM type are also completely classified [2, 9, 12]. They are exactly the rings of Theorem B, that is, the invariant rings $R = \mathbb{C}[\![x, y]\!]^G$ under the action of a finite group G. In this case, $\mathbb{C}[\![x, y]\!]$ is a representation generator for R, that is, contains as direct summands all the indecomposable MCM R-modules.

The main result of this section is a generalization of Theorem B. Again, the proof relies on the process of projectivization, which was described in the introduction.

Theorem 6 *Let* $(R, \mathfrak{m})$ *be a* d*-dimensional CM local ring of finite CM type. Let* M *be a representation generator for* R *(in particular,* M *has a free direct summand). Then* $A := \operatorname{End}_R(M)^{\mathrm{op}}$ *has global dimension at most* $\max\{2, d\}$*. If* $d \geq 2$*, then equality holds.*

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Proof First assume that $d \geq 2$. Let N be a finitely generated left A-module. Take the first $d-1$ steps in a projective resolution of N over A:

$$\mathbf{P}_\bullet : P_{d-1} \xrightarrow{\varphi_{d-1}} \cdots \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \tag{2.1}$$

with $\operatorname{coker} \varphi_1 = N$. For each i, we can use [3, II.2] to write $P_i = \operatorname{Hom}_R(M, M_i)$, where M_i is a direct summand of a direct sum of copies of M. In particular, each M_i is a MCM R-module. Moreover, each φ_i can be written as $\operatorname{Hom}_R(M, f_i)$ for R-homomorphisms $f_i \colon M_i \to M_{i-1}$. This gives the following sequence of MCM R-modules and homomorphisms:

$$\mathbf{C}_\bullet : M_{d-1} \xrightarrow{f_{d-1}} \cdots \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0. \tag{2.2}$$

Since M has a free direct summand and $\operatorname{Hom}_R(M, \mathbf{C}_\bullet)$ is exact, it follows that in fact $\mathbf{C}_\bullet$ is exact. Put $M_d = \ker(f_{d-1})$. Then M_d is a MCM R-module by the depth lemma, and left-exactness of Hom gives an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M, M_d) \longrightarrow P_{d-1} \xrightarrow{\varphi_{d-1}} \cdots \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0. \tag{2.3}$$

Since M_d is MCM, $\operatorname{Hom}_R(M, M_d)$ is A-projective, and N has projective dimension at most d.

To see that the global dimension of A is exactly d, take N to be a simple A-module. Then N has finite length as an R-module, so is of depth zero. A projective resolution of N is in particular an exact sequence of MCM R-modules, and so a projective resolution of length less than d would contradict the depth lemma. Thus N has projective dimension exactly d. Since the global dimension of A is the maximum of the projective dimensions of the simple modules, this finishes the case $d \geq 2$.

If $d < 2$, then we can repeat the first part of the argument, simply taking an A-projective resolution of length 1. The remainder of the proof is the same, showing that $\operatorname{gldim} A \leq 2$. ∎

One can also state the proof above in terms of the adjoint pair $g_* = \operatorname{Hom}_R(M, -)$ and $g^* = - \otimes_A M$. In this context, the fact that $\mathbf{C}_\bullet$ is exact comes down to the facts that (i) M is a projective A-module, so g^* is an exact functor, and (ii) $g_* g^*$ is the identity on projective A-modules.

Following Iyama [14], we extend the definition of representation dimension to rings of positive Krull dimension.

Definition 7 Let T be a complete regular local ring and let R be a T-algebra, finitely generated and free as a T-module. Let $\mathcal{C}$ be the category of R-modules which are finitely generated free T-modules. Let

$$\operatorname{repdim} R = \inf_{N \in \mathcal{C}} \{\operatorname{gldim} \operatorname{End}_R(R \oplus R^* \oplus N)\},$$

where $R^* := \operatorname{Hom}_T(R, T)$.

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Proposition 8 *Let R be a CM complete local ring of finite CM type. Then* $\operatorname{repdim} R \leq \max\{2, \dim R\}$.

Proof By Cohen's structure theorem, R is a finitely generated module over some complete regular local ring T. The MCM R-modules are precisely the R-modules that are free over T. Finally, $R^* \cong \omega_R$ is the canonical module for R, which is MCM. Theorem 6 then shows that $\operatorname{End}_R(R \oplus \omega_R \oplus N)$ has global dimension at most $\max\{2, \dim R\}$, where N is the direct sum of the remaining indecomposable MCM R-modules. ∎

Remark 9 As mentioned in the Introduction, Auslander proved in [1] that if Λ is an Artin algebra of finite representation type, with additive generator M, then $\Gamma := \operatorname{End}_\Lambda(M)$ has representation dimension two. In fact, he showed that Γ is what is now called an *Auslander algebra*, that is, Γ has global dimension two and dominant dimension two. We say that a ring A has *dominant dimension* at least t if, in a minimal injective resolution $0 \to A \to I^\bullet$ of A, the j-th injective module I^j is also projective for $j < t$.

One checks easily that the proof given in [3, VI.5] applies verbatim to CM local rings of finite CM type, and shows that, in the situation of Theorem 6, $A = \operatorname{End}_R(M)$ has dominant dimension two. To see that one cannot hope for higher dominant dimension, consider a three-dimensional CM local ring R of finite CM type, *e.g.* Example 11 or 12. Let M be the direct sum of the indecomposable MCM R-modules, and $A = \operatorname{End}_R(M)$. If A has dominant dimension > 2, then the minimal injective resolution of A is

$$0 \longrightarrow A \longrightarrow \operatorname{Hom}_R(M, I^0) \longrightarrow \operatorname{Hom}_R(M, I^1) \longrightarrow \operatorname{Hom}_R(M, I^2)$$

where each $\operatorname{Hom}_R(M, I^j)$ is a projective-injective A-module. It follows that each I^j is an injective R-module. By the equivalence of categories between $\operatorname{add}(M)$ and projective A-modules then, the minimal injective resolution of M over R is

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2,$$

and applying $\operatorname{Hom}_R(M, -)$ to this injective resolution preserves exactness. This implies that $\operatorname{Ext}^1_R(M, M) = 0$, which is quite false in Examples 11 and 12.

3 Connections with Non-Commutative Crepant Resolutions

Though it is a purely algebraic statement, Theorem 6 is closely related to geometric statements about resolution of singularities. Recent work of M. Van den Bergh [20, 21] has revealed unexpected connections between the (geometric) resolutions of certain rational singularities and the algebraic properties of certain endomorphism rings over their coordinate rings. To make this connection more precise, we quote the following definition of Van den Bergh:

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Definition 10 Let R be a Gorenstein normal domain. A *non-commutative crepant resolution* of R is an R-algebra $A = \mathrm{End}_R(M)$, for a finitely generated reflexive R-module M, such that A is an MCM R-module and $\mathrm{gldim}\, A_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}\, R$.

Non-commutative crepant resolutions were introduced to solve a problem related to their geometric counterparts. A (*geometric*) *crepant resolution* of a scheme X is a projective morphism $f\colon Y \to X$, with Y regular, such that $f^*\omega_X = \omega_Y$, where ω denotes the canonical bundle. A. Bondal and D. Orlov [4] conjecture that if X has a geometric crepant resolution, then any two have equivalent bounded derived categories of coherent sheaves. M. Kapranov and E. Vasserot [16] verify the conjecture of Bondal–Orlov for two-dimensional quotient singularities $\mathbb{C}^2/G$; they show that any geometric crepant resolution is derived equivalent to the non-commutative crepant resolution given by $\mathrm{End}_R(\mathbb{C}[\![x, y]\!])$.

Van den Bergh pushes this point of view into dimension three. He proves [20] that a three-dimensional Gorenstein normal $\mathbb{C}$-algebra with terminal singularities has a non-commutative crepant resolution if and only if it has a geometric crepant resolution, and furthermore that the two crepant resolutions are derived equivalent, establishing the conjecture of Bondal–Orlov in this case. In fact, he conjectures [20, Conjecture 4.6] that all crepant resolutions of a given Gorenstein scheme X, non-commutative as well as geometric, are derived equivalent. He also gives some further examples in which non-commutative crepant resolutions exist.

Theorem 6 implies that if R is a Gorenstein local ring of finite CM type, containing a field and having dimension two or three, then R has a non-commutative crepant resolution. The completion of such a ring is the analytic local ring of one of the simple hypersurface singularities (see, for example, [24]), the MCM modules of which are known. One can thus check that for each of the simple singularities, the endomorphism ring of a representation generator is indeed an MCM module.

In fact, Theorem 6 gives a little more: the reflexive module in the definition of the non-commutative crepant resolution can actually be taken to be MCM. This is consistent with all the other known examples of non-commutative crepant resolutions [20, Remark 4.4].

It is worth pointing out that Theorem 6 also implies that graded Gorenstein local rings of finite CM type (containing $\mathbb{C}$) have rational singularities in dimensions 2 and 3. Van den Bergh shows [20, Proposition 3.3] that if R is a positively graded Gorenstein algebra over a field, with an isolated singularity, and R has a noncommutative crepant resolution of singularities, then R has at most rational singularities. Of course, this is already known for the Gorenstein local rings of finite CM type, following from the complete classification of such rings [5, 12, 17].

Non-commutative crepant resolutions are not yet defined for non-Gorenstein rings. It is tempting to accept Definition 10 verbatim for all CM normal domains, and look to Theorem 6 as a source of non-commutative crepant resolutions in this context as well. This optimism is quickly tempered by the following two examples (the only known non-Gorenstein CM local rings of of finite CM type and dimension ≥ 3).

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Example 11 Let $R = k[[x^2, xy, y^2, yz, z^2, xz]]$, where k is an algebraically closed field of characteristic zero. By [24, 16.10], R has finite CM type. The indecomposable MCM R-modules are the free module of rank one, the canonical module $\omega \cong (x^2, xy, xz)$, and $M := \operatorname{syz}_1^R(\omega)$, which has rank 2. By Theorem 6, $A := \operatorname{End}(R \oplus \omega \oplus M)$ has global dimension 3. However, $\operatorname{depth}_R A = 2$ (this can be easily checked with, say, Macaulay2 [10]). The culprit is M: both $\operatorname{Hom}_R(M, R)$ and $\operatorname{Hom}_R(M, M)$ have depth 2.

Removing M, however, eliminates the problem. Observe that R is a ring of invariants of $k[[x, y, z]]$ under an action of $\mathbb{Z}_2$. Therefore $\operatorname{End}_R(k[[x, y, z]])$ is isomorphic to the twisted group ring $k[[x, y, z]] * \mathbb{Z}_2$, and the twisted group ring has global dimension 3 by [24, Ch. 10]. Finally, since $k[[x, y, z]] \cong R \oplus \omega$ as an R-module, we see that $R \oplus \omega$ gives a noncommutative crepant resolution of R, and exhibits $\operatorname{repdim} R \leq 3$.

Example 12 Let $R = k[[x, y, z, u, v]]/I$, where I is generated by the 2×2 minors of the matrix $\left(\begin{smallmatrix} x & y & u \\ y & z & v \end{smallmatrix}\right)$. Then R has finite CM type [24, 16.12]. The only indecomposable nonfree MCM R-modules are, up to isomorphism,

- the canonical module $\omega \cong (u, v)R$;
- $M := \operatorname{syz}_1^R(\omega)$, isomorphic to the ideal $(x, y, u)R$;
- $N := \operatorname{syz}_2^R(\omega)$, rank two and 6-generated;
- $L := M^\vee$, the canonical dual of M, isomorphic to the ideal $(x, y, z)R$.

In particular, $\omega^* = \operatorname{Hom}_R(\omega, R)$ is isomorphic to M. Then by Theorem 6, $A := \operatorname{End}_R(R \oplus \omega \oplus M \oplus N \oplus L)$ has global dimension 3. Again, A fails to be MCM as an R-module, since none of L^*, N^*, and $\operatorname{Hom}_R(\omega, M)$ are MCM. In this example, $\operatorname{End}_R(R \oplus \omega)$ and $\operatorname{End}_R(R \oplus M)$ are the only endomorphism rings of the form $\operatorname{End}_R(D)$, with D nonfree MCM, that are themselves MCM. I do not know whether the endomorphism ring $\operatorname{End}_R(R \oplus \omega)$ has finite global dimension.

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