J. Austral. Math. Soc. 20 (Series A) (1975), 33–37.

DISPERSIVE AND EXPLOSIVE MAPPINGS

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(Received 9 April 1973)

Communicated by E. S. Barnes

Abstract

Let Q, R be rational numbers and real numbers respectively. We use $V(F)$ and $W(F)$ to denote finite dimensional inner product spaces over F. Given $V(Q)$, we use $V(R)$ for the smallest inner space over R containing $V(Q)$. It is known that an R-homomorphism of $V(R)$ to $W(R)$ is continuous. We prove that if a Q-homomorphism f: $V(Q) \to W(Q)$ cannot be extended to an R-homomorphism $\bar{f}$: $V(R) \to W(R)$, then f is dispersive, i.e., given any $v_0 \in V(Q)$ and $\varepsilon > 0$, the image set f$[D(v_0, \varepsilon)]$, where $D(v_0, \varepsilon) = [v: v \in V(Q), \| v - v_0 \| < \varepsilon]$, is not bounded. It is also shown that some Q-homomorphism f: $V(Q) \to W(Q)$ can be explosive in the sense that for any $v_0 \in V(Q)$ and $\varepsilon > 0$, the set $f[D[v_0, \varepsilon)]$ is dense in $W(Q)$. As a particular case of dispersive and explosive Q-homomorphisms, we show that the algebraic number field isomorphism f: $Q(a) \to Q(\beta)$, where $f(a) = \beta$ and $a \neq \beta$ or $\bar{\beta}$ ($\bar{\beta}$ being complex conjugates of β) is explosive.

1. Introduction

Let Q, R, C denote rational numbers, real numbers, and complex numbers respectively. Analysts have produced abundant results on linear operators over R (R-homomorphisms) while numerically we can cope with only a handful of matrices with rational entries which represent Q-homomorphisms. It is worthwhile to know the possible consequences of mistaking a Q-homomorphism for an R-homomorphism.

It is well known that an R-homomorphism of a finite dimensional inner product space into an inner product space is continuous. In this paper we show that a Q-homomorphism of an inner product space into an inner product space can be "dispersive" and "explosive". Throughout this paper, $V(F)$ and $W(F)$ denote inner product spaces over F. Clearly any $V(Q)$ can be extended to $V(R)$ such that

$$V(R) = \{\sum a_i \underset{\sim}{v}_i : a_i \in R, \underset{\sim}{v}_i \in V(Q)\}$$

with inner product defined by

$$(\sum_i a_i \underset{\sim}{v}_i, \sum_j b_j \underset{\sim}{v}_j) = \sum_{i,j} a_i b_j (\underset{\sim}{v}_i, \underset{\sim}{v}_j),$$

where $(\underset{\sim}{v}_i, \underset{\sim}{v}_j)$ is the inner product of $\underset{\sim}{v}_i, \underset{\sim}{v}_j$ in $V(Q)$.

A mapping $f: V(F) \to W(F)$ is said to be *dispersive* if given any $\underset{\sim}{v}_0 \in V$ and $\varepsilon > 0$, the image set $f[D(\underset{\sim}{v}_0, \varepsilon)]$, where

$$D(\underset{\sim}{v}_0, \varepsilon) = \{\underset{\sim}{v} : \underset{\sim}{v} \in V(F), \|\underset{\sim}{v} - \underset{\sim}{v}_0\| < \varepsilon\},$$

is not bounded, and a dispersive mapping is said to be *explosive* if $f[D(\underset{\sim}{v}_0, \varepsilon)]$ is dense in $W(F)$.

THEOREM 1. *If a Q-homomorphism $f: V(Q) \to W(Q)$ cannot be extended to an R-homomorphism $\bar{f}: V(R) \to W(R)$, then f is dispersive.*

THEOREM 2. *Let $f: V(Q) \to W(Q)$ be Q-homomorphism. If there exist*

$$r_{ij} \in R, \underset{\sim}{v}_j \in V(Q), \underset{\sim}{e}_i \in W(R), \qquad i = 1, 2, \cdots m, \; j = 1, 2, \cdots, n,$$

such that $\{e_1, e_2, \cdots, e_m\}$ is a basis for $W(R)$ and

$$\sum_{j=1}^{n} r_{ij} \underset{\sim}{v}_j = \underset{\sim}{0}, \; \underset{\sim}{e}_i = \sum_{j=1}^{n} r_{ij} f(\underset{\sim}{v}_j) \neq \underset{\sim}{0}, \qquad i = 1, 2, \cdots, m,$$

then f is explosive.

As a particular case of Theorems 1 and 2, we give

THEOREM 3. *Let $f: Q(\alpha) \to Q(\beta)$ be an algebraic number field isomorphism, where $f(\alpha) = \beta$ and $\alpha \neq \beta$ or $\bar{\beta}$, then f is explosive.*

2. A lemma

To prove results, we need

LEMMA 1. *Let $V(Q)$, $W(Q)$ be given. Suppose there exist*

$$\underset{\sim}{v}_i \in V(Q), \underset{\sim}{w}_i \in W(Q), r_i \in R, \qquad i = 1, 2, \cdots, n,$$

such that

$$\sum_{i=1}^{n} r_i \underset{\sim}{v}_i = \underset{\sim}{0} \text{ and } \sum_{i=1}^{n} r_i \underset{\sim}{w}_i = \underset{\sim}{e} \neq \underset{\sim}{0},$$

then given $M > \varepsilon > 0$, $\exists\, a_1, a_2, \cdots, a_n \in Q$ and a positive integer k, such that

$$\left\| \sum_{i=1}^{n} a_i \underset{\sim}{v}_i \right\| < \varepsilon, \tag{2.1}$$

$$\left\| \sum_{i=1}^{n} a_i \underset{\sim}{w}_i \right\| > M, \tag{2.2}$$

and

$$\left\| \sum_{i=1}^{n} a_i \underset{\sim}{w}_i - k\underset{\sim}{e} \right\| < \varepsilon. \tag{2.3}$$

PROOF. Let k be a positive integer satisfying

$$k \left\| \underset{\sim}{e} \right\| > M + \sum_{i=1}^{n} \left\| \underset{\sim}{w}_i \right\|$$

and

$$k\varepsilon > \sum_{i=1}^{n} \{ \| v_i \| + \| w_i \| \}.$$

Now for $i = 1, 2, \cdots, n$, there exist integers p_i, q_i such that

$$r_i = \frac{p_i}{q_i} + \frac{\delta_i}{q_i^2}, \ |\delta_i| < 1, \ q_i > k,$$

where $\delta_i = 0$ if r_i is rational, otherwise δ_i is real and irrational. Let $a_i = kp_i/q_i$. Then

$$\left\| \sum_{i=1}^{n} a_i \underset{\sim}{v}_i \right\| = \left\| \sum_{i=1}^{n} \frac{k\delta_i}{q_i^2} \underset{\sim}{v}_i \right\| \leqq \frac{1}{k} \sum_{i=1}^{n} \left\| \underset{\sim}{v}_i \right\| < \varepsilon$$

implies (2.1) is satisfied.

Next we see that

$$\left\| \sum_{i=1}^{n} a_i \underset{\sim}{w}_i \right\| = \left\| k \sum_{i=1}^{n} r_i \underset{\sim}{w}_i - \sum_{i=1}^{n} \frac{k\delta_i}{q_i^2} \underset{\sim}{w}_i \right\|$$

$$\geqq \left| k \left\| \underset{\sim}{e} \right\| - \sum_{i=1}^{n} \left\| \underset{\sim}{w}_i \right\| \right| > M.$$

This proves (2.2).

Lastly we have

$$\left\| \sum_{i=1}^{n} a_i \underset{\sim}{w}_i - k\underset{\sim}{e} \right\| = \left\| \sum_{i=1}^{n} \frac{k\delta_i}{q_i^2} \underset{\sim}{w}_i \right\| < \left(\frac{1}{k} \right) \sum_{i=1}^{n} \left\| \underset{\sim}{w}_i \right\| < \varepsilon.$$

This proves (2.3).

3. Proof of theorems

Before we prove the theorems, we should mention that as f is a $\boldsymbol{Q}$–homomorphism, $f[D(\underset{\sim}{v}_0, \varepsilon)]$ is not bounded if and only if $f[D(\underset{\sim}{0}, \varepsilon)]$ is not bounded and $f[D(\underset{\sim}{v}_0, \varepsilon)]$ is dense in $W(\boldsymbol{Q})$ if and only if $f[D(\underset{\sim}{0}, \varepsilon)]$ is dense in $W(\boldsymbol{Q})$.

PROOF OF THEOREM 1. If f cannot be extended to an $\boldsymbol{R}$-homomorphism, then $\bar{f}(\sum_{i=1}^{n} r_i \underset{\sim}{v}_i)$ cannot be well defined and so for some

https://doi.org/10.1017/S1446788700023892 Published online by Cambridge University Press

$$r_1, r_2, \cdots, r_n, s_1, s_2, \cdots, s_n \in \boldsymbol{R},$$

$$\sum_{i=1}^{n} r_i \underset{\sim}{v}_i = \sum_{i=1}^{n} s_i \underset{\sim}{v}_i \text{ but } \sum_{i=1}^{n} r_i f(\underset{\sim}{v}_i) \neq \sum_{i=1}^{n} s_i f(\underset{\sim}{v}_i)$$

Hence

$$\sum_{i=1}^{n} (r_i - s_i)\underset{\sim}{v}_i = \underset{\sim}{0}, \text{ while } \sum_{i=1}^{n} (r_i - s_i) f(\underset{\sim}{v}_i) \neq 0.$$

By Lemma 1, now, with r_i replaced by $r_i - s_i$, if $M > \varepsilon > 0$, $\exists\, a_1, a_2, \cdots, a_n \in \boldsymbol{Q}$ such that

$$\Big\| \sum_{i=1}^{n} a_i \underset{\sim}{v}_i \Big\| < \varepsilon \text{ but } \Big\| \sum_{i=1}^{n} a_i f(\underset{\sim}{v}_i) \Big\| > M.$$

This essentially proves Theorem 1.

PROOF OF THEOREM 2. Take any $\underset{\sim}{w} \in \boldsymbol{W}(\boldsymbol{Q})$. Then $\underset{\sim}{w} = \sum_{i=1}^{m} s_i \underset{\sim}{e}_i$ for some $s_1, s_2, \cdots, s_n \in \boldsymbol{R}$. Theorem 2 is proved if we can show that given $\varepsilon > 0$, $\exists\, \underset{\sim}{v} \in \boldsymbol{V}$ such that $\| \underset{\sim}{v} \| < \varepsilon$ and $\| f(\underset{\sim}{v}) - \underset{\sim}{w} \| < \varepsilon$.

Lemma 1 ensures that for each $i = 1, 2, \cdots, m$, $\exists\, a_{i_1}, a_{i_2}, \cdots a_{i_n} \in \boldsymbol{Q}$ and integer $k_i > |s_i|$ such that

$$\Big\| \sum_{j=1}^{n} a_{ij} \underset{\sim}{v}_j \Big\| < \frac{\varepsilon}{2m}$$

and

$$\Big\| \sum_{j=1}^{n} a_{ij} f(\underset{\sim}{v}_j) - k_i \underset{\sim}{e}_i \Big\| < \frac{\varepsilon}{2m}.$$

Since $k_i > |s_i|$, we can find $b_i \in \boldsymbol{Q}$ such that

$$|b_i| < 1 \text{ and } |b_i k_i - s_i| \, \| e_i \| < \frac{\varepsilon}{2m}.$$

Putting

$$\underset{\sim}{v} = \sum_{i=1}^{m} b_i \sum_{j=1}^{n} a_{ij} \underset{\sim}{v}_j,$$

we see that

$$\| \underset{\sim}{v} \| \leqq \sum_{i=1}^{m} |b_i| \Big\| \sum_{j=1}^{n} a_{ij} \underset{\sim}{v}_j \Big\| < \frac{m\varepsilon}{2m} < \varepsilon,$$

and that

$$\| f(\underset{\sim}{v}) - \underset{\sim}{w} \| \leqq \sum_{i=1}^{m} |b_i| \Big\| \sum_{j=1}^{n} a_{ij} \underset{\sim}{v}_j - k_i \underset{\sim}{e}_i \Big\| + \sum_{i=1}^{m} |b_i k_i - s_i| \, \| \underset{\sim}{e}_i \|$$

$$< \frac{m\varepsilon}{2m} + \frac{m\varepsilon}{2m} = \varepsilon.$$

This proves Theorem 2.

PROOF OF THEOREM 3. We may regard $Q(\alpha)$ and $Q(\beta)$ as inner product spaces over Q. So we let $V(Q) = Q(\alpha)$ and $W(Q) = Q(\beta)$. First suppose α, β are both real. Then $W(R) = R$ may be regarded as a 1-dimensional inner product space over R. Remembering $\alpha \neq \beta$ and $f(\alpha) = \beta$, we have

$$1 \circ \alpha + (-\alpha) \circ 1 = 0 \text{ but } e_1 = 1 \circ f(\alpha) + (-\alpha) f(1) \neq 0.$$

Also, $\{e_1\}$ is a basis for $W(R)$. Here f is clearly explosive by Theorem 2.

Suppose now β is not real. Then $W(R) = C$ may be regarded as a 2-dimensional inner product space over R. It is easy to see that every element in $Q(\alpha)$ can be expressed as $r_1 + r_2\alpha$, where $r_1, r_2 \in R$. Hence for some $r_1, r_2 \in R$, we have

$$\alpha^2 + r_1 + r_2\alpha = 0 = \alpha^3 + r_1\alpha + r_2\alpha^2.$$

If in addition

$$\beta^2 + r_1 + r_2\beta = 0 = \beta^3 + r_1\beta + r_2\beta^2,$$

then, as $\bar{\beta}^2 + r_1 + r_2\bar{\beta} = 0$, we have

$$0 = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \bar{\beta} \\ \alpha^2 & \beta^2 & \bar{\beta}^2 \end{vmatrix} = (\alpha - \beta)(\beta - \bar{\beta})(\bar{\beta} - \alpha).$$

This contradicts $\alpha\bar{\alpha} \neq \beta\bar{\beta}$ and $\beta \neq \bar{\beta}$. Putting $e_1 = \beta^2 + r_1 + r_2\beta$ and $e_2 = \beta^3 + r_1\beta + r_2\beta^2$, remembering β is not real and C is 2-dimensional, we see that $\{e_1, e_2\}$ is a basis for $W(R)$. It now follows from Theorem 2 that f is explosive.

Acknowledgement

The author would like to thank his colleagues R. Berghout and J. Giles for their help in the preparation of this paper.

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