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ON AN ESTIMATE OF THE PARTIAL SUMS OF VILENKIN-FOURIER

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ABSTRACT. We show that the partial sums S_nf of the Vilenkin-Fourier series of $f \in L^1$ are of exponential type off any set where the Hardy-Littlewood maximal function of f is bounded. It then follows that $S_{n_k}f(x) = o(\log\log n_k)$ a.e. for any lacunary sequence $\{n_k\}$. Our results are Vilenkin-Fourier series analogues of those of R. A. Hunt [1].

1. **Introduction.** Let $\{p_i\}_{i\geq 0}$ be a sequence of integers with $p_i \geq 2$, and $G = \prod_{i=0}^{\infty} Z_{p_i}$ be the direct product of cyclic groups of order p_i. For $x = \{x_k\} \in G$, define $\phi_k(x) = \exp(2\pi i x_k / p_k)$, $k = 0, 1, 2, \ldots$. The set of characters of G consists of all finite products of $\{\phi_k\}$, which we enumerate in the following manner. Let $m_0 = 1$, $m_k = \prod_{i=0}^{k-1} p_i$, $k = 1, 2, \ldots$. Express each nonnegative integer n as a finite sum $n = \sum_{k=0}^{\infty} \alpha_k m_k$, where $0 \leq \alpha_k < p_k$, and let $\chi_n = \prod_{k=0}^{\infty} \phi_k^{\alpha_k}$. For the case $p_i = 2$, $i = 0, 1, \ldots$, G is the dyadic group, $\{\phi_k\}$ are the Rademacher functions and $\{\chi_n\}$ are the Walsh functions. In general, the system $(G, \{\chi_n\})$ is a realization of the multiplicative Vilenkin system studied in [5]. In this paper, there is no restriction on the orders $\{p_i\}$.

We consider Fourier series with respect to $\{\chi_n\}$. Let μ be the Haar measure on G normalized by $\mu(G) = 1$. For $f \in L^1$, let

$$S_nf(x) = \int_G f(t) \sum_{j=0}^{n-1} \chi_j(x-t)\, d\mu(t), \qquad n = 1, 2, \ldots$$

be the n^{th} partial sum of the Vilenkin-Fourier series of f. It is shown in [6] that there are absolute constants C and C_p such that, for $n = 1, 2, \ldots$,

$$\mu\{|S_nf| > y\} \leq Cy^{-1}\|f\|_1, \quad f \in L^1,\ y > 0, \tag{1.1}$$

and

$$\|S_nf\|_p \leq C_p\|f\|_p, \quad f \in L^p, \quad 1 < p < \infty. \tag{1.2}$$

In this paper, we give a refinement of the above estimates and show that for $f \in L^1$, S_nf is of exponential type off any set where the Hardy-Littlewood maximal function of f is bounded.

Before we define the Hardy-Littlewood maximal function that is appropriate for the study of Vilenkin-Fourier series, we introduce the following notation. We identify G

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with the unit interval $(0,1)$ by associating with each $\{x_i\} \in G$, $0 \le x_i < p_i$, the point $\sum_{i=0}^{\infty} x_i m_{i+1}^{-1} \in (0,1)$. If we disregard the countable set of p_i-rationals, this mapping is one-one, onto and measure-preserving. Let $\{G_k\}$ be the sequence of subgroups of G defined by

$$G_0 = G, \quad G_k = \prod_{i=0}^{k-1}\{0\} \times \prod_{i=k}^{\infty} Z_{p_i}, \quad k = 1,2,\dots .$$

On the interval $(0,1)$, cosets of G_k are intervals of the form $\left(jm_k^{-1}, (j+1)m_k^{-1}\right)$, $j = 0,1,\dots, m_k - 1$. A subset I of a coset $x + G_k$, $x \in G$, $k = 0,1,\dots$, is called a generalized interval if I is a union of cosets of G_{k+1}, and I is an interval when $x + G_k$ is considered as a circle. The collection of all generalized intervals is denoted by $\mathcal{J}$.

For $f \in L^1$, the Hardy-Littlewood maximal function of f is defined by

$$Mf(x) = \sup_{\substack{x \in I \\ I \in \mathcal{J}}} \frac{1}{\mu(I)} \int_I |f| \, d\mu.$$

This maximal function was first introduced by P. Simon in [3]. He also showed that there are absolute constants C and C_p such that

$$\mu\{Mf > y\} \le Cy^{-1}\|f\|_1, \quad f \in L^1,\ y > 0, \tag{1.3}$$

and

$$\|Mf\|_p \le C_p \|f\|_p, \quad f \in L^p,\ 1 < p \le \infty.$$

(See also [7].)

We obtain the following Vilenkin-Fourier series analogues of results of R. A. Hun [1]. (See also Muckenhoupt [2]).

THEOREM 1. *There is an absolute constant C such that, for $n = 1,2,\dots$,*

$$\mu\{Mf \le y, |S_n f| > \lambda y\} \le Ce^{-\lambda/C},\ f \in L^1,\ y > 0,\ \lambda > 0. \tag{1.4}$$

THEOREM 2. *Let $\{n_k\}_{k\ge 1}$ be a lacunary sequence, i.e., there is $\alpha > 1$ such that $n_{k+1} > \alpha n_k$, $k = 1,2,\dots$. Then there is an absolute constant C such that*

$$\mu\left\{\sup_k \frac{|S_{n_k} f|}{\log\log n_k} > y\right\} \le Cy^{-1}\|f\|_1,\ f \in L^1,\ y > 0. \tag{1.5}$$

Moreover, $S_{n_k} f(x) = o(\log\log n_k)$ a.e. for $f \in L^1$.

For the full sequence of partial sums, there is the following analogue of a result for trigonometric series. (See [8, I, pp. 65–66].)

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THEOREM 3. *There is an absolute constant C such that*

$$\mu\left\{\sup_{n\geq 2}\frac{|S_nf|}{\log n} > y\right\} \leq Cy^{-1}\|f\|_1, \quad f\in L^1,\ y>0.$$

Moreover, $S_nf(x) = o(\log\ n)$ a.e. for $f \in L^1$.

The constants C in the above theorems are independent of the orders $\{p_i\}$.

Theorem 2 is a consequence of Theorem 1. As it is shown in [1], (1.3) and the uniform exponential estimates in Theorem 1 imply that

$$\sup_k \frac{|S_{n_k}f(x)|}{\log\log n_k} < \infty \quad \text{a.e.}$$

A theorem of E. M. Stein [4] then yields (1.5). Since polynomials in $\{\chi_n\}$ are dense in L^1, the "o" result follows. Theorem 3 can be obtained from Theorem 1 in a similar manner.

Our proof of Theorem 1 consists of adapting the method used in [1] to the Vilenkin system. In what follows C will denote an absolute constant which may vary from line to line.

2. **Proof of Theorem 1.** We recall some properties of Vilenkin-Fourier series. Let $S_n^*f = \bar{\chi}_n S_n(f\chi_n)$ be the n^{th} modified partial sum, $n = 1, 2, \ldots$. It is shown in [6] that if $n = \sum_{k=0}^{\infty}\alpha_k m_k$, $0 \leq \alpha_k < p_k$, then

$$S_n^*f = \sum_{k=0}^{\infty} S^*_{\alpha_k m_k}f \tag{2. 1}$$

and

$$S^*_{\alpha_k m_k}f(x) = \frac{1}{\mu(G_k)}\int_{x+G_k} f(t)\phi^{-\alpha_k}(x-t)\Big(\sum_{j=0}^{\alpha_k-1}\phi_k^j(x-t)\Big)\,d\mu(t).$$

(The sum on the right is interpreted to be zero if $\alpha_k = 0$.) $S^*_{\alpha_k m_k}f$ can be expressed in terms of conjugate functions, defined by

$$H_kf(x) = \frac{1}{2}\frac{1}{\mu(G_k)}\int_{(x+G_k)\cap\{x_k\neq t_k\}} f(t)\cot\big(\pi(x_k-t_k)/p_k\big)\,d\mu(t), \tag{2. 2}$$

$f\in L^1$, $x = \{x_k\} \in G$. We have

$$\begin{aligned} S^*_{\alpha_k m_k}f(x) =& \frac{\alpha_k}{\mu(G_k)}\int_{(x+G_k)\cap\{x_k=t_k\}} f(t)\,d\mu(t)\\ &+\frac{1}{2}\phi_k^{-\alpha_k}(x)\frac{1}{\mu(G_k)}\int_{(x+G_k)\cap\{x_k\neq t_k\}} f(t)\phi_k^{\alpha_k}(t)\,d\mu(t)\\ &-\frac{1}{2}\frac{1}{\mu(G_k)}\int_{(x+G_k)\cap\{x_k\neq t_k\}} f(t)d\mu(t)\\ &+i\phi_k^{-\alpha_k}(x)H_k(f\phi_k^{\alpha_k})(x) - iH_kf(x). \end{aligned} \tag{2. 3}$$

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Because of these special properties of the modified partial sums, we shall prove Theorem 1 by establishing (1.4) with S_n replaced by S_n^*.

Let $n = \sum_{k=0}^{\infty} \alpha_k m_k$, $0 \leq \alpha_k < p_k$, $f \in L^1$ and $y > \|f\|_1$. Applying the modified Calderón-Zygmund decomposition lemma [6, Lemma 2] to the function f and the value $3y$, we obtain a collection $\mathcal{C} = \{I_j\}$ of disjoint generalized intervals such that

$$3y < \frac{1}{\mu(I_j)} \int_{I_j} |f|\, d\mu \leq 9y, \quad I_j \in \mathcal{C}$$

and

$$|f(x)| \leq 3y \text{ for a.e. } x \notin \bigcup_j I_j \equiv \Omega.$$

We write $\mathcal{C} = \bigcup_{k=0}^{\infty} \mathcal{C}_k$, where each $I_j \in \mathcal{C}_k$ is a union of cosets of G_{k+1} and is a proper subset of a coset of G_k.

Let $I_j \in \mathcal{C}_k$ and I_j be contained in the coset $x + G_k$. If $\mu(I_j) \geq \mu(G_k)/3$, define $3I_j = x + G_k$. If $\mu(I_j) < \mu(G_k)/3$, consider $x + G_k$ as a circle, and define $3I_j$ to be the interval in this circle which has the same center as I_j and has measure $\mu(3I_j) = 3\mu(I_j)$. If $x \in 3I_j$, then

$$Mf(x) \geq \frac{1}{\mu(3I_j)} \int_{3I_j} |f|\, d\mu \geq \frac{1}{3\mu(I_j)} \int_{I_j} |f|\, d\mu > y.$$

Hence, if we let $\Omega^* = \bigcup_j (3I_j)$, we have $\{Mf \leq y\} \subset {}^c\Omega^*$.

Next we decompose f as $f = g + b$ with

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \Omega \\ a_{ij} + b_{kj}\phi_k^{-\alpha_k}(x) & \text{if } x \in I_j \in \mathcal{C}_k, \end{cases}$$

where a_{kj}, b_{kj} are constants chosen in such a way that

$$\int_{I_j} f\, d\mu = \int_{I_j} (a_{kj} + b_{kj}\phi_k^{-\alpha_k})\, d\mu,$$

and

$$\int_{I_j} f\phi_k^{\alpha_k}\, d\mu = \int_{I_j} (a_{kj} + b_{kj}\phi_k^{-\alpha_k})\phi_k^{\alpha_k}\, d\mu.$$

It is shown in [6, Lemma 2] that g and $b = f - g$ satisfy

(2.4) $$|g| \leq Cy \text{ a.e.},$$

(2.5) $$b(x) = 0 \text{ if } x \notin \Omega,$$

(2.6) $$\int_{I_j} b\, d\mu = 0 \text{ for every } I_j \in \mathcal{C},$$

(2.7) $$\int_{I_j} b\phi_k^{\alpha_k}\, d\mu = 0 \text{ for every } I_j \in \mathcal{C}_k\ k = 0, 1, \dots,$$

and

(2.8) $$\int_{I_j} |b|\, d\mu \leq Cy\mu(I_j) \text{ for every } I_j \in \mathcal{C}.$$

To estimate $S_n^* g$, we use the following exponential estimate for L^∞ functions.

THEOREM 4. *There is an absolute constant C such that, for $n = 1, 2, \ldots$,*

$$\mu\{|S_n f| > y\} \le Ce^{-y/(C\|f\|_\infty)}, \quad f \in L^\infty,\ y > 0.$$

PROOF. Applying the Marcinkiewicz interpolation theorem [8, II, p. 112] to (1.1) and the case $p = 2$ of (1.2), we obtain the case $1 < p < 2$ of (1.2) with $C_p = O\big(1/(p-1)\big)$ as $p \longrightarrow 1$. By duality, we get (1.2) for $2 < p < \infty$ with $C_p = O(p)$ as $p \longrightarrow \infty$. Theorem 4 then follows from an extrapolation theorem [8, II, p. 119].

We now return to the proof of Theorem 1. From (2.4) and Theorem 4, we have

$$\mu\{|S_n^* g| > \lambda y/2\} \le Ce^{-\lambda/C}.$$

Since $S_n^* f = S_n^* g + S_n^* b$, Theorem 1 will be proved if we show

$$\mu\{x \in {}^c\Omega^* : |S_n^* b| > \lambda y/2\} \le Ce^{-\lambda/C}. \tag{2.9}$$

To do this we expand $S_n^* b$ in terms of the conjugate functions as in (2.1) and (2.3). For $x \notin \Omega^*$, it follows from (2.5), (2.6) and (2.7) that the first three terms in (2.3) vanish, and we are left with

$$S_n^* b(x) = i \sum_{k=0}^{\infty} \big\{\phi_k^{-\alpha_k}(x) H_k(b\phi_k^{\alpha_k})(x) - H_k b(x)\big\}.$$

(See the explanation in [6] pp. 317–318.) (2.9) will be proved if we show that the measures of the sets

$$A = \Big\{x \in {}^c\Omega^* : \sum_{k=0}^{\infty} |H_k(b\phi_k^{\alpha_k})(x)| > \lambda y/4\Big\}$$

and

$$B = \Big\{x \in {}^c\Omega^* : \sum_{k=0}^{\infty} |H_k b(x)| > \lambda y/4\Big\}$$

are bounded by $Ce^{-\lambda/C}$.

For the first set, we have

$$\mu(A) \le 4(\lambda y)^{-1} \sum_{k=0}^{\infty} \int_{{}^c\Omega^*} \chi_A(x) |H_k(b\phi_k^{\alpha_k})(x)|\, d\mu(x). \tag{2.10}$$

If $x \notin \Omega^*$, it follows from (2.2), (2.5), (2.6) and (2.7) that

$$H_k(b\phi_k^{\alpha_k})(x) = \frac{1}{2} \frac{1}{\mu(G_k)} \sum_{\substack{I_j \subset x + G_k \\ I_j \in \mathcal{C}_k}} \int_{I_j} b(t) \phi_k^{\alpha_k}(t) \times \left\{\cot\Big(\frac{\pi(x_k - t_k)}{p_k}\Big) - \cot\Big(\frac{\pi(x_k - t_k^j)}{p_k}\Big)\right\} d\mu(t),$$

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where $t^j = \{t^j_k\}_{k\geq 0}$ is any fixed point in I_j. (See [6], p. 318). Let I be any coset of G_k. Fubini's theorem gives

$$\int_{{}^c\Omega^*\cap I} \chi_A(x)|H_k(b\phi_k^{\alpha_k})(x)|\,d\mu(x)$$

$$(2.11)\qquad \leq \frac{1}{2}\frac{1}{\mu(I)} \sum_{\substack{I_j\subset I\\ I_j\in \mathcal{C}_k}} \int_{I_j} |b(t)| \int_{I\cap\ {}^c(3I_j)} \chi_A(x)$$

$$\times \left|\cot\Big(\frac{\pi(x_k-t_k)}{p_k}\Big) - \cot\Big(\frac{\pi(x_k-t^j_k)}{p_k}\Big)\right| d\mu(x)\,d\mu(t).$$

Let $3^{\ell+1}I_j = 3(3^\ell I_j)$, $\ell = 1,2,\ldots$. If $3I_j \neq I$, write $I\cap\ {}^c(3I_j) = \bigcup_{\ell=1}^{L_j-1} I\cap (3^{\ell+1}I_j\setminus 3^\ell I_j)$, where $L_j = \min\{\ell \geq 1 : 3^\ell I_j = I\}$. For $1 \leq \ell \leq L_j - 1$, $x \in I\cap (3^{\ell+1}I_j\setminus 3^\ell I_j)$ and $t, t^j \in I_j$, we have

$$\begin{aligned}
&\left|\cot\Big(\frac{\pi(x_k-t_k)}{p_k}\Big) - \cot\Big(\frac{\pi(x_k-t^j_k)}{p_k}\Big)\right| \\
&\quad = \left|\sin\Big(\frac{\pi(t_k-t^j_k)}{p_k}\Big)\Big/\Big\{\sin\Big(\frac{\pi(x_k-t_k)}{p_k}\Big)\sin\Big(\frac{\pi(x_k-t^j_k)}{p_k}\Big)\Big\}\right| \\
&\quad \leq C\mu(I_j)\mu(I)/\big(\mu(3^{\ell-1}I_j)\big)^2 \\
&\quad \leq C3^{-\ell}\mu(I)/\mu(3^{\ell+1}I_j).
\end{aligned}$$

Summing over ℓ, substituting into (2.11) and using (2.8), we obtain

$$\begin{aligned}
&\int_{{}^c\Omega^*\cap I} \chi_A(x)|H_k(b\phi_k^{\alpha_k})(x)|\,d\mu(x) \\
&\quad \leq Cy \sum_{\substack{I_j\subset I\\ I_j\in \mathcal{C}_k}} \int_{I_j} \sum_{\ell=1}^{\infty} 3^{-\ell} \frac{1}{\mu(3^{\ell+1}I_j)} \int_{3^{\ell+1}I_j} \chi_A(x)\,d\mu(x)\,d\mu(t) \\
&\quad \leq Cy \sum_{\substack{I_j\subset I\\ I_j\in \mathcal{C}_k}} \int_{I_j} M\chi_A(t)\,d\mu(t),
\end{aligned}$$

since the average of χ_A over $3^{\ell+1}I_j$ is bounded by $M\chi_A(t)$, $t \in I_j$. We now sum over all cosets I of G_k and then over all k. From (2.10) we obtain

$$\mu(A) \leq C\lambda^{-1}\int_G M\chi_A\,d\mu.$$

Since $M\chi_A \leq 1$, $\int_G M\chi_A\,d\mu = \int_0^1 \mu\{M\chi_A > y\}\,dy$. By (1.3), $\mu\{M\chi_A > y\} \leq \min\{1, Cy^{-1}\mu(A)\}$. Hence

$$\begin{aligned}
\mu(A) &\leq C\lambda^{-1}\Big\{\int_0^{\mu(A)} dy + \int_{\mu(A)}^1 Cy^{-1}\mu(A)\,dy\Big\} \\
&= C\lambda^{-1}\mu(A)\big(1 - C\log\mu(A)\big).
\end{aligned}$$

If $\mu(A) = 0$, there is nothing to prove. Otherwise, dividing by $\mu(A)$ and rearranging, we obtain

$$\mu(A) \leq Ce^{-\lambda/C}.$$

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The estimate of $\mu(B)$ is similar, using (2.6) instead of (2.7). This completes the proof of Theorem 1.

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