RUIN IN THE PERTURBED COMPOUND POISSON RISK PROCESS UNDER INTEREST FORCE

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Abstract

In this paper, we study ruin in a perturbed compound Poisson risk process under stochastic interest force and constant interest force. By using the technique of stochastic control, we show that the ruin probability in the perturbed risk model is always twice continuously differentiable provided that claim sizes have continuous density functions. In the perturbed risk model, ruin may be caused by a claim or by oscillation. We decompose the ruin probability into the sum of two ruin probabilities; one is the probability that ruin is caused by a claim and the other is the probability that ruin is caused by oscillation. Integrodifferential equations for these ruin probabilities are derived when the interest force is constant. When the claim sizes are exponentially distributed, explicit solutions of the ruin probabilities are derived from the integrodifferential equations. Numerical examples are given to illustrate the effects of diffusion volatility and interest force on the ruin probabilities.

Keywords: Compound Poisson risk process; diffusion; Brownian motion; jump diffusion process; ruin probability; Hamilton–Jacobi–Bellman equation; viscosity solution; confluent hypergeometric function; Kummer's confluent hypergeometric equation

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1. Introduction

Let

$$U_t = ct - \sum_{k=1}^{N(t)} Y_k + \sigma W_t = ct - S_t + \sigma W_t, \qquad t \geq 0, \tag{1.1}$$

be a compound Poisson risk process perturbed by a diffusion or a jump diffusion process, where $c > 0$ is the rate of premium; $\{Y_k, \ k = 1, 2, \ldots\}$ is a sequence of independent and identically distributed nonnegative random variables, denoting claim sizes; $\{N(t), \ t \geq 0\}$ is a Poisson process with rate $\lambda > 0$, representing the number of claims up to time t; $\{W_t, \ t \geq 0\}$ is a standard Brownian motion; $\sigma > 0$ is a constant, representing the diffusion volatility parameter; and $S_t = \sum_{k=1}^{N(t)} Y_k$ is the compound Poisson process. In addition, $\{Y_k, \ k = 1, 2, \ldots\}$, $\{N(t), \ t \geq 0\}$, and $\{W_t, \ t \geq 0\}$ are independent. As pointed out in Dufresne and Gerber (1991), the perturbed compound Poisson risk process adds an uncertainty to premium income or an additional uncertainty to aggregate claims.

Assume that the surplus U_t is invested continuously at a constant interest force $\delta > 0$ and let X_t denote the surplus at time t under the constant interest force δ with an initial surplus $u \geq 0$; thus,

$$X_t = e^{\delta t}\left(u + \int_0^t e^{-\delta s}\, dU_s\right), \quad t \geq 0, \qquad X_0 = u. \tag{1.2}$$

More generally, if the surplus U_t is invested continuously at a stochastic interest force and the deterministic function δt in (1.2) is another jump diffusion process δ_t with

$$\delta_t = \delta t + \sigma_R W_{R,t} + \sum_{i=1}^{N_R(t)} \ln(1 + X_i) = \delta t + \sigma_R W_{R,t} + S_{R,t}$$

or, equivalently, the deterministic function $e^{\delta t}$ in (1.2) is replaced by a stochastic process e^{δ_t} given by

$$e^{\delta_t} = e^{\delta t + \sigma_R W_{R,t}} \prod_{i=1}^{N_R(t)} (1 + X_i),$$

then the surplus at time t under the stochastic interest force δ_t with an initial surplus $u \geq 0$ is given by

$$X_t = e^{\delta_t}\left(u + \int_0^t e^{-\delta_s}\, dU_s\right), \quad t \geq 0, \qquad X_0 = u. \tag{1.3}$$

Here σ_R is a constant; $\{W_{R,t},\ t \geq 0\}$ is a standard Brownian motion; $\{N_R(t),\ t \geq 0\}$ is a Poisson process with rate $\lambda_R \geq 0$, $\{X_i,\ i = 1, 2, \ldots\}$ is a sequence of independent and identically distributed random variables such that $1 + X_i > 0$ for all $i = 1, 2, \ldots$; $\{W_{R,t},\ t \geq 0\}$, $\{N_R(t),\ t \geq 0\}$, and $\{X_i,\ i = 1, 2, \ldots\}$ are independent; and $S_{R,t} = \sum_{i=1}^{N_R(t)} \ln(1 + X_i)$ is a compound Poisson process. Furthermore, the process $\{\delta_t,\ t \geq 0\}$ is assumed to be independent of the process $\{U_t,\ t \geq 0\}$ and, by convention, $\sum_{i=1}^0 = 0$ and $\prod_{i=1}^0 = 1$.

The risk process (1.3) has been studied by Paulsen and Gjessing (1997) and expressed in the form of their Equation (2.4). The risk process (1.2) is a special case of the risk process (1.3) when $\sigma_R = 0$ and $\lambda_R = 0$.

Denote the ruin time of the risk process (1.3) by T, i.e. $T = \inf\{t: X_t < 0\}$ or $T = \infty$ if $X_t \geq 0$ for all $t \geq 0$. Define the ruin probability with an initial surplus $u \geq 0$ by

$$\psi(u) = \Pr\{T < \infty \mid X_0 = u\} = \Pr\{X_t < 0 \text{ for some } t \geq 0 \mid X_0 = u\}.$$

In this perturbed risk model, ruin may occur in two different situations. In one situation, ruin is caused by a claim and, in the other, ruin is caused by oscillation. We let

$$T_s = \begin{cases} \inf\{t: X_t < 0,\ X_h > 0,\ 0 < h < t\}, \\ \infty \quad \text{if } X_t \geq 0 \text{ for all } t \geq 0, \end{cases}$$

i.e. T_s is the time at which ruin is caused by a claim. Furthermore, we let

$$T_d = \begin{cases} \inf\{t: X_t = 0,\ X_h > 0,\ 0 < h < t\}, \\ \infty \quad \text{if } X_t \geq 0 \text{ for all } t \geq 0, \end{cases}$$

i.e. T_d is the time at which ruin is caused by oscillation. Then $T = \min\{T_s, T_d\}$. Moreover, we denote the ruin probabilities in the two situations by

$$\psi_s(u) = \Pr\{T_s < \infty \mid X_0 = u\}$$

and

$$\psi_d(u) = \Pr\{T_d < \infty \mid X_0 = u\},$$

respectively.

It is obvious that the ruin probability $\psi(u)$ can be decomposed as follows:

$$\psi(u) = \psi_s(u) + \psi_d(u), \qquad u \geq 0. \tag{1.4}$$

In addition, it follows from the oscillating nature of the sample paths of X_t that

$$\psi_d(0) = \psi(0) = 1 \quad \text{and} \quad \psi_s(0) = 0. \tag{1.5}$$

See Dufresne and Gerber (1991) for a detailed discussion of (1.4) and (1.5) in the perturbed compound Poisson risk process (1.1).

Given that ruin has occurred, the probability that it is due to a claim is given by $\Pr\{T_s < \infty \mid T < \infty\} = \psi_s(u)/\psi(u)$ and the probability that it is due to oscillation is given by $\Pr\{T_d < \infty \mid T < \infty\} = \psi_d(u)/\psi(u)$.

Ruin in the perturbed compound Poisson process has been studied extensively in the literature; see, for example, Dufresne and Gerber (1991), Gerber and Landry (1998), Paulsen and Gjessing (1997), Yang and Zhang (2001), and references therein. Recently, ruin under interest force has attracted increasing attention in the risk theory literature. For instance, ruin in the compound Poisson risk process under constant interest force has been studied in Asmussen (2000), Cai and Dickson (2002), Paulsen and Gjessing (1997), Sundt and Teugels (1995), and references therein.

In the study of ruin probability, a commonly used method is to first derive integrodifferential or differential equations satisfied by the ruin probability and then to discuss the properties and solutions of the ruin probability based on the equations. For example, let $\phi(u)$ be the ruin probability in the perturbed compound Poisson process without interest force or in the risk process (1.1), i.e. $\phi(u) = \Pr\{u + U_t < 0 \text{ for some } t \geq 0\}$. Using a heuristic argument, Dufresne and Gerber (1991) derived an integrodifferential equation (their Equation (2.1)) for $\phi(u)$ under the assumption that $\phi(u)$ is twice continuously differentiable. The same argument and assumption were used in Gerber and Landry (1998), and references therein, in another study of ruin in the risk process (1.1). However, these references did not explain under what conditions $\phi(u)$ is twice continuously differentiable.

For the ruin probability $\psi(u)$ in the general risk process (1.3), Paulsen and Gjessing (1997) derived, in their Theorem 2.1, the integrodifferential equation for $\psi(u)$ by using Itô's formula under the assumption that $\psi(u)$ is twice continuously differentiable with a bounded first derivative. Again, Paulsen and Gjessing (1997) did not discuss under what conditions $\psi(u)$ is twice continuously differentiable.

Theoretically, it is necessary to discuss the differentiability of the ruin probability when one derives integrodifferential or differential equations for it. Such a discussion for the ruin probability in the compound Poisson risk process can be found in Grandell (1991), who used renewal and differential arguments. Furthermore, Wang and Wu (2001) and Cai (2004) have studied the differentiability of the ruin probability $\psi(u)$ in the risk process (1.3) when $\lambda_R = 0$.

They first used a renewal argument to derive the integral equations for $\psi(u)$, and then gave the conditions under which $\psi(u)$ is twice, or more times, continuously differentiable. Their conditions involve the differentiability of the density function of the claim size Y_1 and relationships between the parameters λ, δ, and σ. Unfortunately, neither the methods of Wang and Wu (2001) and Cai (2004) nor those of Grandell (1991) apply to the differentiability of the ruin probability in (1.3) in general cases.

In this paper, we first use the technique of stochastic control and the result of Hipp and Plum (2003) to prove that the ruin probability $\psi(u)$ in the risk process (1.3) is always twice continuously differentiable, provided that the claim size Y_1 has a continuous density function, and to give the integrodifferential equation for $\psi(u)$. We then discuss ruin probabilities in the perturbed compound Poisson risk process with constant interest force in detail. In this case, we derive integrodifferential equations for the ruin probability $\psi(u)$ using the Hamilton–Jacobi–Bellman (HJB) equation and the ruin probabilities $\psi_s(u)$ and $\psi_d(u)$ using a differential argument and the Itô formula. When claim sizes are exponentially distributed, explicit solutions of these ruin probabilities are given. Numerical examples are given to illustrate these results and the effects of diffusion volatility and interest force on the ruin probabilities.

2. The HJB equation and differentiability of the ruin probability

In this section, using stochastic control techniques and the result of Hipp and Plum (2003), we prove that the ruin probability $\psi(u)$ in the risk process (1.3) is always twice continuously differentiable provided that the claim sizes have a continuous density function.

We first describe the ideas of the proof. It is well known in stochastic control theory that, in many cases, a value function is not smooth enough to satisfy the HJB equation in the classical or usual sense. A weak formulation of the solution to the HJB equation has been used in the literature. A commonly used weak formulation is called the viscosity solution, proposed by Crandall and Lions (1983). For more detailed discussions of viscosity solutions, see Crandall *et al.* (1992) and Fleming and Soner (1993). Here, by following the paper of Hipp and Plum (2003), we first show that the survival probability in the risk process (1.3) satisfies the HJB equation in a weak formulation sense. It is well known that, as in Hipp and Plum (2003), the HJB equation has a unique solution. Thus, the solution must be the survival probability in our problem. We then use the result in Hipp and Plum (2003), which states that the solution to the corresponding HJB equation is twice continuously differentiable. Therefore, the ruin probability, as a function of the initial surplus, is twice continuously differentiable.

We then give the detailed arguments of the proof. In doing so, we describe the risk model (1.3) in terms of stochastic differential equations, as follows. The dynamics of the surplus process X_t in (1.3) can be expressed as

$$
\begin{aligned}
\mathrm{d}X_t &= (c + \delta_t X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t - \mathrm{d}S_t, & X_0 &= u, \\
\mathrm{d}\delta_t &= \delta\,\mathrm{d}t + \sigma_\mathrm{R}\,\mathrm{d}W_{\mathrm{R},t} + \mathrm{d}S_{\mathrm{R},t}, & \delta_0 &= \delta_0.
\end{aligned}
\tag{2.1}
$$

Instead of considering the original problem, we first consider the following problem. We assume that an insurance company pays dividends continuously at a rate of $\alpha(X_\alpha(t))$ at time t. Suppose that the dynamics of the surplus process $X_\alpha(t)$ is given by

$$
\begin{aligned}
\mathrm{d}X_\alpha(t) &= (c + \delta_t X_\alpha(t) - \alpha(X_\alpha(t)))\,\mathrm{d}t + \sigma\,\mathrm{d}W_t - \mathrm{d}S_t, & X_\alpha(0) &= u, \\
\mathrm{d}\delta_t &= \delta\,\mathrm{d}t + \sigma_\mathrm{R}\,\mathrm{d}W_{\mathrm{R},t} + \mathrm{d}S_{\mathrm{R},t}, & \delta_0 &= \delta_0.
\end{aligned}
$$

Assume that Y_1 and X_1 have finite expectations and continuous distributions. We know that the surplus process $X_\alpha(t)$ has the same distribution as $\hat{X}_\alpha(t)$, where the surplus process $\hat{X}_\alpha(t)$ is given by

$$d\hat{X}_\alpha(t) = (c + \delta_t \hat{X}_\alpha(t) - \alpha(\hat{X}_\alpha(t))) \, dt + \sqrt{\sigma^2 + \sigma_R^2 \hat{X}_\alpha^2(t)} \, dB_t - dS_t + \hat{X}_\alpha(t^-) \, dS_{R,t},$$

$$\hat{X}_\alpha(0) = u; \tag{2.2}$$

cf. Equation (2.5) of Paulsen and Gjessing (1997). Here $\{B_t, \ t \geq 0\}$ is a standard Brownian motion independent of the compound Poisson processes $\{S_t, \ t \geq 0\}$ and $\{S_{R,t}, \ t \geq 0\}$.

As before, we define the ruin probability associated with the risk process (2.2) as

$$\psi_\alpha(u) = \Pr\{\hat{X}_\alpha(t) < 0 \text{ for some } t \geq 0 \mid \hat{X}_\alpha(0) = u\}.$$

The objective of the insurance company is to minimize the ruin probability $\psi_\alpha(u)$ by choosing the optimal dividend strategy. This is a stochastic control problem. Let $R_\alpha(u) = 1 - \psi_\alpha(u)$ be the survival probability corresponding to dividend policy α. Then, by the dynamic programming principle, the value function $R_\alpha(u)$ satisfies the following HJB equation, where a prime denotes differentiation:

$$\sup_\alpha \{\lambda \, E[R_\alpha(u - Y_1) - R_\alpha(u)] + \lambda_R \, E[R_\alpha(u(1 + X_1)) - R_\alpha(u)]$$

$$+ (c + \delta u - \alpha(u))R_\alpha'(u) + \tfrac{1}{2}(\sigma^2 + u^2 \sigma_R^2)R_\alpha''(u)\} = 0. \tag{2.3}$$

It is obvious that the optimal dividend policy is for the insurance company to pay no dividends, or to set $\alpha = 0$. By the verification theorem, the HJB equation (2.3) becomes

$$\lambda \, E[R_0(u - Y_1) - R_0(u)] + \lambda_R \, E[R_0(u(1 + X_1)) - R_0(u)]$$

$$+ (c + \delta u)R_0'(u) + \tfrac{1}{2}(\sigma^2 + u^2 \sigma_R^2)R_0''(u) = 0. \tag{2.4}$$

As mentioned earlier, it is well known that (2.4) has a unique continuous solution. Furthermore, $R_0(u)$ is the survival probability of the surplus process $\hat{X}_0(t) = \hat{X}(t)$ given by

$$d\hat{X}(t) = (c + \delta_t \hat{X}(t)) \, dt + \sqrt{\sigma^2 + \sigma_R^2 \hat{X}^2(t)} \, dB_t - dS_t + \hat{X}(t^-) \, dS_{R,t},$$

$$\hat{X}(0) = u.$$

Again, $\hat{X}(t)$ and X_t in (2.1) have the same distribution and, therefore, $R_0(u)$ and $\psi_0(u)$ are respectively the survival and ruin probabilities for model (2.1). Hence, $R_0(u) = R(u)$ and $\psi_0(u) = \psi(u)$.

We have the following result, similar to Theorem 5 of Hipp and Plum (2003), showing that $R(u) = 1 - \psi(u)$ is twice continuously differentiable.

Theorem 2.1. *Assume that the claim size Y_1 has a continuous density $f(y)$. Then (2.4) has a unique solution $R(u)$ that is nonnegative, nondecreasing, concave with $R(\infty) = 1$, continuous on $[0, \infty)$, and twice continuously differentiable on $(0, \infty)$.*

Proof. Equation (2.4) has a form similar to Equation (10) of Hipp and Plum (2003), and the proof of Theorem 5 of Hipp and Plum (2003) applies to it. We thus omit the proof of Theorem 2.1, and refer the reader to Hipp and Plum (2003) for details.

Remark 2.1. Paulsen and Gjessing (1997) assumed that $\psi(u)$ is twice continuously differentiable in order to use Itô's formula to obtain the integrodifferential equation satisfied by $\psi(u)$. Here, using the result of Hipp and Plum (2003), we prove that $\psi(u)$ is always twice continuously differentiable provided that the claim sizes have continuous density functions. This result also greatly relaxes the conditions of Wang and Wu (2001) and Cai (2004) on differentiability of the ruin probability $\psi(u)$.

3. Integrodifferential equations for ruin probabilities

In this section, we study the risk process X_t given in (1.2) and consider ruin in the perturbed compound Poisson risk process under constant interest force. We first give the integrodifferential equation for $\psi(u)$ using the HJB equation (2.4) directly and then, using a differential argument and Itô's formula, we derive integrodifferential equations for $\psi_s(u)$ and $\psi_d(u)$. The differential argument is a common method used in ruin theory; see Grandell (1991) for the method used in the compound Poisson risk process and Dufresne and Gerber (1991) for the method used in the perturbed compound Poisson risk process.

Throughout the paper, we denote the distribution function of Y_1 by F, with $F(0) = 0$, and the tail of a distribution function B by $\overline{B}(x) = 1 - B(x)$.

Theorem 3.1. (Integrodifferential equation for $\psi(u)$.) *Assume that F has a continuous density function. Then, for any $u > 0$, $\psi(u)$ satisfies the integrodifferential equation*

$$\tfrac{1}{2}\sigma^2\psi''(u) + (\delta u + c)\psi'(u) + \lambda\overline{F}(u) = \lambda\psi(u) - \lambda\int_0^u \psi(u - y)\,\mathrm{d}F(y) \qquad (3.1)$$

with the following boundary conditions:

$$\psi(\infty) = 0, \qquad \psi(0) = 1, \qquad \tfrac{1}{2}\sigma^2\psi''(0^+) + c\psi'(0^+) = 0. \qquad (3.2)$$

Proof. Let $\lambda_R = 0$ and $\sigma_R = 0$ and note that $R_0(u) = 1 - \psi_0(u) = 1 - \psi(u)$ in the HJB equation (2.4). We then have

$$\lambda\psi(u) - \lambda\,\mathrm{E}[\psi(u - Y_1)] = (c + \delta u)\psi'(u) + \tfrac{1}{2}\sigma^2\psi''(u). \qquad (3.3)$$

Furthermore,

$$
\begin{aligned}
\mathrm{E}[\psi(u - Y_1)] &= \int_0^\infty \psi(u - y)\,\mathrm{d}F(y) \\
&= \int_0^u \psi(u - y)\,\mathrm{d}F(y) + \int_u^\infty \psi(u - y)\,\mathrm{d}F(y) \\
&= \int_0^u \psi(u - y)\,\mathrm{d}F(y) + \overline{F}(u),
\end{aligned}
$$

which, together with (3.3), implies (3.1).

In addition, the boundary condition $\psi(\infty) = 0$ follows from $\psi(u) \leq \phi(u)$ and $\phi(\infty) = 0$, the boundary condition $\psi(0) = 1$ follows from (1.5), and the last boundary condition follows from letting $u \downarrow 0$ in (3.1).

We point out that (3.1) can also be derived using Theorem 2.1 of Paulsen and Gjessing (1997). However, their theorem holds under the assumption that $\psi(u)$ is twice continuously differentiable and has a bounded first derivative.

Furthermore, we point out that the arguments in Section 2 for the differentiability of $\psi(u)$ do not apply to $\psi_s(u)$ and $\psi_d(u)$ and, hence, that HJB equations similar to (2.4) are not available for $\psi_s(u)$ and $\psi_d(u)$. The reason is that, to use the stochastic control method, we must have a system of stochastic differential equations for $\psi(u)$: the corresponding systems of stochastic differential equations for $\psi_s(u)$ and $\psi_d(u)$ are unknown. In other words, we cannot separate the system of stochastic differential equations for $\psi(u)$ into two parts.

However, using a differential argument and Itô's formula, we can derive integrodifferential equations for $\psi_s(u)$ and $\psi_d(u)$ under the assumption that both $\psi_s(u)$ and $\psi_d(u)$ are twice continuously differentiable. We conjecture that $\psi_s(u)$ and $\psi_d(u)$ are twice continuously differentiable provided that F has a continuous density function, as we did for $\psi(u)$.

Theorem 3.2. (Integrodifferential equation for $\psi_s(u)$.) *Assume that $\psi_s(u)$ is twice continuously differentiable. Then, for any $u > 0$, $\psi_s(u)$ satisfies the integrodifferential equation*

$$\tfrac{1}{2}\sigma^2\psi_s''(u) + (\delta u + c)\psi_s'(u) + \lambda\overline{F}(u) = \lambda\psi_s(u) - \lambda\int_0^u \psi_s(u - y)\,dF(y) \qquad (3.4)$$

with the following boundary conditions:

$$\psi_s(\infty) = 0, \qquad \psi_s(0) = 0, \qquad \tfrac{1}{2}\sigma^2\psi_s''(0^+) + c\psi_s'(0^+) = -\lambda. \qquad (3.5)$$

Proof. Let

$$h(t) = ue^{\delta t} + c\int_0^t e^{\delta s}\,ds + \sigma\int_0^t e^{\delta s}\,dW_s - u. \qquad (3.6)$$

Consider the risk process X_t, defined by (1.2), in an infinitesimal time interval $(0, t]$. Since $N(t)$ is a Poisson process, there are three possible cases.

(i) There are no claims in $(0, t]$ and, thus,

$$X_t = ue^{\delta t} + c\int_0^t e^{\delta s}\,ds + \sigma\int_0^t e^{\delta s}\,dW_s = u + h(t).$$

(ii) There is exactly one claim in $(0, t]$, with claim amount y, and

   (a) $y < u + h(t)$, i.e. ruin does not occur and, thus,

$$X_t = ue^{\delta t} + c\int_0^t e^{\delta s}\,ds + \sigma\int_0^t e^{\delta s}\,dW_s - y = u + h(t) - y;$$

   (b) $y > u + h(t)$, i.e. ruin occurs due to the claim; or

   (c) $y = u + h(t)$, i.e. ruin occurs due to oscillation (the probability that this case occurs is 0).

(iii) There is more than one claim in $(0, t]$.

Thus, considering cases (i), (ii), and (iii) and noticing that, in case (ii)(b), $\psi_s(u + h(t) - y) = 1$ if $y > u + h(t)$, we have

$$\psi_s(u) = (1 - \lambda t) \, \mathrm{E}[\psi_s(u + h(t))] + \lambda t \, \mathrm{E}\left[\int_0^{u+h(t)} \psi_s(u + h(t) - y) \, \mathrm{d}F(y)\right]$$

$$+ \lambda t \, \mathrm{E}\left[\int_{u+h(t)}^{\infty} \psi_s(u + h(t) - y) \, \mathrm{d}F(y)\right] + o(t)$$

$$= (1 - \lambda t) \, \mathrm{E}[\psi_s(u + h(t))] + \lambda t \, \mathrm{E}\left[\int_0^{u+h(t)} \psi_s(u + h(t) - y) \, \mathrm{d}F(y)\right]$$

$$+ \lambda t \, \mathrm{E}[\overline{F}(u + h(t))] + o(t)$$

or, equivalently,

$$\lambda t \, \mathrm{E}[\psi_s(u + h(t))] = \mathrm{E}[\psi_s(u + h(t))] - \psi_s(u) + \lambda t \, \mathrm{E}\left[\int_0^{u+h(t)} \psi_s(u + h(t) - y) \, \mathrm{d}F(y)\right]$$

$$+ \lambda t \, \mathrm{E}[\overline{F}(u + h(t))] + o(t). \tag{3.7}$$

If we let $Y(t) = u + h(y)$ then (3.6) implies that

$$\mathrm{d}Y(t) = (u\delta + c)\mathrm{e}^{\delta t} \, \mathrm{d}t + \sigma \mathrm{e}^{\delta t} \, \mathrm{d}W_t, \qquad Y(0) = u.$$

By Itô's formula, we have

$$\mathrm{d}\psi_s(u + h(t)) = \mathrm{d}\psi_s(Y(t))$$

$$= ((u\delta + c)\mathrm{e}^{\delta t} \psi_s'(Y(t)) + \tfrac{1}{2}\sigma^2 \mathrm{e}^{2\delta t} \psi_s''(Y(t))) \, \mathrm{d}t + \sigma \mathrm{e}^{\delta t} \psi_s'(Y(t)) \, \mathrm{d}W_t$$

or, equivalently,

$$\psi_s(u + h(t)) = \psi_s(Y(t))$$

$$= \psi_s(u) + \int_0^t ((u\delta + c)\mathrm{e}^{\delta x} \psi_s'(Y(x)) + \tfrac{1}{2}\sigma^2 \mathrm{e}^{2\delta x} \psi_s''(Y(x))) \, \mathrm{d}x$$

$$+ \int_0^t \sigma \mathrm{e}^{\delta x} \psi_s'(Y(x)) \, \mathrm{d}W_x,$$

which implies that

$$\mathrm{E}[\psi_s(u + h(t))]$$

$$= \psi_s(u) + \int_0^t ((u\delta + c)\mathrm{e}^{\delta x} \, \mathrm{E}[\psi_s'(Y(x))] + \tfrac{1}{2}\sigma^2 \mathrm{e}^{2\delta x} \, \mathrm{E}[\psi_s''(Y(x))]) \, \mathrm{d}x. \tag{3.8}$$

Therefore, by dividing by t on both sides of (3.7), letting $t \to 0$, and using (3.8), we obtain

$$\lambda \psi_s(u) = (\delta u + c)\psi_s'(u) + \tfrac{1}{2}\sigma^2 \psi_s''(u) + \lambda \int_0^u \psi_s(u - y) \, \mathrm{d}F(y) + \lambda \overline{F}(u),$$

which implies (3.4).

In addition, the boundary condition $\psi_s(\infty) = 0$ follows from $\psi_s(u) \le \psi(u)$ and $\psi(\infty) = 0$, the boundary condition $\psi_s(0) = 0$ follows from (1.5), and the last boundary condition follows from letting $u \downarrow 0$ in (3.4).

We note that $\psi(u)$ and $\psi_s(u)$ satisfy the same second-order integrodifferential equations, but have different boundary conditions.

Theorem 3.3. (Integrodifferential equation for $\psi_d(u)$.) *Assume that $\psi_d(u)$ is twice continuously differentiable. Then, for any $u > 0$, $\psi_d(u)$ satisfies the integrodifferential equation*

$$\frac{1}{2}\sigma^2\psi_d''(u) + (\delta u + c)\psi_d'(u) = \lambda\psi_d(u) - \lambda\int_0^u \psi_d(u - y)\,\mathrm{d}F(y) \tag{3.9}$$

with the following boundary conditions:

$$\psi_d(\infty) = 0, \qquad \psi_d(0) = 1, \qquad \frac{1}{2}\sigma^2\psi_d''(0^+) + c\psi_d'(0^+) = \lambda. \tag{3.10}$$

Proof. Consider cases (i), (ii), and (iii) in the proof of Theorem 3.2 and notice that, in case (ii)(b), $\psi_d(u + h(t) - y) = 0$ if $y > u + h(t)$. We see that

$$\psi_d(u) = (1 - \lambda t)\,\mathrm{E}[\psi_d(u + h(t))] + \lambda t\,\mathrm{E}\left[\int_0^{u+h(t)} \psi_d(u + h(t) - y)\,\mathrm{d}F(y)\right]$$

$$+ \lambda t\,\mathrm{E}\left[\int_{u+h(t)}^{\infty} \psi_d(u + h(t) - y)\,\mathrm{d}F(y)\right] + o(t)$$

$$= (1 - \lambda t)\,\mathrm{E}[\psi_d(u + h(t))] + \lambda t\,\mathrm{E}\left[\int_0^{u+h(t)} \psi_d(u + h(t) - y)\,\mathrm{d}F(y)\right] + o(t),$$

which implies (3.9) by the same arguments as for (3.1).

In addition, the boundary condition $\psi_d(\infty) = 0$ follows from $\psi_d(u) \leq \psi(u)$ and $\psi(\infty) = 0$, the boundary condition $\psi_d(0) = 1$ follows from (1.5), and the last boundary condition follows from letting $u \downarrow 0$ in (3.9).

We remark that the integrodifferential equations (3.1), (3.4), and (3.9) satisfy relation (1.4), as do the boundary conditions (3.2), (3.5), and (3.10).

4. Ruin with exponential claim sizes

Paulsen and Gjessing (1997) derived explicit solutions of $\psi(u)$ when claim sizes are exponentially distributed. In this section, we derive explicit expressions for $\psi_s(u)$ and $\psi_d(u)$ when claim sizes are exponentially distributed. We first derive the third-order differential equations satisfied by the ruin probabilities.

Corollary 4.1. *Under the conditions of Theorems 3.1, 3.2, and 3.3, if F is an exponential distribution with a density function $f(x) = \beta e^{-\beta x}$, $x > 0$, $\beta > 0$, then, for any $u > 0$, $\psi(u)$, $\psi_s(u)$, and $\psi_d(u)$ satisfy the third-order differential equations*

$$\begin{cases} \frac{1}{2}\sigma^2\psi'''(u) + (\delta u + c + \frac{1}{2}\beta\sigma^2)\psi''(u) + (\beta(\delta u + c) - \lambda + \delta)\psi'(u) = 0, \\ \frac{1}{2}\sigma^2\psi_s'''(u) + (\delta u + c + \frac{1}{2}\beta\sigma^2)\psi_s''(u) + (\beta(\delta u + c) - \lambda + \delta)\psi_s'(u) = 0, \\ \frac{1}{2}\sigma^2\psi_d'''(u) + (\delta u + c + \frac{1}{2}\beta\sigma^2)\psi_d''(u) + (\beta(\delta u + c) - \lambda + \delta)\psi_d'(u) = 0, \end{cases} \tag{4.1}$$

with the following boundary conditions:

$$\psi(\infty) = 0, \qquad \psi(0) = 1, \qquad \frac{1}{2}\sigma^2\psi''(0^+) + c\psi'(0^+) = 0,$$

$$\psi_s(\infty) = 0, \qquad \psi_s(0) = 0, \qquad \frac{1}{2}\sigma^2\psi_s''(0^+) + c\psi_s'(0^+) = -\lambda, \tag{4.2}$$

$$\psi_d(\infty) = 0, \qquad \psi_d(0) = 1, \qquad \frac{1}{2}\sigma^2\psi_d''(0^+) + c\psi_d'(0^+) = \lambda. \tag{4.3}$$

Proof. When F is exponential, (3.9) can be re-expressed as

$$\lambda \psi_{\mathrm{d}}(u) - (\delta u + c)\psi_{\mathrm{d}}'(u) - \tfrac{1}{2}\sigma^2 \psi_{\mathrm{d}}''(u) = \lambda \int_0^u \psi_{\mathrm{d}}(u - y)\,\mathrm{d}F(y)$$

$$= \lambda \beta \mathrm{e}^{-\beta u} \int_0^u \mathrm{e}^{\beta y} \psi_{\mathrm{d}}(y)\,\mathrm{d}y. \qquad (4.4)$$

Taking derivatives with respect to u on both sides of (4.4), we obtain

$$\lambda \psi_{\mathrm{d}}'(u) - \delta \psi_{\mathrm{d}}'(u) - (\delta u + c)\psi_{\mathrm{d}}''(u) - \tfrac{1}{2}\sigma^2 \psi_{\mathrm{d}}''(u)$$

$$= -\beta \left[\lambda \beta \mathrm{e}^{-\beta u} \int_0^u \mathrm{e}^{\beta y} \psi_{\mathrm{d}}(y)\,\mathrm{d}y \right] + \lambda \beta \psi_{\mathrm{d}}(u),$$

which, together with (4.4), gives

$$(\lambda - \delta)\psi_{\mathrm{d}}'(u) - (\delta u + c)\psi_{\mathrm{d}}''(u) - \tfrac{1}{2}\sigma^2 \psi_{\mathrm{d}}''(u)$$

$$= -\beta[\lambda \psi_{\mathrm{d}}(u) - (\delta u + c)\psi_{\mathrm{d}}'(u) - \tfrac{1}{2}\sigma^2 \psi_{\mathrm{d}}''(u)] + \lambda \beta \psi_{\mathrm{d}}(u),$$

implying that the equation for $\psi_{\mathrm{d}}(u)$ in (4.1) holds.

The boundary conditions for $\psi_{\mathrm{d}}(u)$ follow from (3.10). The third-order differential equations for $\psi(u)$ and $\psi_{\mathrm{s}}(u)$ can be derived similarly.

We note that, when the claim sizes are exponentially distributed, the three ruin probabilities $\psi(u)$, $\psi_{\mathrm{s}}(u)$, and $\psi_{\mathrm{d}}(u)$ satisfy the same third-order differential equation, but have different boundary conditions. The third-order differential equations in Corollary 4.1, together with the boundary conditions, enable us to obtain explicit solutions for the ruin probabilities. To do so, we recall some results about the confluent hypergeometric function and Kummer's confluent hypergeometric equation.

Assume that $y \equiv y(x)$ is twice differentiable. The following second-order differential equation is called Kummer's confluent hypergeometric equation:

$$x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (b - x)\frac{\mathrm{d}y}{\mathrm{d}x} - ay = 0. \qquad (4.5)$$

If y_1 and y_2 are any two linearly independent solutions then the general solution of (4.5) is given by

$$y(x) = Ay_1(x) + By_2(x),$$

where A and B are arbitrary coefficients that may depend on a and b but not on x, and can be determined from the boundary conditions satisfied by $y(x)$.

Denote the confluent hypergeometric function by $M(a, b; x)$, i.e.

$$M(a, b; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\left(\frac{x^n}{n!}\right),$$

where $(a)_n = a(a + 1) \cdots (a + n - 1)$ for $n = 1, 2, \ldots$, and $(a)_0 = 1$.

$M(a, b; x)$ is the simplest solution of Kummer's confluent hypergeometric equation (4.5) and is absolutely convergent for all real or complex values of a, b, and x, excluding $b = 0, -1, -2, \ldots$. Moreover, it is known that the confluent hypergeometric function satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x} M(a, b; x) = \frac{a}{b} M(a + 1, b + 1; x); \tag{4.6}$$

see Seaborn (1991) or Slater (1960) for details.

The third-order differential equations in Corollary 4.1 can be reduced to the form of Kummer's confluent hypergeometric equation using suitable function transformations. We employ the same transforms used for $\psi(u)$ in Paulsen and Gjessing (1997), since $\psi_\mathrm{s}(u)$ and $\psi_\mathrm{d}(u)$ satisfy the same third-order differential equation as does $\psi(u)$. To do so, let $u = z - \alpha$ and $\psi_\mathrm{s}'(u) = \mathrm{e}^{-\beta z} g(z)$, where

$$\alpha = \frac{c}{\delta} - \frac{\beta \sigma^2}{2\delta}.$$

Then the equation for $\psi_\mathrm{s}(u)$ in (4.1) becomes

$$\tfrac{1}{2}\sigma^2 g''(z) + \delta z g'(z) + (\delta - \lambda) g(z) = 0. \tag{4.7}$$

Furthermore, let $z^2 = -(\sigma^2/\delta)x$ and $g(z) = h(x)$. Then (4.7) becomes the following Kummer confluent hypergeometric equation, with $a = \frac{1}{2} - \frac{1}{2}\lambda/\delta$ and $b = \frac{1}{2}$:

$$x h''(x) + \left(\frac{1}{2} - x\right) h'(x) - \left(\frac{1}{2} - \frac{\lambda}{2\delta}\right) h(x) = 0. \tag{4.8}$$

Thus, by Equation (1.3.3) of Slater (1960),

$$y_1(x) = \mathrm{e}^x U(b - a, b; -x) = \mathrm{e}^x U\left(\frac{\lambda}{2\delta}, \frac{1}{2}; -x\right)$$

is a solution of (4.8), where $U(a, b; x)$ is the second form of the confluent hypergeometric function. Furthermore, by Equation (1.2.17) of Slater (1960),

$$y_0(x) = x^{1-b} \mathrm{e}^x M(1 - a, 2 - b; -x) = \sqrt{x}\,\mathrm{e}^x M\left(\frac{1}{2} + \frac{\lambda}{2\delta}, \frac{3}{2}; -x\right)$$

is a solution of (4.8) and y_0 is linearly independent of y_1.

It is easy to see that if $y_0 \equiv y_0(x)$ is a solution of (4.8), then so is $\sqrt{-1}\,y_0 =: \mathrm{i}y_0$. Hence,

$$y_2 \equiv y_2(x) = \mathrm{i}y_0(x) = \sqrt{-x}\,\mathrm{e}^x M\left(\frac{1}{2} + \frac{\lambda}{2\delta}, \frac{3}{2}; -x\right)$$

is also a solution of (4.8) linearly independent of y_1, since y_1 and y_0 are linearly independent. Therefore, the general solution to (4.8) is given by

$$h(x) = A_\mathrm{s} y_1(x) + B_\mathrm{s} y_2(x)$$
$$= A_\mathrm{s} \mathrm{e}^x U\left(\frac{\lambda}{2\delta}, \frac{1}{2}; -x\right) + B_\mathrm{s} \sqrt{-x}\,\mathrm{e}^x M\left(\frac{1}{2} + \frac{\lambda}{2\delta}, \frac{3}{2}; -x\right),$$

where A_s and B_s are arbitrary constants.

Thus,

$$g(z) = h(x) = A_s e^{-\delta z^2/\sigma^2} U\left(\frac{\lambda}{2\delta}, \frac{1}{2}; \frac{\delta z^2}{\sigma^2}\right)$$
$$+ B_s \frac{\sqrt{\delta} z}{\sigma} e^{-\delta z^2/\sigma^2} M\left(\frac{1}{2} + \frac{\lambda}{2\delta}, \frac{3}{2}; \frac{\delta z^2}{\sigma^2}\right),$$

which gives

$$\psi_s'(u) = e^{-\beta z} g(z) = A_s e^{-\beta(u+\alpha)} \exp\left\{-\frac{\delta(u+\alpha)^2}{\sigma^2}\right\} U\left(\frac{\lambda}{2\delta}, \frac{1}{2}; \frac{\delta(u+\alpha)^2}{\sigma^2}\right)$$
$$+ B_s e^{-\beta(u+\alpha)} \frac{\sqrt{\delta}(u+\alpha)}{\sigma} \exp\left\{-\frac{\delta(u+\alpha)^2}{\sigma^2}\right\}$$
$$\times M\left(\frac{1}{2} + \frac{\lambda}{2\delta}, \frac{3}{2}; \frac{\delta(u+\alpha)^2}{\sigma^2}\right)$$
$$= A_s^* h_1(u) + B_s^* h_2(u),$$

where $A_s^* = A_s e^{-\beta\alpha}$ and $B_s^* = B_s \sqrt{\delta} e^{-\beta\alpha}/\sigma$ are arbitrary constants,

$$h_1(u) = \exp\left\{-\left(\beta u + \frac{\delta(u+\alpha)^2}{\sigma^2}\right)\right\} U\left(\frac{\lambda}{2\delta}, \frac{1}{2}; \frac{\delta(u+\alpha)^2}{\sigma^2}\right), \tag{4.9}$$

and

$$h_2(u) = (u+\alpha) \exp\left\{-\left(\beta u + \frac{\delta(u+\alpha)^2}{\sigma^2}\right)\right\} M\left(\frac{1}{2} + \frac{\lambda}{2\delta}, \frac{3}{2}; \frac{\delta(u+\alpha)^2}{\sigma^2}\right). \tag{4.10}$$

Hence,

$$\psi_s(u) = -\int_u^\infty \psi_s'(y)\,dy = -A_s^* H_1(u) - B_s^* H_2(u), \tag{4.11}$$

where

$$H_1(u) = \int_u^\infty h_1(y)\,dy, \qquad H_2(u) = \int_u^\infty h_2(y)\,dy,$$

and the constants A_s^* and B_s^* can be determined from the boundary conditions satisfied by $\psi_s(u)$.

To determine the coefficients A_s^* and B_s^*, note that $H_1'(u) = -h_1(u)$ and $H_2'(u) = -h_2(u)$ and, hence, $H_1''(u) = -h_1'(u)$ and $H_2''(u) = -h_2'(u)$. Furthermore, from (4.6), (4.9), (4.10), and

$$\frac{d}{dx} U(a, b; x) = -a U(a+1, b+1; x),$$

we obtain

$$h_1'(0) = -\left(\beta + \frac{2\delta\alpha}{\sigma^2}\right) e^{-\delta\alpha^2/\sigma^2} U\left(\frac{\lambda}{2\delta}, \frac{1}{2}; \frac{\delta\alpha^2}{\sigma^2}\right)$$
$$- \frac{\lambda\alpha}{\sigma^2} e^{-\delta\alpha^2/\sigma^2} U\left(1 + \frac{\lambda}{2\delta}, \frac{3}{2}; \frac{\delta\alpha^2}{\sigma^2}\right)$$

and

$$h_2'(0) = e^{-\delta\alpha^2/\sigma^2} M\left(\frac{1}{2} + \frac{\lambda}{2\delta}, \frac{3}{2}; \frac{\delta\alpha^2}{\sigma^2}\right)$$

$$- \alpha\left(\beta + \frac{2\delta\alpha}{\sigma^2}\right) e^{-\delta\alpha^2/\sigma^2} M\left(\frac{1}{2} + \frac{\lambda}{2\delta}, \frac{3}{2}; \frac{\delta\alpha^2}{\sigma^2}\right)$$

$$+ \frac{2\alpha^2(\delta + \lambda)}{3\sigma^2} e^{-\delta\alpha^2/\sigma^2} M\left(\frac{3}{2} + \frac{\lambda}{2\delta}, \frac{5}{2}; \frac{\delta\alpha^2}{\sigma^2}\right).$$

Thus,

$$H_1''(0) = -h_1'(0) = \left[\left(\beta + \frac{2\delta\alpha}{\sigma^2}\right) U\left(\frac{\lambda}{2\delta}, \frac{1}{2}; \frac{\delta\alpha^2}{\sigma^2}\right) + \frac{\lambda\alpha}{\sigma^2} U\left(1 + \frac{\lambda}{2\delta}, \frac{3}{2}; \frac{\delta\alpha^2}{\sigma^2}\right)\right] e^{-\delta\alpha^2/\sigma^2}$$

and

$$H_2''(0) = -h_2'(0) = \left[\alpha\left(\beta + \frac{2\delta\alpha}{\sigma^2}\right) - 1\right] M\left(\frac{1}{2} + \frac{\lambda}{2\delta}, \frac{3}{2}; \frac{\delta\alpha^2}{\sigma^2}\right) e^{-\delta\alpha^2/\sigma^2}$$

$$- \frac{2\alpha^2(\delta + \lambda)}{3\sigma^2} M\left(\frac{3}{2} + \frac{\lambda}{2\delta}, \frac{5}{2}; \frac{\delta\alpha^2}{\sigma^2}\right) e^{-\delta\alpha^2/\sigma^2}.$$

Moreover,

$$H_1'(0) = -h_1(0) = -e^{-\delta\alpha^2/\sigma^2} U\left(\frac{\lambda}{2\delta}, \frac{1}{2}; \frac{\delta\alpha^2}{\sigma^2}\right)$$

and

$$H_2'(0) = -h_2(0) = -\alpha e^{-\delta\alpha^2/\sigma^2} M\left(\frac{1}{2} + \frac{\lambda}{2\delta}, \frac{3}{2}; \frac{\delta\alpha^2}{\sigma^2}\right).$$

From (4.2) and (4.11), the last two boundary conditions for $\psi_s(u)$ reduce to

$$A_s^* H_1(0) + B_s^* H_2(0) = 0,$$
$$\tfrac{1}{2}\sigma^2(A_s^* H_1''(0) + B_s^* H_2''(0)) + c(A_s^* H_1'(0) + B_s^* H_2'(0)) = \lambda.$$

Thus, by solving the equations, we obtain

$$A_s^* = \frac{\lambda H_2(0) e^{\delta\alpha^2/\sigma^2}}{D_1} \quad \text{and} \quad B_s^* = \frac{-\lambda H_1(0) e^{\delta\alpha^2/\sigma^2}}{D_1},$$

where

$$D_1 = H_1(0)\frac{\sigma^2}{2} M\left(\frac{1}{2} + \frac{\lambda}{2\delta}, \frac{3}{2}; \frac{\delta\alpha^2}{\sigma^2}\right) + H_1(0)\frac{\alpha^2(\delta + \lambda)}{3} M\left(\frac{3}{2} + \frac{\lambda}{2\delta}, \frac{5}{2}; \frac{\delta\alpha^2}{\sigma^2}\right)$$

$$+ H_2(0)\frac{\lambda\alpha}{2} U\left(1 + \frac{\lambda}{2\delta}, \frac{3}{2}; \frac{\delta\alpha^2}{\sigma^2}\right).$$

Similarly, the ruin probability $\psi_d(u)$ takes the form

$$\psi_d(u) = -\int_u^\infty \psi_d'(y)\,dy = -A_d H_1(u) - B_d H_2(u), \tag{4.12}$$

where the constants A_d and B_d can be determined from the boundary conditions satisfied by $\psi_d(u)$. In doing so, by (4.3) and (4.12), the last two boundary conditions for $\psi_d(u)$ reduce to

$$A_d H_1(0) + B_d H_2(0) = -1,$$
$$\tfrac{1}{2}\sigma^2(A_d H_1''(0) + B_d H_2''(0)) + c(A_d H_1'(0) + B_d H_2'(0)) = -\lambda.$$

Thus, the coefficients A_d and B_d for $\psi_d(u)$ are given by

$$A_d = \frac{1}{D_1}\left(-\lambda H_2(0)e^{\delta\alpha^2/\sigma^2} - \frac{1}{2}\sigma^2 M\left(\frac{1}{2} + \frac{\lambda}{2\delta}, \frac{3}{2}; \frac{\delta\alpha^2}{\sigma^2}\right)\right.$$
$$\left. - \frac{1}{3}\alpha^2(\delta + \lambda)M\left(\frac{3}{2} + \frac{\lambda}{2\delta}, \frac{5}{2}; \frac{\delta\alpha^2}{\sigma^2}\right)\right)$$

and

$$B_d = \frac{1}{D_1}\left(-\frac{1}{2}\lambda\alpha U\left(1 + \frac{\lambda}{2\delta}, \frac{3}{2}; \frac{\delta\alpha^2}{\sigma^2}\right) + \lambda H_1(0)e^{\delta\alpha^2/\sigma^2}\right).$$

To end the paper, we use numerical examples to illustrate applications of the explicit solutions and the effects of the interest force δ and the diffusion volatility σ on the ruin probabilities $\psi(u)$, $\psi_s(u)$, and $\psi_d(u)$.

Example 4.1. Assume that the claim sizes are exponentially distributed and that $c = 103$, $\lambda = 100$, $\beta = 1$, and $\sigma = 0.8$. We calculate the values of $\psi(u)$, $\psi_s(u)$, and $\psi_d(u)$, using the formulae derived above, for $\delta = 0.105, 0.085, 0.065, 0.045, 0$. When $\delta = 0$, by Equations (6.4) and (6.17) of Dufresne and Gerber (1991), we have

$$\psi(u) = c_1 e^{-r_1 u} + c_2 e^{-r_2 u},$$
$$\psi_s(u) = c_1^s e^{-r_1 u} + c_2^s e^{-r_2 u},$$
$$\psi_d(u) = c_1^d e^{-r_1 u} + c_2^d e^{-r_2 u},$$

where

$$c_1 = \frac{r_1 - \beta}{\beta}\frac{r_2}{r_1 - r_2}, \qquad c_2 = \frac{r_2 - \beta}{\beta}\frac{r_1}{r_2 - r_1},$$
$$c_1^s = c_1 - c_1^d, \qquad c_2^s = c_2 - c_2^d,$$
$$c_1^d = \frac{r_1 - \beta}{r_1 - r_2}, \qquad c_2^d = \frac{r_2 - \beta}{r_2 - r_1},$$

and r_1 and r_2 are the solutions to the equation $\lambda/(\beta - r) + \frac{1}{2}\sigma^2 r = c$.

In Tables 1–3 we present the values of these ruin probabilities for selected values of u. The calculations were performed using MATHEMATICA®. From the tables, we see that the ruin probabilities $\psi(u)$ and $\psi_d(u)$ decrease as u and δ increase and that the dependence of $\psi(u)$ on δ is strongest for small values of u. On the other hand, the behavior of $\psi_s(u)$ differs from that of both $\psi(u)$ and $\psi_d(u)$. For example, $\psi_s(u)$ is not a decreasing function of u and δ.

Example 4.2. Assume that the claim sizes are exponentially distributed and that $c = 103$, $\lambda = 100$, $\beta = 1$, and $\delta = 0.03$. We calculate the values of $\psi(u)$, $\psi_s(u)$, and $\psi_d(u)$, using the formulae derived above, for $\sigma = 1.2, 1.0, 0.8, 0.6, 0.4$, and present the ruin probabilities

TABLE 1: Effects of interest force on the ruin probability $\psi(u)$.

u	$\psi(u)$				
	$\delta = 0.105$	$\delta = 0.085$	$\delta = 0.065$	$\delta = 0.045$	$\delta = 0$
0	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
0.001	0.987 379	0.987 993	0.988 672	0.989 446	0.991 984
0.005	0.963 270	0.965 056	0.967 035	0.969 284	0.976 671
0.01	0.955 781	0.957 931	0.960 313	0.963 021	0.971 914
0.055	0.952 201	0.954 526	0.957 100	0.960 027	0.969 640
1	0.910 692	0.915 029	0.919 835	0.925 300	0.943 257
5	0.749 126	0.761 087	0.774 390	0.789 574	0.839 816
10	0.576 690	0.595 926	0.617 522	0.642 414	0.726 320
50	0.034 760	0.046 777	0.063 926	0.089 242	0.227 340

TABLE 2: Effects of interest force on the ruin probability $\psi_s(u)$.

u	$\psi_s(u)$				
	$\delta = 0.105$	$\delta = 0.085$	$\delta = 0.065$	$\delta = 0.045$	$\delta = 0$
0	0.000 000	0.000 000	0.000 000	0.000 000	0.000 000
0.001	0.262 479	0.263 091	0.263 768	0.264 539	0.267 069
0.005	0.761 860	0.763 641	0.765 613	0.767 855	0.775 218
0.01	0.913 326	0.915 469	0.917 843	0.920 543	0.929 408
0.055	0.949 251	0.951 569	0.954 135	0.957 053	0.966 636
1	0.907 871	0.912 195	0.916 986	0.922 434	0.940 335
5	0.746 806	0.758 730	0.771 991	0.787 128	0.837 215
10	0.574 903	0.594 080	0.615 609	0.640 424	0.724 070
50	0.034 652	0.046 632	0.063 728	0.088 966	0.226 636

TABLE 3: Effects of interest force on the ruin probability $\psi_d(u)$.

u	$\psi_d(u)$				
	$\delta = 0.105$	$\delta = 0.085$	$\delta = 0.065$	$\delta = 0.045$	$\delta = 0$
0	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
0.001	0.724 900	0.724 902	0.724 904	0.724 907	0.724 915
0.005	0.201 410	0.201 415	0.201 422	0.201 429	0.201 453
0.01	0.042 455	0.042 462	0.042 470	0.042 478	0.042 506
0.055	0.002 950	0.002 957	0.002 965	0.002 974	0.003 004
1	0.002 821	0.002 834	0.002 849	0.002 866	0.002 922
5	0.002 320	0.002 357	0.002 399	0.002 446	0.002 601
10	0.001 787	0.001 846	0.001 913	0.001 990	0.002 250
50	0.000 108	0.000 145	0.000 198	0.000 276	0.000 704

for selected values of u in Tables 4–6. These calculations were also performed using MATH-EMATICA.

From the tables, we see that the ruin probabilities $\psi(u)$ and $\psi_d(u)$ increase as σ increases and that the dependence of $\psi(u)$ on σ is strongest for small values of u, as in Example 4.1. Furthermore, the behavior of $\psi_s(u)$ again differs from that of both $\psi(u)$ and $\psi_d(u)$.

TABLE 4: Effects of volatility on the ruin probability $\psi(u)$.

	$\psi(u)$				
u	$\sigma = 1.2$	$\sigma = 1.0$	$\sigma = 0.8$	$\sigma = 0.6$	$\sigma = 0.4$
0	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
0.001	0.995 211	0.993 313	0.990 116	0.984 354	0.974 003
0.005	0.981 630	0.976 884	0.971 236	0.966 081	0.964 039
0.01	0.972 601	0.968 560	0.965 372	0.963 986	0.963 808
0.055	0.962 845	0.962 682	0.962 568	0.962 479	0.962 416
1	0.930 389	0.930 200	0.930 045	0.929 924	0.929 837
5	0.803 475	0.803 104	0.802 800	0.802 563	0.802 393
10	0.665 251	0.664 723	0.664 289	0.663 951	0.663 708
50	0.117 848	0.117 389	0.117 012	0.116 720	0.116 511

TABLE 5: Effects of volatility on the ruin probability $\psi_s(u)$.

	$\psi_s(u)$				
u	$\sigma = 1.2$	$\sigma = 1.0$	$\sigma = 0.8$	$\sigma = 0.6$	$\sigma = 0.4$
0	0.000 000	0.000 000	0.000 000	0.000 000	0.000 000
0.001	0.128 448	0.179 399	0.265 207	0.419 898	0.697 769
0.005	0.491 507	0.618 604	0.769 801	0.907 567	0.961 699
0.01	0.730 629	0.838 269	0.922 886	0.959 069	0.963 057
0.055	0.955 419	0.957 999	0.959 586	0.960 800	0.961 669
1	0.923 929	0.925 706	0.927 164	0.928 302	0.929 115
5	0.797 896	0.799 224	0.800 313	0.801 162	0.801 770
10	0.660 632	0.661 511	0.662 231	0.662 793	0.663 193
50	0.117 030	0.116 821	0.116 650	0.116 516	0.116 421

TABLE 6: Effects of volatility on the ruin probability $\psi_d(u)$.

	$\psi_d(u)$				
u	$\sigma = 1.2$	$\sigma = 1.0$	$\sigma = 0.8$	$\sigma = 0.6$	$\sigma = 0.4$
0	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
0.001	0.866 763	0.813 914	0.724 909	0.564 456	0.276 234
0.005	0.490 123	0.358 280	0.201 435	0.058 514	0.002 340
0.01	0.241 972	0.130 291	0.042 486	0.004 917	0.000 751
0.055	0.007 426	0.004 683	0.002 982	0.001 679	0.000 747
1	0.006 460	0.004 494	0.002 881	0.001 622	0.000 722
5	0.005 579	0.003 880	0.002 487	0.001 401	0.000 623
10	0.004 619	0.003 212	0.002 058	0.001 158	0.000 515
50	0.000 818	0.000 568	0.000 362	0.000 204	0.000 090

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