

On the real Common Chords of a Point Circle and Ellipse.

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(1) If O be a given point in the plane of a given conic, the mutual relationship between point and conic is marked, first and foremost, by the existence of a certain determinate straight line (which is always real) known as the polar of O with respect to the conic. Next following the polar in natural order of sequence, come a certain pair of determinate straight lines:—

The single real pair of common chords of the conic and a point circle at O .

(2) As in the generality of cases, it will be best to rely upon analysis for discovery of facts, and then to look to geometry for elucidation.

(3) Consider the conic represented by the equation

$$x^2 + y^2 = (ax + by + c)(a'x + b'y + c') \quad \dots \quad \dots \quad (\text{A})$$

which really involves five constants, for either c or c' may be put = 1. It represents the locus of a point which moves in such a manner that the square of its distance from the origin varies as the product of its perpendicular distances upon the fixed straight lines

$$\left. \begin{array}{l} ax + by + c = 0 \\ a'x + b'y + c' = 0 \end{array} \right\} \quad \dots \quad \dots \quad \dots \quad (\text{B})$$

(4) These two straight lines (B) must be regarded as the real common chords of the conic and a point circle at O . They cannot intersect the conic in real points, and consequently lie outside the conic. Since any circle theoretically intersects a conic in four points lying two and two upon three pairs of common chords, and (A) may be written in the form

$$y^2 - i^2x^2 = (ax + by + c)(a'x + b'y + c'), \quad (i^2 = -1)$$

therefore $y \pm ix$ is one pair of imaginary chords; and of the other two pairs, one pair must be real and one imaginary.

(5) It may here be noted that the straight lines (B) are equally inclined to either axis of the conic following the general property of the chords of intersection of a circle and conic. For the coefficient of xy vanishes in (A) when $ab' + b'a = 0$; so that perhaps a more convenient form might therefore be taken as

$$x^2 + y^2 = (\lambda x + \mu y + \nu)(\lambda x - \mu y + \nu')$$

Again, when (B) represents coincident straight lines, the origin becomes a focus and the coincident common chords become the corresponding directrix.

(6) The geometrical connexion of the foregoing is as follows:— If the conic be reciprocated with respect to any point O we get a second conic. The foci of the first reciprocate into certain straight lines $\Delta\Delta', \delta\delta'$; while the property that the product of the perpendiculars from the foci upon any tangent to the first conic is constant reciprocates into the property that the (distance OP)² of any point on the second conic from O varies as the product of the perpendiculars from P on $\Delta\Delta', \delta\delta'$. (By using Salmon's Theorem.) Thus from Art. (3) we see that $\Delta\Delta', \delta\delta'$ are the real common chords of intersection of a point circle at O with the second conic.

(7) It will be convenient to designate the above pair of straight lines the Delta lines of the conic corresponding to O .

(8) Properties of the Delta lines are therefore easily found by reciprocating focal properties; as, for instance, the property that the tangent at any point is equally inclined to the focal distances becomes by reciprocation the part of any tangent intercepted by the Delta lines is divided at the point of contact into two segments which subtend equal or supplementary angles at O .

(9) The Delta lines intersect upon the polar of O . Also if Σ, σ be the poles of the Delta lines, then $C\Sigma, C\sigma$ are equally inclined to either axis. (The points Σ, σ correspond to the directrices of the reciprocated conic.)

FIGURE 25.

(10) Let UU' be a straight line outside an ellipse meeting the ellipse in the imaginary points ω, ω' . Then \mathcal{T} the middle point of

$\omega\omega'$ is real, and is found by drawing the tangent at P parallel to UU' and producing CP to meet UU' in T .

Also ωT (or $=\omega'T$) is given by the equation

$$\frac{CT^2}{CP^2} + \frac{\omega T^2}{CD^2} = 1$$

where since $CT > CP$ ωT^2 is negative. Through T draw $TO \perp r$ to UU' (*i.e.*, parallel to the normal at P). Then if a point circle at O pass through ω, ω'

$$OT^2 + \omega T^2 = 0,$$

hence
$$\frac{CT^2}{CP^2} - \frac{OT^2}{CD^2} = 1.$$

Hence O lies on the concentric hyperbola passing through P whose conjugate diameter is $=CD$ and is perpendicular to CD . This is obviously the confocal hyperbola through P (for at their point of intersection CP is a common semi-diameter, and the conjugate semi-diameter $= (SP \cdot SP')^{\frac{1}{2}}$ in each case though in perpendicular directions).

(11) Since in Fig. 25 O may lie on either side of UU' , we see that to any line outside an ellipse correspond two determinate points the point-circles at which have the given line as their common chord with the ellipse.

(12) To determine therefore the Delta lines corresponding to any point O with respect to a given ellipse, we have the following construction. Draw the confocal hyperbola through O intersecting the ellipse at the extremities of the equi diameters PCP', pCp' ; draw OT parallel to the normal at P to meet CP produced in T ; and through T draw UU' perpendicular to OT or parallel to the tangent at P . Then UU' is one of the Delta lines required: and a similar construction gives the other.

(13) If upon CP, Cp points Q, q be taken respectively such that $CQ \cdot CT = CP^2 = Cp^2 = Cq \cdot Ct$, then Q, q are obviously the points in which the tangent at O to the confocal hyperbola meets CP, Cp respectively. Consequently Q, O, q are in a straight line; and UU', uu' intersect on the polar of O . See Art. (9).

(14) Various geometrical properties may be noted. By reciprocation we find that if R be any point on $\triangle\triangle'$ (see Fig. 26a), and $R\Sigma$ intersect the curve in Q, Q' , then OQ, OQ' each divide the angle ΣOR into parts whose sines are in the ratio $e:1$, and consequently the range $\{RQ\Sigma Q'\}$ is harmonic, as we should expect. This interesting result may be otherwise stated: If BC be the fixed base of a triangle whose variable vertex P describes a straight line, the locus of Q taken on PC such that $\sin QBC : \sin QBP$ is constant is a conic.

(15) Again (in Fig. 26b), if the tangent at Q meet $\triangle\triangle'$ in Z , and $Q\Sigma$ meet $\triangle\triangle'$ in R (and the curve in Q'), ZOR is a right angle. Whence ZQ' is the tangent at Q' ; and the conjugate points Z, R subtend a right angle at O . Thus given a straight line outside an ellipse, the circle described upon any pair of conjugate points lying on that straight line passes through two fixed points—the point circles at which have the given straight line as their chord of intersection with the ellipse.

(16) The pedal of an ellipse with respect to O is found by reciprocation to be the locus of a point whose distances from three points (one of which is O and the others the feet of the perpendiculars from O on the Delta lines) are connected by a linear relation (bicircular quartic).

(17) Since in Fig. 26b, OQ^2 varies as the product of perpendiculars from Q on Delta lines, OQ is a maximum (therefore OQ normal at Q) when this product is constant for two consecutive positions of Q . From the properties of the rectangular hyperbola we know that this is the case when the tangent at Q intercepted between the Delta lines is bisected at the point of contact. Thus we are led to the conclusion that for any pair of Delta lines the tangent to the ellipse can assume four positions in which the intercepted part is bisected by the point of contact, and consequently the points Z, Z' are then equi-distant from O .

(18) I have already shown in the Reprint to the *Educational Times*, Vol. LIV., Appendix I, that when $\triangle\triangle'$ touches the ellipse and consequently Σ lies on the curve, σ in this case becomes the Frégier point of Σ and $\delta\delta'$ the Frégier line.

(19) To the best of my belief the late Professor Wolstenholme, in his monumental collection of problems (containing as it does such a vast store of interesting matter relating to Conics), does not indicate any construction for the Delta lines, or specify their equations.

Dr. James Booth was evidently aware of their existence, and, in the special case only in which O lies on the axis, employs their properties in his "New Geometrical Methods," Vol. I

Dr. Taylor, in his "Geometry of Conics," mentions their existence, but gives no construction.

I conclude with the following analytical notes.

I. To find the value of λ for which the equation

$$(x - a)^2 + (y - \beta)^2 - \lambda \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) = 0 \quad \dots \quad (A)$$

represents straight lines; in other words, to find the common chords of the ellipse and a point-circle at a, β .

The discriminant must $= 0$, or

$$\left(1 - \frac{\lambda}{a^2} \right) \left(1 - \frac{\lambda}{b^2} \right) (\alpha^2 + \beta^2 + \lambda) - \left(1 - \frac{\lambda}{a^2} \right) \beta^2 - \left(1 - \frac{\lambda}{b^2} \right) \alpha^2 = 0.$$

It will be found that one value of $\lambda = 0$, as we should *a priori* expect since the lines $(y - \beta) = \pm i(x - a)$ form one pair of common chords.

The other two values of λ are given by the equation

$$\lambda^2 + \lambda(\alpha^2 + \beta^2 - a^2 - b^2) - (b^2\alpha^2 + a^2\beta^2 - a^2b^2) = 0. \quad \dots \quad (B)$$

Note if a, β lies on the ellipse, another value of $\lambda = 0$, and the third value of $\lambda = a^2 + b^2 - \alpha^2 - \beta^2$, and the real common chords are

$$(x - \alpha^2) + (y - \beta)^2 = (\alpha^2 + b^2 - a^2 - \beta^2) \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right).$$

which may be thrown into the form

$$(a^2 - b^2) \left\{ \frac{x\alpha}{a^2} + \frac{y\beta}{b^2} - 1 \right\} \left\{ \frac{x\alpha}{a^2} - \frac{y\beta}{b^2} - \frac{a^2 + b^2}{a^2 - b^2} \right\} = 0$$

the factors representing the tangent at (α, β) and the Frégier line corresponding to α, β .

Again, if α, β lies on the orthocyclic circle the two values of λ are $= \pm (b^2\alpha^2 + a^2\beta^2 - a^2b^2)$

From (A) the lines will be real or not, according as the parallels through the origin are real or not: that is, according as

$$\left(1 - \frac{\lambda}{a^2}\right)x^2 + \left(1 - \frac{\lambda}{b^2}\right)y^2 = 0$$

has real or imaginary roots.

Since this may be written

$$y^2 = \frac{a^2 - \lambda}{\lambda - b^2} \cdot \frac{b^2}{a^2} x^2$$

we see that λ must lie between b^2 and a^2 .

Now in (B) if we substitute $\lambda = a^2$ and $\lambda = b^2$ we get $(a^2 - b^2)\alpha^2$ and $(b^2 - a^2)\beta^2$ respectively, the first of which is positive and the second negative. This shows that (B) has always a real root between b^2 and a^2 .

II. If P be a point on the ellipse whose excentric angle is α , the coordinates of P are $a\cos\alpha, b\sin\alpha$. It can be easily verified that the semi-axes of the confocal hyperbola through P are

$$c\cos\alpha, c\sin\alpha \quad \text{where} \quad c^2 = a^2 - b^2.$$

For $\frac{a^2\cos^2\alpha}{c^2\cos^2\alpha} - \frac{b^2\sin^2\alpha}{c^2\sin^2\alpha} = 1$, and $c^2\cos^2\alpha + c^2\sin^2\alpha = c^2$.

If O be any point on this confocal hyperbola its coordinates may be taken as $c\cos\alpha\sec\phi$ and $c\sin\alpha\tan\phi$.

Now I have actually verified that the equations

$$\begin{aligned} (x - c\cos\alpha\sec\phi)^2 + (y - c\sin\alpha\tan\phi)^2 \\ = \left\{ \frac{c}{a}x\cos\alpha + \frac{c}{b}y\sin\alpha - (a\sec\phi - b\tan\phi) \right\} \\ \times \left\{ \frac{c}{a}x\cos\alpha - \frac{c}{b}y\sin\alpha - (a\sec\phi + b\tan\phi) \right\} \end{aligned}$$

and $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)(a^2\sin^2\alpha + b^2\cos^2\alpha) = 0$

are identically equal.

This shows that the real chords of the point-circle at the point whose coordinates are $c \cos a \sec \phi$ $c \sin a \tan \phi$ are the straight lines

$$\left(\frac{c}{a} x \cos a - a \sec \phi \right) \pm \left(\frac{c}{b} y \sin a + b \tan \phi \right),$$

which expressions are probably new.