VARIATION OF CONGRUENCES OF CURVES OF AN ORTHOGONAL ENNUPLE IN A RIEMANNIAN SPACE

T. K. PAN

1. Introduction. Consider any three congruences of an orthogonal ennuple at a point of a Riemannian space. When one congruence is moved by local and a second congruence is moved by parallel displacement in the direction of the third congruence, the rate of change of cosine of the angle between the first two congruences is well known as Ricci's coefficient of rotation and has been extensively studied. It is the purpose of this note to investigate the corresponding rate of change, when the third congruence is replaced by an arbitrary one, in connection with parallelism and equidistance of congruences as studied by Miss Peters [**2**; **3**].

The notation of Eisenhart [**1**] will be used for the most part.

2. Definition. A congruence of curves is uniquely determined by a vector field, which at a point is tangent to the curve of the congruence through the point. Let $\lambda_{h|}$ $(h = 1, \ldots, n)$, be the unit tangents to n congruences of an orthogonal ennuple in a Riemannian space V_n, whose first fundamental form $g_{ij}\, dx^i\, dx^j$ is positive definite. We assume that the components $\lambda_{h|}{}^i$ and the coefficients g_{ij} are real analytic functions of the coordinates x's.

Let

$$u^i = \sum_{m=1}^{n} c_m \lambda_{m|}{}^i$$

be an arbitrary but fixed congruence of curves C, where $c_h = u^i \lambda_{h|i} = \cos\theta_h$, θ_h being the angle which C makes with the congruence $\lambda_{h|}$. The unit tangent vector to C is given by

$$\xi^i = \sum_{m=1}^{n} c_m \lambda_{m|}{}^i \Big/ \left(\sum_{m=1}^{n} c_m{}^2\right)^{\frac{1}{2}}. \tag{2.1}$$

When $\lambda_{h|}$ is displaced locally and $\lambda_{k|}$ parallelly along C, the rate at which the cosine of the angle between them changes is measured by p_{hk}, where

$$p_{hk} = \lambda_{h|i,j}\, \lambda_{k|}{}^i \xi^j = \sum_{m=1}^{n} c_m \gamma_{hkm} \Big/ \left(\sum_{m=1}^{n} c_m{}^2\right)^{\frac{1}{2}}, \tag{2.2}$$

γ_{hkm} being Ricci's coefficients of rotation of the orthogonal ennuple.

The vanishing of the partial derivatives of p_{hk} with respect to the c's requires

$$c_1 : c_2 : \ldots : c_n = \gamma_{hk1} : \gamma_{hk2} : \ldots : \gamma_{hkn}. \tag{2.3}$$

Substituting (2.3) into (2.2) we find that the extreme of p_{hk} as ξ^i varies is equal to

Received September 20, 1952; in revised form November 25, 1952.

https://doi.org/10.4153/CJM-1953-058-7 Published online by Cambridge University Press

(2.4) $$\gamma_{hk} = \left(\sum_{m=1}^{n} \gamma_{hkm}{}^2\right)^{\frac{1}{2}}.$$

The preceding result (2.4) is obtained with the assumption that p_{hk} is a function of the c's. Suppose p_{hk} is independent of the c's. Then from (2.2) we have

(2.5) $$\gamma_{hkm} = 0 \qquad (m = 1, \ldots, n),$$

and consequently $p_{hk} = 0$. If zero is considered as the extreme of zero's, this extreme value can be obtained also by the substitution of (2.5) into (2.4). Hence the formula (2.4) is valid whether p_{hk} is a function of the c's or not.

We call γ_{hk} *the variation of* $\lambda_h|$ *with respect to* $\lambda_k|$. When m in (2.1) does not take the value k, we call γ_{hk} *the subvariation of* $\lambda_h|$ *with respect to* $\lambda_k|$.

3. Properties. Since $\gamma_{hkm} + \gamma_{khm} = 0$ and

$$g_{ij}\left\{\sum_{m=1}^{n} \gamma_{hkm}\,\lambda_m|^i \Big/ \left(\sum_{m=1}^{n} \gamma_{hkm}{}^2\right)^{\frac{1}{2}}\right\}\left\{\sum_{m=1}^{n} \gamma_{hkm}\lambda_m|^j \Big/ \left(\sum_{m=1}^{n} \gamma_{hkm}{}^2\right)^{\frac{1}{2}}\right\} = -1,$$

it is evident that *the variation of any two orthogonal congruences with respect to each other are equal and the corresponding directions of displacement are coincident but opposite in sense.*

The variation of any congruence with respect to itself is zero, since $\gamma_{hhm} = 0$ for $m = 1, \ldots, n$. The variation of a congruence with respect to another congruence of an ennuple, that is γ_{hk}, $h \neq k$, is zero if and only if the rate of change p_{hk} is zero along every congruence of the ennuple and hence along any congruence of curves in the V_n. In both cases the curve of displacement is arbitrary.

All the congruences of an orthogonal ennuple are normal if and only if

(3.1) $$\gamma_{hkm} = 0 \qquad (h, k, m = 1, \ldots, n; h \neq k \neq m \neq h).$$

To such an ennuple corresponds an n-ply orthogonal system of hypersurfaces. Substitution of (3.1) into (2.4) gives

(3.2) $$\gamma_{hk} = (\gamma_{hkk}{}^2 + \gamma_{khh}{}^2)^{\frac{1}{2}} = (1/r_{hk}{}^2 + 1/r_{kh}{}^2)^{\frac{1}{2}} = (1/b_{hk}{}^2 + 1/b_{kh}{}^2)^{\frac{1}{2}},$$

where $1/r_{hk}$ and $1/b_{hk}$ are respectively [**2,** pp. 108–109] the angular and distantial spreads of the congruence $\lambda_h|$ with respect to the congruence $\lambda_k|$. Hence we have

THEOREM 3.1. *In an orthogonal ennuple of normal congruences, a congruence has zero variation with respect to a second if and only if each congruence is equidistant with respect to the other or each congruence is parallel with respect to the other.*

When the variation (subvariation) of a congruence of an orthogonal ennuple with respect to another congruence of the ennuple is numerically equal to a coefficient of rotation, we call the former congruence a *principal* (*subprincipal*)

https://doi.org/10.4153/CJM-1953-058-7 Published online by Cambridge University Press

congruence with respect to the latter congruence. Hence we have from (3.2).

THEOREM 3.2. *A congruence of an orthogonal ennuple of normal congruences is a principal congruence with respect to another congruence of the ennuple if and only if one is parallel along the other.*

Let $n - 1$ mutually orthogonal congruences $\lambda_\alpha|$ $(\alpha = 1, \ldots, n-1)$, orthogonal to a normal congruence $\lambda_n|$, be canonical with respect to the latter. Necessary and sufficient conditions for this [**1,** p. 128] are $\gamma_{n\alpha\beta} = 0$, $\alpha \neq \beta$. Such a normal congruence is normal to the hypersurface V_{n-1} determined by the $\lambda_\alpha|$'s, which are the lines of curvature of the V_{n-1}.

The subvariation $\gamma_{\alpha n}$ of $\lambda_\alpha|$ with respect to $\lambda_n|$ is then equal to $|\gamma_{\alpha n\alpha}|$. Hence we have

THEOREM 3.3. *A congruence of an orthogonal ennuple is a subprincipal congruence with respect to a congruence of the ennuple if the former is any one of the $n - 1$ principal directions in the hypersurface normal to the latter.*

The difference between the squares of the variation and the subvariation of $\lambda_h|$ with respect to $\lambda_n|$ is found from (2.4) to be

$$\sum_{m=1}^{n} \gamma_{hnm}{}^2 - \sum_{\alpha=1}^{n-1} \gamma_{hn\alpha}{}^2 = \gamma_{hnn}{}^2.$$

Let μ^i denote the angular spread vector of $\lambda_h|$ with respect to $\lambda_n|$, that is $\mu^i = \lambda_h|_{i,j}\lambda_n|^j$. The projection of the vector μ^i in the direction $\lambda_n|$ is called the tendency of $\lambda_h|$ in that direction. Its value is

$$\mu^i \lambda_n|_i = \lambda_h|_{i,j}\lambda_n|^i\lambda_n|^j = \gamma_{hnn}, \tag{3.3}$$

which is equal to zero for $h = 1, \ldots, n-1$ if and only if $\lambda_n|$ is a congruence of geodesics [**1,** pp. 100]. Hence we have

THEOREM 3.4. *The difference between the squares of the variation and the subvariation of a congruence of an orthogonal ennuple with respect to another congruence of the ennuple is the square of the tendency of the former in the direction of the latter. The variation and the subvariation of each of $n - 1$ congruences of an orthogonal ennuple with respect to the remaining one congruence of the ennuple are equal if and only if the latter is a congruence of geodesics.*

Let the congruence $\lambda_h|$ of an orthogonal ennuple be normal. Then [**1,** p. 114] we have $\gamma_{h\alpha\beta} = \gamma_{h\beta\alpha}$ $(\alpha, \beta = 1, \ldots, n;\ \alpha \neq \beta \neq h)$. Consequently, equation (2.4) reduces to

$$\gamma_{hk}{}^2 = \gamma_{hkh}{}^2 + \sum_{m=1}^{n} \gamma_{mhk}{}^2 = \gamma_{hkh}{}^2 + 1/r_{hk}{}^2, \tag{3.4}$$

since $\gamma_{hhk} = 0$. Hence we have

THEOREM 3.5. *The square of the variation of a normal congruence of an orthogonal ennuple with respect to another congruence of the ennuple differs from the*

https://doi.org/10.4153/CJM-1953-058-7 Published online by Cambridge University Press

square of the angular spread of the same by the square of the tendency of the latter congruence in the direction of the former congruence. The variation and the angular spread of a normal congruence of an orthogonal ennuple with respect to any one congruence of the ennuple are numerically equal if and only if the tendency of the latter in the direction of the former is zero.

Equation (3.4) indicates that the vanishing of γ_{hk} implies the vanishing of γ_{hkh} and $1/r_{hk}$ and conversely. Hence we have

THEOREM 3.6. *If the variation of a normal congruence of an orthogonal ennuple with respect to another congruence of the ennuple is zero, then the former is parallel along the latter and the tendency of the latter in the direction of the former is zero. Conversely, if a normal congruence of an orthogonal ennuple is parallel along another congruence of the ennuple, whose tendency in the direction of the normal congruence is zero, then the variation of the normal congruence with respect to the latter congruence is zero.*

An immediate consequence of the preceding theorem is that *a normal congruence of geodesics of an orthogonal ennuple is parallel along a congruence of the ennuple if and only if the variation of the normal congruence with respect to it is zero.*

By summing over h in (2.4) we obtain

$$\sum_{h=1}^{n} \gamma_{hk}^{2} = \sum_{h=1}^{n} 1/r_{kh}^{2}, \tag{3.5}$$

where $1/r_{kk}$ denotes the first curvature of $\lambda_{k|}$. Note that equation (3.5) holds for general orthogonal ennuple of congruences. Hence we have

THEOREM 3.7. *The curves of a congruence of an orthogonal ennuple are parallel along the curves of all congruences of the ennuple if and only if the variation of the congruence with respect to each congruence of the ennuple is zero. The curves of a congruence of an orthogonal ennuple are geodesics if the variation of the congruence with respect to each congruence of the ennuple is zero.*

If $\gamma_{hk} = 0$ for $h, k = 1, \ldots, n$, then we have from (3.5)

$$1/r_{kh} = 0, \quad 1/r_{kk} = 0 \qquad (h, k = 1, \ldots, n).$$

Consequently, all the congruences of the ennuple consists of geodesics and the curves of each congruence are parallel along the curves of all congruences of the ennuple [**3,** p. 565] and hence parallel along the curves of any congruence in the V_n. Thus we obtain from (3.5)

THEOREM 3.8. *The variation of each congruence of an orthogonal ennuple with respect to every other congruence of the ennuple is zero if and only if the ennuple consists of congruences of geodesics and each congruence of the ennuple is parallel along every other congruence of the ennuple and hence parallel along every congruence in the V_n.*

https://doi.org/10.4153/CJM-1953-058-7 Published online by Cambridge University Press

REFERENCES

1. L. P. Eisenhart, *Riemannian geometry* (Princeton, 1949).

2. R. M. Peters, *Parallelism and equidistance in Riemannian geometry*, Amer. J. Math., *57* (1935), 103–111.

3. R. M. Peters, *Parallelism and equidistance of congruences*, Amer. J. Math., *59* (1937), 564–574.

University of California and
University of Oklahoma

https://doi.org/10.4153/CJM-1953-058-7 Published online by Cambridge University Press