TOTAL DIGRAPHS

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1. Introduction. The line-graph $\mathcal{L}(G)$ of an ordinary graph G is that graph whose points can be put in one-to-one correspondence with the lines of G in such a way that two points of $\mathcal{L}(G)$ are adjacent if and only if the corresponding lines of G are adjacent. This concept originated with Whitney [5]. $\mathcal{L}(G)$ has the property that its (point) chromatic number equals the line chromatic number of G, where the point (line) chromatic number of graph is the minimum number of colors required to color the points (lines) of the graph such that adjacent points (lines) are colored differently. Behzad [1] defines the total chromatic number of a graph G as the minimum number of colors needed to color both the points and lines of G so that two adjacent elements (i.e., two points, two lines, or a point and a line) have different colors. The total graph $\mathcal{T}(G)$ of a graph G is that graph whose points can be put in one-to-one correspondence with the points and lines of G in such a way that two points of $\mathcal{T}(G)$ are adjacent if and only if the corresponding elements of G are adjacent (see[1]). Analogous to the situation with line-graphs, it follows that the chromatic number of $\mathcal{T}(G)$ equals the total chromatic number of G.

Harary and Norman [2] generalized the definition of line-graph to the directed case thereby introducing the "line-digraph". It is the object of this paper to extend the concept of total graph in a similar and natural way to the directed case and to develop some of the properties of the "total digraph".

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We begin by presenting some definitions. (For all terms not defined here, the reader is directed to [3].) In the line $v_i v_j$ we say that the point v_i is <u>adjacent to</u> v_j and v_j is <u>adjacent from</u> v_i. We also say that the line $x = v_i v_j$ is <u>adjacent to</u> the line $y = v_j v_k$, v_i is <u>adjacent to</u> x, and x is <u>adjacent to</u> v_j.

If to the line $x = v_i v_j$ in a digraph D we insert a new point v_{ij} and lines $x' = v_i v_{ij}$ and $x'' = v_{ij} v_j$, then the path $v_i, x', v_{ij}, x'', v_j$ is called a <u>carrier path</u> to x. The point v_{ij} is itself a carrier since its indegree $id(v_{ij})$ and outdegree $od(v_{ij})$ are both 1. Also, if uv and vw are lines of D, then the line uw is referred to as a <u>transitive line</u> of D.

We now define the <u>total digraph</u> $\mathcal{T}(D)$ of a digraph D to be the digraph whose points are in one-to-one correspondence with the points and lines of D, and such that the point u is adjacent to the point v in $\mathcal{T}(D)$ if and only if in D the element corresponding to u is adjacent to the element corresponding to v. A digraph T is a <u>total digraph</u> if there exists a digraph D such that $T = \mathcal{T}(D)$. (See Figure 1 for examples.)

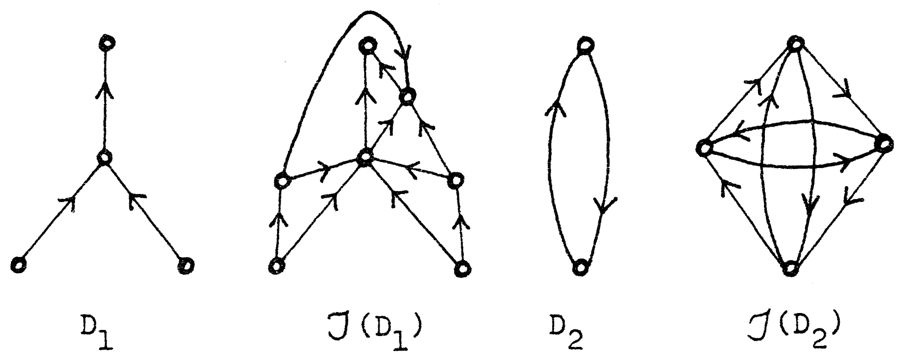

Figure 1

If D has p points and q lines, it follows immediately that $\mathcal{T}(D)$ has $p + q$ points. Since every line of $\mathcal{T}(D)$ must correspond to either a line in D (there are q of these), two

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adjacent lines in D (this is given by $\sum_{i=1}^{p} id(v_i)\, od(v_i))$, a line adjacent to a point in D (there are q of these), or a point adjacent to a line in D (there are also q of these), we see that $\mathcal{J}(D)$ has precisely $3q + \sum_{i=1}^{p} id(v_i)\, od(v_i)$ lines. Another invariant of interest which can be computed in a straightforward manner is the number of cyclic triples (cycles of length 3); in fact, if a digraph D has c cyclic triples and n pairs of mutually adjacent points, then $\mathcal{J}(D)$ has $2c + 4n$ cyclic triples.

2. An Alternative Approach to Total Digraphs. In this section we give a characterization of total digraphs utilizing the so-called subdivision digraph.

The subdivision digraph $\mathcal{S}(D)$ of a digraph D is that digraph obtained from D by replacing each line $v_i v_j$ of D with a new point v_{ij} and the two lines $v_i v_{ij}$ and $v_{ij} v_j$. A digraph S is called a subdivision digraph if there exists a digraph D such that $S = \mathcal{S}(D)$. Necessary and sufficient conditions for a digraph to be a subdivision digraph are given next.

THEOREM 1. A connected digraph S is a subdivision digraph if and only if (a) S is a cycle of even length n, $n \geq 4$, or (b) every semipath joining two noncarriers (distinct or not) has even length.

Proof of necessity. Since only a cycle can be subdivided to yield a cycle and since the subdivision process doubles the length of a cycle, it follows that the only cycles which are subdivision digraphs must be of even length $n \geq 4$. If S is a subdivision digraph which is not a cycle, then (b) follows directly.

Proof of sufficiency. If (a) holds, i.e., if S is a cycle of even length $n \geq 4$, then let D be a cycle of length $n/2$, and then $S = \mathcal{S}(D)$. If S is not a cycle, we distinguish two cases. (i) If S is simply a semipath with points (in order) $v_0, v_1, \ldots, v_{2k}$, then let D be the digraph consisting of the points v_{2i} $(0 \leq i \leq k)$ and having v_{2i} adjacent to (from) v_{2i+2} if and only if v_{2i} is adjacent to (from) v_{2i+1} in S; then $S = \mathcal{S}(D)$.

(ii) If S is not a semipath, then S contains at least one noncarrier v which is not a terminal point. Let V be the set of all points of S which are connected to v with a semipath of even length. This set is well-defined since if a point u of V were connected to v by both an even and an odd semipath, this would imply the noncarrier v is connected to itself by a semipath of odd length, contradicting our hypothesis. Now if V is taken to be the point set of a digraph D where a point v_i is adjacent to a point v_j in D if and only if there is a path of length two from v_i to v_j, it is then easy to see that $S = \mathcal{S}(D)$.

As an analogue to the square of an ordinary graph [4], the <u>square</u> D^2 of a digraph D is defined as that digraph whose points are those of D and such that a point u is adjacent to a point v in D^2 if and only if u is connected to v by a path of length one or two in D.

We are now in a position to state the principal result of this section.

THEOREM 2. A digraph T is a total digraph if and only if there exists a subdivision digraph S such that S^2 is isomorphic to T.

<u>Proof.</u> We show that for any digraph D, $[\mathcal{S}(D)]^2 = \mathcal{T}(D)$. Each of the digraphs $\mathcal{S}(D)$, $[\mathcal{S}(D)]^2$, and $\mathcal{T}(D)$ has $p+q$ points, where D has p points and q lines. By the definition of $\mathcal{S}(D)$, each line v_iv_j of D is replaced by the carrier path $v_i, v_iv_{ij}, v_{ij}, v_{ij}v_j, v_j$ in $\mathcal{S}(D)$. Furthermore, $[\mathcal{S}(D)]^2$ differs from $\mathcal{S}(D)$ only by the addition of new transitive lines of two types: lines $v_{ij}v_{jk}$ and lines v_iv_j added to each carrier path from v_i to v_j. If we correspond the points v_{ij} of $[\mathcal{S}(D)]^2$ to lines v_iv_j in D and points v_i of $[\mathcal{S}(D)]^2$ to themselves in D, we readily see that $[\mathcal{S}(D)]^2 = \mathcal{T}(D)$, inasmuch as a point u is then adjacent to a point v in $[\mathcal{S}(D)]^2$ if and only if the element of D corresponding to u is adjacent to the element of D corresponding to v.

Thus if T is a total digraph, i.e., $T = \mathcal{T}(D)$ for some digraph D , then simply take $S = \mathfrak{S}(D)$, and by the above discussion $S^2 = T$. Likewise, if $S = \mathfrak{S}(D)$ and $S^2 = T$, then T is a total digraph, namely $T = \mathfrak{T}(D)$.

3. Some Properties of Total Digraphs. In this section we investigate the connectedness properties of D, $\mathfrak{S}(D)$, and $\mathfrak{T}(D)$.

Unlike ordinary graphs, there are many ways in which a digraph may be "connected". Again following the notation of [3] , we categorize digraphs as follows: C_3 is the class of all strong digraphs, C_2 the class of all strictly unilateral digraphs, C_1 the strictly weak digraphs, and C_0 is the class of disconnected digraphs.

Clearly, the addition to or replacement of a line of D with a transitive line or a carrier path does not affect the existence of a path (or semipath) between points of D , i.e., does not alter the connectedness category of D . This implies that $\mathfrak{S}(D)$ and D^2 belong to category C_i if and only if D belongs to C_i , $i = 0, 1, 2, 3$. Since $\mathfrak{T}(D)$ is isomorphic to $[\mathfrak{S}(D)]^2$, it also follows that D and $\mathfrak{T}(D)$ must belong to the same connectedness category as well.

From the preceding remarks, it is now evident that if A is a strong component of D , then $\mathfrak{S}(A)$ and $\mathfrak{T}(A)$ are strong components of $\mathfrak{S}(D)$ and $\mathfrak{T}(D)$ respectively, and in addition each line v_iv_j not contained in any strong component of D gives rise to a trivial component v_{ij} in $\mathfrak{S}(D)$ and $\mathfrak{T}(D)$.

We are now in a position to establish a result concerning point bases and contrabases of a digraph D and its total digraph $\mathfrak{T}(D)$. To do this we make use of the condensation digraph D* . It is not difficult to show that the number of points of indegree (outdegree) zero in $[\mathfrak{T}(D)]^*$ equals the number of points of indegree (outdegree) zero in D* . This fact coupled with the well-known result ([3] , Chapter 4) that the number of points in a point basis (contrabasis) of a digraph D equals the number of points of indegree (outdegree) zero in D* allows us to conclude that point bases (contrabases) of D and $\mathfrak{T}(D)$ must have the same number of elements.

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Another feature that a digraph and its total digraph share concerns the existence of cycles. If D contains a cycle, then certainly $\mathcal{J}(D)$ does also. On the other hand, suppose $\mathcal{J}(D)$ has a cycle $v_1 v_2 \ldots v_n v_1$. Then the points and lines of D corresponding to the points v_j are similarly oriented and produce a cycle (or cycles) in D. This result along with the others in this section are summarized below.

THEOREM 3. Let D be a digraph and $\mathcal{J}(D)$ its total digraph. Then

(a) D and $\mathcal{J}(D)$ belong to the same connectedness category.

(b) The cardinalities of point bases (contrabases) of D and $\mathcal{J}(D)$ are equal.

(c) D is acyclic (has no cycles) if and only if $\mathcal{J}(D)$ is acyclic.

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