Appendix E

Mass matrices and mixing

E.1 K^o and $\bar{K}^o$

A phenomenological description of the time development of an electrically charged meson $|P\rangle$ at rest is given by the equation

$$i\frac{d}{dt}|P\rangle = [m - (i/2)\,\Gamma]\,|P\rangle \tag{E.1}$$

with its solution

$$|P(t)\rangle = |P(0)\rangle e^{-imt-(1/2)\Gamma t}$$

Here, m is the meson mass, Γ is the decay rate and $1/\Gamma$ is the mean life of the meson.

Electrically neutral mesons, for example $K^o(d\bar{s})$ and $B^o(d\bar{b})$, which have a distinct antimeson, in this example $\bar{K}^o(s\bar{d})$ and $\bar{B}^o(b\bar{d})$, can mix so that (E.1) becomes two coupled equations. For K^o and $\bar{K}^o$ these are

$$i\frac{d}{dt}\begin{pmatrix}|K^o\rangle\\|\bar{K}^o\rangle\end{pmatrix} = \begin{pmatrix}m-(i/2)\,\Gamma & -p^2\\ -q^2 & m-(i/2)\,\Gamma\end{pmatrix}\begin{pmatrix}|K^o\rangle\\|\bar{K}^o\rangle\end{pmatrix} \tag{E.2}$$

p^2 and q^2 are two complex numbers. We can regard the 2×2 mass matrix as an 'effective' Hamiltonian H_{weak}. The equality of the diagonal elements of H_{weak} is guaranteed by *CPT* invariance. The weak interaction generates the off-diagonal elements

$$\langle K^o|H_{weak}|\bar{K}^o\rangle = -p^2,\ \langle\bar{K}^o|H_{weak}|K^o\rangle = -q^2.$$

Contributions to p^2 and q^2 are illustrated in Fig. E.1.

By substitution into (E.2) it can be seen that the eigenstates of H_{weak} are

$$|K_S\rangle = N[p|K^o\rangle + q|\bar{K}^o\rangle] \tag{E.3}$$

and

$$|K_L\rangle = N[p|K^o\rangle - q|\bar{K}^o\rangle] \tag{E.4}$$

with eigenvalues $m - i\Gamma/2 - pq$ and $m - i\Gamma/2 + pq$ respectively. $N = (|p|^2 + |q|^2)^{-1/2}$ is a normalising factor. We choose the sign of the square root, $pq = \sqrt{p^2q^2}$, so that $\mathrm{Im}(pq)$ is positive; then K_L has a longer mean life than K_S.

The mass difference $\Delta m = 2\mathrm{Real}(pq)$ (from experiment $\Delta m \approx 3\times 10^{-12}$ MeV). We shall identify m with the mean mass of K_S and K_L. The mean lives are

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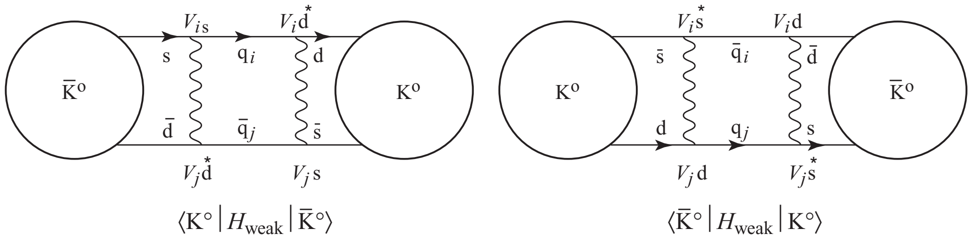

Figure E.1 Quark diagrams illustrating how the weak interaction with W bosons generates mixing. q_i, and q_j are any of the (2/3)e charged quarks u, c or t. The mixing matrix elements are proportional to the products of the four KM factors in the diagrams.

$\tau_{\mathrm{L}} \approx \dfrac{1}{\Gamma - 2\,\mathrm{Im}(pq)}$ and $\tau_{\mathrm{S}} = \dfrac{1}{\Gamma + 2\,\mathrm{Im}(pq)}$ (from experiment $\tau_{\mathrm{L}} \approx 5 \times 10^{-8}$ s, $\tau_{\mathrm{S}} \approx 10^{-10}\,s$.) The subscripts L and S refer to the long and short lives.

From lattice estimations of the bound state wave functions and other QCD modifications, p^2 and q^2 can be calculated by perturbation theory in the weak interaction. Fig.E.1 illustrates the fact that because some of the KM factors V_{is}, etc. are complex numbers, p and q are not equal. As a consequence neither $|\mathrm{K_L}\rangle$ nor $|\mathrm{K_S}\rangle$ is an eigenstate of *CP*. See Section (18.4).

E.2 B$^{\mathrm{o}}$ and $\bar{\mathrm{B}}^{\mathrm{o}}$

The neutral B meson pair B$^{\mathrm{o}}$ and $\bar{\mathrm{B}}^{\mathrm{o}}$ mix by the same mechanism as the neutral K mesons. The parameters m, Γ, p^2 and q^2 take, of course, different values.

For the B pair Im(pq) is much smaller than Γ so that the two mean lives are almost equal. There are two particles of different mass:

$$|\mathrm{B_L}\rangle = N[p|\mathrm{B^o}\rangle + q|\bar{\mathrm{B}}^{\mathrm{o}}\rangle],$$
$$|\mathrm{B_H}\rangle = N[p|\mathrm{B^o}\rangle - q|\bar{\mathrm{B}}^{\mathrm{o}}\rangle].$$

The subscripts L and H refer to their masses: light and heavy.

For B$^{\mathrm{o}}\bar{\mathrm{B}}^{\mathrm{o}}$ mixing it is a fortunate circumstance that the top quark $\mathrm{q}_i = \mathrm{t}$, $\bar{\mathrm{q}}_j = \bar{\mathrm{t}}$ gives the dominant contribution to p^2 and q^2, p^2 is proportional to $(V_{\mathrm{tb}}V_{\mathrm{td}}^*)^2$ and q^2 is proportional to $(V_{\mathrm{tb}}^*V_{\mathrm{td}})^2$ (see Fig. E. 1) Calculations result in the expressions

$$\begin{aligned} p &= \sqrt{m_{\mathrm{B}} m_{\mathrm{t}}}\,\frac{G_F}{4\pi} f_{\mathrm{B}} F_{\mathrm{tt}} V_{\mathrm{tb}} V_{\mathrm{td}}^*, \\ q &= \sqrt{m_{\mathrm{B}} m_{\mathrm{t}}}\,\frac{G_F}{4\pi} f_{\mathrm{B}} F_{\mathrm{tt}} V_{\mathrm{tb}}^* V_{\mathrm{td}}. \end{aligned} \tag{E.5}$$

(Donoghue *et al.*, 1992, p. 395.)

All other contributions are smaller by factors of $(m_{\mathrm{c}}/m_{\mathrm{t}})^2$, m_{B} is the B meson mass, $f_{\mathrm{B}} \approx 0.3$ GeV is its 'leptonic decay constant' and F_{tt} is a dimensionless number, real to a very good approximation.

With F_{tt} real, $\mathrm{Im}(pq) = 0$, and $\mathrm{B_L}$ and $\mathrm{B_H}$ have the same mean life. Within experimental error this is seen to be so. Also $|p| = |q|$ and $p = |p|\mathrm{e}^{\mathrm{i}\beta}$ $q = |p|\mathrm{e}^{-\mathrm{i}\beta}$.

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(See the unitarity triangle, Fig. 18.2). Hence

$$|B_L\rangle = \frac{1}{\sqrt{2}}\left[e^{i\beta}\left|B^o\right\rangle + e^{-i\beta}\left|\bar{B}^o\right\rangle\right]$$
$$|B_H\rangle = \frac{1}{\sqrt{2}}\left[e^{i\beta}\left|B^o\right\rangle - e^{-i\beta}\left|\bar{B}^o\right\rangle\right]. \quad \text{(E.6)}$$

A B_L meson or a B_H meson, at rest, develop independently with time

$$|B_L(t)\rangle = |B_L\,(o)\rangle\, e^{-i(m-\Delta m/2)t - t/2\tau},$$
$$|B_H(t)\rangle = |B_H\,(o)\rangle\, e^{-i(m+\Delta m/2)t - t/2\tau}.$$

After some algebra it then follows that an initial B^o or $\bar{B}^o$ develops in time into a mixture denoted by

$$|B^o_{phy}(t)\rangle = \left[\cos\left(\frac{\Delta mt}{2}\right)|B^o\rangle + ie^{-2i\beta}\sin\left(\frac{\Delta mt}{2}\right)|\bar{B}^o\rangle\right]e^{-imt-t/2\tau}$$
$$|\bar{B}^o_{phy}(t)\rangle = \left[ie^{2i\beta}\sin\left(\frac{\Delta mt}{2}\right)|B^o\rangle + \cos\left(\frac{\Delta mt}{2}\right)|\bar{B}^o\rangle\right]e^{-imt-t/2\tau.} \quad \text{(E.7)}$$

If the meson decays at time t, to a final state $|f\rangle$ the decay amplitude for an initial B^o will be

$$\langle f|B^o_{phy}(t)\rangle = \left[\cos\left(\frac{\Delta mt}{2}\right)A_f + ie^{-2i\beta}\sin\left(\frac{\Delta mt}{2}\right)\bar{A}_f\right]e^{-imt-t/2\tau}$$

and an initial B^o

$$\langle f|\bar{B}^o_{phy}(t)\rangle = \left[ie^{2i\beta}\sin\left(\frac{\Delta mt}{2}\right)A_f + \cos\left(\frac{\Delta mt}{2}\right)\bar{A}_f\right]e^{-imt-t/2\tau}. \quad \text{(E.8)}$$

$A_f = \langle f|B^o_{phy}\rangle$ and $\bar{A}_f = \langle f|\bar{B}^o_{phy}\rangle$ are the amplitudes for the decays $B^o \to f$ and $\bar{B}^o \to f$. If the charge parity (CP) of f is $+1$ then it does not couple to the $CP = -1$ state $(B^o - \bar{B}^o)$; hence $A_f = \bar{A}_f$. The decay rates are then

$$\text{Rate}\left(B^o_{phy}(t) \to f\right) = |A_f|^2 e^{-t/\tau}[1 + \sin(2\beta)\sin(mt)]$$
$$\text{Rate}\left(\bar{B}^o_{phy}(t) \to f\right) = |A_f|^2 e^{-t/\tau}[1 - \sin(2\beta)\sin(mt)]. \quad \text{(E.9)}$$

If f has $CP = -1$ the same expression results but with the $+$ and $-$ signs interchanged.

At Cleo, Babar and Belle, B^o and $\bar{B}^o$ mesons are produced in pairs. If one undergoes a leptonic decay with a negative charge lepton it must have been a $\bar{B}^o$, its partner, at that instant is a B^o and it is the time dependence of this second decay that is measured.

Similarly a positive charge lepton identifies a B^o decay that leaves its partner an initial $\bar{B}^o$. This procedure is called tagging. The mass difference Δm and $\sin 2\beta$ are measured by tracking the time dependence of tagged mesons.

The formulae for p^2 and q^2 for K^o, $\bar{K}^o$ follow the same pattern as for B decays but the top quark contributions are highly suppressed by very small KM factors. c and u quarks contribute significantly and the simplicity for B mesons is lost.