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Abstract

We give a large family of weighted projective planes, blown up at a smooth point, that do not have finitely generated Cox rings. We then use the method of Castravet and Tevelev to prove that the moduli space $\overline{M}_{0,n}$ of stable n-pointed genus-zero curves does not have a finitely generated Cox ring if n is at least 13.

1. Introduction

We work over an algebraically closed field k of characteristic zero. In their recent article [CT15], Castravet and Tevelev proved that the moduli space $\overline{M}_{0,n}$ does not have a finitely generated Cox ring when $n \geqslant 134$. They reduced the non-finite generation problem from the case of moduli spaces to the case of weighted projective planes blown up at the identity element t_0 of the torus. Examples of such weighted projective planes have been studied previously by many algebraists, as the Cox rings of these blowups appear as symbolic algebras of monomial prime ideals. Goto *et al.* [GNW94] gave an infinite family of weighted projective planes $\mathbb{P}(a, b, c)$, such that $\mathrm{Bl}_{t_0}\mathbb{P}(a, b, c)$ does not have a finitely generated Cox ring. The smallest such example, $\mathbb{P}(25, 29, 72)$, was used by Castravet and Tevelev to get the bound $n = 134$.

We extend these results by giving a large family of weighted projective planes $\mathbb{P}(a, b, c)$, such that the blowup $\mathrm{Bl}_{t_0}\mathbb{P}(a, b, c)$ does not have a finitely generated Cox ring. This family includes all examples of Goto *et al.*, but also weighted projective planes with smaller numbers, such as $\mathbb{P}(7, 15, 26)$, $\mathbb{P}(7, 22, 17)$, $\mathbb{P}(12, 13, 17)$ (Table 1 lists more such examples). More generally, we study projective toric surfaces X_Δ of Picard number 1, such that the blowup $\mathrm{Bl}_{t_0}X_\Delta$ does not have a finitely generated Cox ring.

Using the reduction method of Castravet and Tevelev, we prove the following theorem.

Theorem 1.1. *The moduli space $\overline{M}_{0,n}$ does not have a finitely generated Cox ring when $n \geqslant 13$.*

In the terminology of Hu and Keel [HK00], the theorem implies that $\overline{M}_{0,n}$ is not a Mori dream space when $n \geqslant 13$. In contrast, it is known from [HK00] that the variety $\overline{M}_{0,n}$ has a finitely generated Cox ring when $n \leqslant 6$ (see [Cas09] for explicit generators in the case of $\overline{M}_{0,6}$). Remarkably, all log-Fano varieties have finitely generated Cox rings by [BCHM10], but even though $\overline{M}_{0,n}$ is log-Fano for $n \leqslant 6$ that is not the case for $n > 7$. In this way, Hu and Keel's question in [HK00] of whether the Cox ring of $\overline{M}_{0,n}$ is finitely generated now only remains unsettled for $7 \leqslant n \leqslant 12$.

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Table 1. Weighted projective planes $\mathbb{P}(a,b,c)$, $a,b,c \leqslant 30$, with relation $(e,f,-g)$, that satisfy the conditions of Theorem 1.5.

$\mathbb{P}(a,b,c)$	$(e,f,-g)$	$\mathbb{P}(a,b,c)$	$(e,f,-g)$	$\mathbb{P}(a,b,c)$	$(e,f,-g)$
$\mathbb{P}(7,15,26)$	$(1,3,-2)$	$\mathbb{P}(16,25,11)$	$(1,2,-6)$	$\mathbb{P}(22,13,29)$	$(1,5,-3)$
$\mathbb{P}(7,17,29)$	$(1,3,-2)$	$\mathbb{P}(17,13,23)$	$(1,4,-3)$	$\mathbb{P}(22,21,17)$	$(1,3,-5)$
$\mathbb{P}(7,22,17)$	$(1,2,-3)$	$\mathbb{P}(17,16,27)$	$(1,4,-3)$	$\mathbb{P}(23,28,25)$	$(3,2,-5)$
$\mathbb{P}(7,25,19)$	$(1,2,-3)$	$\mathbb{P}(17,21,20)$	$(1,3,-4)$	$\mathbb{P}(24,13,19)$	$(1,4,-4)$
$\mathbb{P}(10,11,27)$	$(1,4,-2)$	$\mathbb{P}(17,25,23)$	$(1,3,-4)$	$\mathbb{P}(24,17,23)$	$(1,4,-4)$
$\mathbb{P}(10,21,13)$	$(1,2,-4)$	$\mathbb{P}(17,29,26)$	$(1,3,-4)$	$\mathbb{P}(24,26,17)$	$(1,3,-6)$
$\mathbb{P}(10,29,17)$	$(1,2,-4)$	$\mathbb{P}(18,23,25)$	$(3,2,-4)$	$\mathbb{P}(26,18,29)$	$(1,5,-4)$
$\mathbb{P}(11,21,25)$	$(3,2,-3)$	$\mathbb{P}(19,11,13)$	$(1,3,-4)$	$\mathbb{P}(27,10,29)$	$(1,6,-3)$
$\mathbb{P}(12,13,17)$	$(1,3,-3)$	$\mathbb{P}(19,22,26)$	$(2,3,-4)$	$\mathbb{P}(27,17,28)$	$(1,5,-4)$
$\mathbb{P}(12,19,23)$	$(1,3,-3)$	$\mathbb{P}(19,26,29)$	$(2,3,-4)$	$\mathbb{P}(27,19,14)$	$(1,3,-6)$
$\mathbb{P}(12,25,29)$	$(1,3,-3)$	$\mathbb{P}(19,27,20)$	$(1,3,-5)$	$\mathbb{P}(27,22,23)$	$(1,4,-5)$
$\mathbb{P}(13,9,29)$	$(1,5,-2)$	$\mathbb{P}(19,29,11)$	$(1,2,-7)$	$\mathbb{P}(27,25,17)$	$(1,3,-6)$
$\mathbb{P}(13,18,25)$	$(3,2,-3)$	$\mathbb{P}(20,21,26)$	$(1,4,-4)$	$\mathbb{P}(29,19,21)$	$(1,4,-5)$
$\mathbb{P}(14,29,25)$	$(3,2,-4)$	$\mathbb{P}(20,22,27)$	$(1,4,-4)$	$\mathbb{P}(29,30,17)$	$(1,3,-7)$

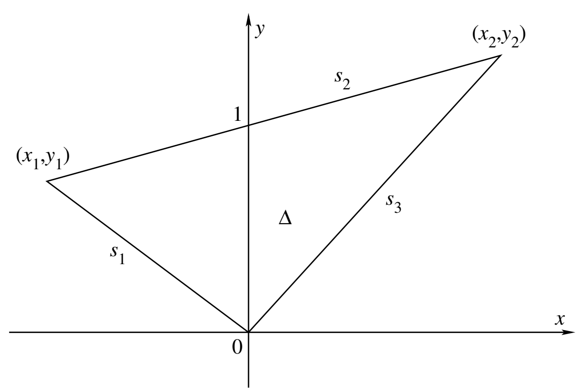

Figure 1. Triangle Δ.

Let us recall that for a normal $\mathbb{Q}$-factorial projective variety X with a finitely generated class group $\mathrm{Cl}(X)$, a Cox ring of X is any multigraded algebra of the form

$$R(X;D_1,\dots,D_r) = \bigoplus_{(m_1,\dots,m_r)\in\mathbb{Z}^r} H^0(X,\mathcal{O}_X(m_1D_1+\cdots+m_rD_r)),$$

where $D_1,\dots,D_r$ are Weil divisors whose classes span $\mathrm{Cl}(X)\otimes\mathbb{Q}$. The finite generation of a Cox ring of X is equivalent to the finite generation of every Cox ring of X, and it has strong implications for the birational geometry of X (see [HK00]). In the language of [HK00], X is a Mori dream space if and only if X has a finitely generated Cox ring.

To construct the toric varieties X_Δ, we start with a triangle $\Delta \subset \mathbb{R}^2$ as shown in Figure 1. The vertices of Δ have rational coordinates, $(0,0)$ is one vertex, and the point $(0,1)$ lies in the interior of the opposite side. Such a triangle is uniquely determined by the slopes of its sides $s_1 < s_2 < s_3$.

The triangle Δ defines a toric variety X_Δ, whose fan is the normal fan of Δ. Let $\mathrm{Bl}_{t_0} X_\Delta$ be the blowup of X_Δ at the identity point of the torus $t_0 \in T \subset X_\Delta$.

Theorem 1.2. *Let the triangle Δ as in Figure 1 be given by rational slopes $s_1 < s_2 < s_3$. The variety $\mathrm{Bl}_{t_0} X_\Delta$ does not have a finitely generated Cox ring if the following two conditions are satisfied.*

(i) *Let*

$$w = \frac{1}{s_2 - s_1} + \frac{1}{s_3 - s_2}.$$

Then $w \leqslant 1$.

(ii) *Let $n = |[s_1, s_2] \cap \mathbb{Z}|$. Then $|(n-1)[s_2, s_3] \cap \mathbb{Z}| = n$ and $ns_2 \notin \mathbb{Z}$.*

The number w in the theorem is the width of Δ: if (x_1, y_1) and (x_2, y_2) are the two non-zero vertices of Δ, then $w = x_2 - x_1$. To explain the second condition, consider a multiple $m\Delta$ that has integral vertices. A column in $m\Delta$ consists of all lattice points with a fixed first coordinate. Then n is the number of lattice points in the second column from the left (i.e. with x-coordinate mx_1+1). The second condition requires that the nth column from the right (i.e. with x-coordinate $mx_2 - (n-1)$) contains exactly n lattice points. Moreover, the $(n+1)$th column from the right should not contain a lattice point on the top edge.

It is easy to construct examples of triangles Δ that satisfy the conditions of Theorem 1.2. In fact, one can find a non-empty open region in $\mathbb{R}^3$, so that any rational point (s_1, s_2, s_3) in that region defines such a triangle.

Different triangles may give rise to isomorphic toric varieties. However, if a triangle Δ exists with the property that $w \leqslant 1$, then the two lattice points $(0,0)$ and $(0,1)$ in Δ are determined by the toric variety X_Δ. (As we will see below, the binomial $1-y$ determined by the two lattice points defines an irreducible curve $C \subset \mathrm{Bl}_{t_0} X_\Delta$ of non-positive self-intersection $C \cdot C \leqslant 0$, hence its class lies on the boundary of the cone of effective curves and is thus uniquely determined up to a scalar multiple. The image of the curve C in X_Δ passes through two of the three torus-fixed points, hence if there were two different curves, their intersection would be positive.) The other triangles that give rise to toric varieties that are isomorphic by a toric morphism are obtained from Δ by applying an integral linear transformation that preserves the two lattice points. These transformations are generated by reflections across the y-axis and shear transformations $(x, y) \mapsto (x, y + ax)$ for $a \in \mathbb{Z}$. The shear transformation adds the integer a to each of the three slopes and does not affect the two conditions of the theorem. The reflection switches the columns from the left with the columns from the right. A triangle and its reflection cannot both satisfy the conditions of Theorem 1.2 for $n \neq 2$. When $n = 2$, Δ satisfies the two conditions if and only if its reflection satisfies them.

Example 1.3. Let

$$s_1 = -\tfrac{2}{3}, \quad s_2 = \tfrac{1}{2}, \quad s_3 = 8.$$

Then the two conditions of the theorem are satisfied with $w = 104/105$ and $n = 1$. The normal fan of Δ has rays generated by

$$v_1 = (2, 3), \quad v_2 = (1, -2), \quad v_3 = (-8, 1),$$

which satisfy the relation

$$15v_1 + 26v_2 + 7v_3 = 0.$$

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Moreover, since v_1, v_2, v_3 generate the lattice $\mathbb{Z}^2$, the toric variety X_Δ is the weighted projective plane $\mathbb{P}(15,26,7)$. Then by Theorem 1.2, $\mathrm{Bl}_{t_0}\mathbb{P}(15,26,7) \cong \mathrm{Bl}_{t_0}\mathbb{P}(26,15,7)$ does not have a finitely generated Cox ring. This last fact also admits a simpler direct proof; see Remark 2.5.

Example 1.4. Take

$$s_1 = -\tfrac{11}{3}, \quad s_2 = -\tfrac{4}{3}, \quad s_3 = \tfrac{2}{3}.$$

Then one checks that the two conditions are satisfied, with $w = 3/7 + 3/6 = 13/14$, $n = 2$. The normal fan of the triangle Δ has rays generated by

$$v_1 = (11,3), \quad v_2 = (-4,-3), \quad v_3 = (-2,3),$$

satisfying the relation

$$6v_1 + 13v_2 + 7v_3 = 0.$$

The vectors v_1, v_2, v_3 generate a sublattice of index 3 in $\mathbb{Z}^2$. It follows that X_Δ is a quotient of $\mathbb{P}(6,13,7)$ by an order-three subgroup of the torus.

As the examples above illustrate, the toric varieties X_Δ are, in general, quotients of weighted projective planes $\mathbb{P}(a,b,c)$ by a finite subgroup of the torus. We would like to know which weighted projective planes $\mathbb{P}(a,b,c)$ correspond to triangles as in Theorem 1.2. Let e, f, g be positive integers, $\gcd(e,f,g) = 1$, such that

$$ae + bf - cg = 0.$$

We call $(e, f, -g)$ a relation for $\mathbb{P}(a,b,c)$.

Theorem 1.5. *Let $\mathbb{P}(a,b,c)$ be a weighted projective plane with relation $(e,f,-g)$. Then the blowup $\mathrm{Bl}_{t_0}\mathbb{P}(a,b,c)$ does not have a finitely generated Cox ring if the following conditions are satisfied.*

(i) *Let*

$$w = \frac{g^2 c}{ab}.$$

Then $w \leqslant 1$.

(ii) *Let n be the number of integers $\delta \leqslant 0$, such that*

$$(b,a) + \delta(e,-f)$$

has non-negative components, both divisible by g. Then there must exist exactly n integers $\gamma \geqslant 0$, such that

$$(n-1)(b,a) + \gamma(e,-f)$$

has non-negative components, both divisible by g. Moreover, $n(b,a) \neq (0,0) \pmod{g}$.

To find whether the theorem applies to a weighted projective plane $\mathbb{P}(a,b,c)$, one has to consider all relations $(e,f,-g)$, possibly after permuting a,b,c. However, there can be at most one such relation satisfying $w \leqslant 1$, even after permuting a,b,c (this is again due to the existence of a curve C of non-positive self-intersection). To find this relation, one only needs to consider the values $g \leqslant \sqrt{ab/c}$ and search for e and f. In any case, finding the relation and checking the two conditions of the theorem are best done on a computer. Using a computer we found 6814 weighted projective planes $\mathbb{P}(a,b,c)$ with $a,b,c \leqslant 100$ that satisfy the conditions of the theorem.[1] The 42 cases with $a,b,c \leqslant 30$ are listed in Table 1.

[1] The list of these projective planes is available on the authors' websites.

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Example 1.6. Consider $\mathbb{P}(19, 11, 13)$, with relation $(e, f, -g) = (1, 3, -4)$. We check that the two conditions hold.

(i) $w = 208/209 < 1$.

(ii) The set of integers $\delta \leqslant 0$ such that

$$(11, 19) + \delta(1, -3)$$

has non-negative components divisible by 4 is $\delta \in \{-3, -7, -11\}$. Hence $n = 3$. Now the set of integers $\gamma \geqslant 0$, such that

$$2(11, 19) + \gamma(1, -3)$$

has non-negative components divisible by 4 is $\gamma \in \{2, 6, 10\}$. Finally, $3(11, 19) \neq (0, 0) \pmod 4$.

It follows that $\mathrm{Bl}_{t_0}\mathbb{P}(19, 11, 13)$ does not have a finitely generated Cox ring.

Example 1.7 [GNW94]. Consider the family of weighted projective planes $\mathbb{P}(7N - 3, 8N - 3, (5N - 2)N)$, where $N \geqslant 4$, $3 \nmid N$. We check that the two conditions are satisfied with relation $(e, f, -g) = (N, N, -3)$. Note that we need $3 \nmid N$ for $\gcd(e, f, g) = 1$.

(i)
$$w = \frac{9(5N - 2)N}{(7N - 3)(8N - 3)} < 1 \quad \text{when } N \geqslant 3.$$

(ii) For $\delta \in \{-2, -5\}$,

$$(8N - 3, 7N - 3) + \delta(N, -N)$$

has non-negative components divisible by 3. Hence $n = 2$. For $\gamma \in \{1, 4\}$,

$$(8N - 3, 7N - 3) + \gamma(N, -N)$$

has non-negative components divisible by 3. Moreover, since $3 \nmid N$,

$$2(8N - 3, 7N - 3) \neq (0, 0) \pmod 3.$$

Similarly, the other family considered by Goto *et al.* in [GNW94], $\mathbb{P}(7N - 10, 8N - 3, 5N^2 - 7N + 1)$, $N \geqslant 5$, satisfies the two conditions with relation $(e, f, -g) = (N, N - 1, -3)$. (In fact, the case $N = 3$ of this family, $\mathbb{P}(11, 21, 25)$, is listed in Table 1.)

Remark 1.8. Our proofs do not work in positive characteristic. The reason is that we use the vanishing of partial derivatives to describe the order of vanishing of a polynomial at a point. This description fails in positive characteristic.

However, more is known in positive characteristic. In positive characteristic, if $\mathrm{Bl}_{t_0}\mathbb{P}(a, b, c)$ contains an irreducible curve $C \neq E$ of negative self-intersection, then it has a finitely generated Cox ring (see [Cut91] Proposition 4 and the discussion preceding it). All our examples of $\mathrm{Bl}_{t_0}\mathbb{P}(a, b, c)$ contain such a curve C of negative self-intersection.

2. Proof of Theorem 1.2

We start the proof using a geometric argument as in [CT15].

The toric variety X_Δ is $\mathbb{Q}$-factorial and $\mathrm{Cl}(X_\Delta) \otimes \mathbb{Q}$ is one-dimensional, with basis the class of the $\mathbb{Q}$-divisor H corresponding to the triangle Δ. Denote $X = \mathrm{Bl}_{t_0} X_\Delta$. Then $\mathrm{Cl}(X) \otimes \mathbb{Q}$ is

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two-dimensional, with basis the classes of the exceptional divisor E and the pullback of H, which we also denote H.

Recall from toric geometry [Ful93] that lattice points in $m\Delta$ correspond to certain torus-invariant sections of $\mathcal{O}_{X_\Delta}(mH)$. We identify a lattice point (i,j) with the monomial x^iy^j considered as a regular function on the torus T. The two lattice points $(0,0)$ and $(0,1)$ define a section $1-y$ of $\mathcal{O}_{X_\Delta}(H)$. We let C be the strict transform of this curve in X. Then C has class $[H-E]$. Since the self-intersection H^2 is equal to twice the area of Δ (which equals w), we get

$$C^2 = H^2 + E^2 = w - 1 \leqslant 0.$$

It follows that C and E are two irreducible curves on X with non-positive self-intersection. Their classes generate the Mori cone of curves of X (which in the case of surfaces coincides with the pseudoeffective cone of X). Its dual, the nef cone of X, is generated by the class of H and the class $D = [H - wE] \in C^{\perp}$.

By a result of Cutkosky [Cut91], the Cox ring of X is finitely generated if and only if there exists an integer $m > 0$, such that some effective divisor in the class mD does not have the curve C as a component. We will fix an m large and divisible enough such that mD is integral, and prove that any section of $\mathcal{O}_X(mD)$ vanishes on C. We may replace m by an integer multiple if necessary. Notice that this implies that although D generates an extremal ray of the nef cone of X it is not semiample, so the Cox ring of X cannot be finitely generated by [HK00, Definition 1.10 and Proposition 2.9].

A divisor in the class mD is defined by a Laurent polynomial (considered as a regular function on the torus T)

$$f(x,y) = \sum_{(i,j)\in m\Delta\cap\mathbb{Z}^2} a_{ij}x^iy^j,$$

that vanishes to order at least $W = mw$ at the point $t_0 = (1,1)$. In other words, all partial derivatives of f of order up to $W-1$ vanish at t_0. Now it suffices to prove that for such f, the coefficient a_{mx_1,my_1} at one of the non-zero vertices of $m\Delta$ is zero. Indeed, this implies that the section defined by f vanishes at the T-fixed point in X_Δ corresponding to the vertex. Similarly, the curve C passes through that fixed point, but since $C \cdot D = 0$, it follows that f must vanish on C.

Remark 2.1. There is a more algebraic argument for the claim in the previous paragraph. We want to prove that $f(x,y)$ vanishes on C, in other words, that $1-y$ divides f. This happens if and only if the column sums

$$c_i = \sum_{(i,j)\in m\Delta\cap\mathbb{Z}^2} a_{ij}$$

all vanish. There are $W+1$ column sums c_i. The derivatives ∂_x^l for $l = 0,\ldots,W-1$ give W linearly independent relations on c_i. If we can find one more linearly independent relation, then $c_i = 0$ for all i. The vanishing of a_{mx_1,my_1} gives such an extra relation.

We first transform the triangle $m\Delta$ by integral translations and shear transformations $(i,j) \mapsto (i, j+ai)$ for $a \in \mathbb{Z}$. The translation operation multiplies f with a monomial, and the shear transformation performs a change of variables on the torus. The two operations do not affect the order of vanishing of f at t_0 or the conditions of the theorem. The shear transformation has the effect of adding the integer a to each of the three slopes s_1, s_2, s_3.

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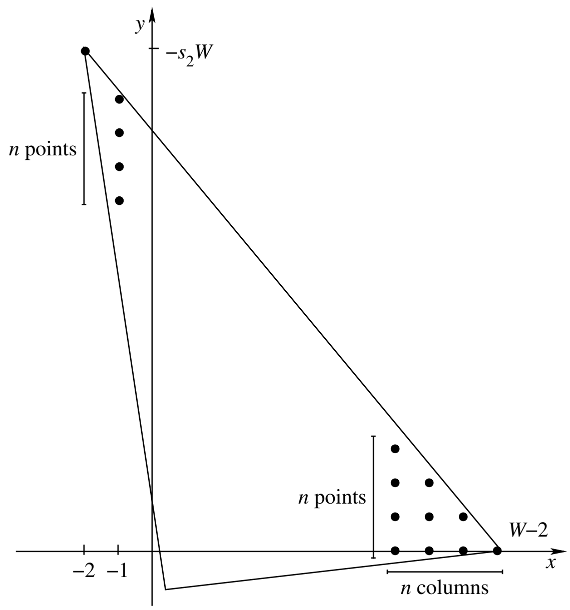

FIGURE 2. Triangle $\tilde{\Delta}$.

Let us start by bringing $m\Delta$ into the form shown in Figure 2. We first apply a shear transformation, so that $-2 < s_2 < -1$. (Note that $s_2 \notin \mathbb{Z}$ by condition (2).) We then translate the triangle so that (mx_1, my_1) moves to a point with x-coordinate -2 and (mx_2, my_2) moves to a point on the x-axis. Call the transformed triangle $\tilde{\Delta}$. Note that the transformations do not change the number of lattice points in the columns. In particular, the second column from the left in $\tilde{\Delta}$ again contains n lattice points.

Consider now the right n columns of the triangle $\tilde{\Delta}$. If $n = 1$, then no more preparations are needed. For $n > 1$, we may assume that the second column from the right contains at least two lattice points. Otherwise apply the reflection $(i, j) \mapsto (-i, j)$ to the original triangle Δ to reduce to the case $n = 1$. Since $-2 \leqslant s_2 < -1$, it follows that the lattice points in the second column from the right have y-coordinates $1, 0, \ldots$. By condition (2), the nth column from the right contains exactly n lattice points, which must have y-coordinates $0, 1, \ldots, n-1$. It now follows that for any $j = 1, \ldots, n$, the jth column from the right contains exactly j lattice points with y-coordinates $0, 1, \ldots, j-1$. In summary, the lattice points in the n columns on the right are

$$(W - n - 1 + i, j), \quad i, j \geqslant 0, \quad i + j < n.$$

Consider a derivative

$$\left(\sum_{i=0}^{n} \alpha_i \partial_x^i \partial_y^{n-i}\right) \partial_x^{W-n-1}$$

of order $W - 1$. This derivative vanishes on all monomials $x^i y^j$ for $(i, j) \in \tilde{\Delta} \cap \mathbb{Z}^2$, except for the monomials with $i < 0$. There are $n + 1$ such monomials, corresponding to lattice points in the left two columns of $\tilde{\Delta}$. We claim that there exist coefficients α_i, such that the derivative, when evaluated at t_0, vanishes on all n monomials corresponding to lattice points in the second column, and it does not vanish on the monomial corresponding to the vertex. This implies that the coefficient in f of the monomial corresponding to the vertex must be zero.

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Consider monomials $x^i y^j$ with $(i,j) \in \tilde{\Delta} \cap \mathbb{Z}^2$ and $i < 0$. Let us first apply ∂_x^{W-n-1} to these monomials. The result is the set of monomials (with non-zero coefficients that we may ignore)

$$x^{-(a+1)} y^{b+n+1}, x^{-a} y^{b+j}, \quad j = 0, \ldots, n-1.$$

Here $a = W - n$ and $b = -s_2 W - n - 1$. Making m (and hence also W) bigger if necessary, we may assume that both $a, b > 0$. Lemma 2.2 below shows that we can choose the desired coefficients α_i if $a(n+1) \neq bn$. This condition is equivalent to $-s_2 \neq 1 + 1/n$, which follows from the assumption that $ns_2 \notin \mathbb{Z}$. $\square$

Lemma 2.2. *Let $n, a, b > 0$ be integers such that $a(n+1) \neq bn$. Consider two sets of monomials:*

$$S_1 = \{x^{-(a+1)} y^{b+n+1}\}, \quad S_2 = \{x^{-a} y^{b+j}\}_{j=0,\ldots,n-1}.$$

Then there exists a derivative

$$D = \sum_{i=0}^{n} \alpha_i \partial_x^{n-i} \partial_y^i,$$

such that D applied to every monomial in S_2 vanishes at $t_0 = (1,1)$, and D applied to the monomial in S_1 does not vanish at t_0.

Proof. The strategy for the proof is as follows. It is fairly easy to write down a derivative D that vanishes when applied to monomials in S_2 and evaluated at t_0. Applying this D to the monomial in S_1 and evaluating at t_0 results in a complicated expression involving binomials. We use Lemma 2.3 below to simplify this expression.

To start, there exists a non-zero derivative

$$\widetilde{D} = \sum_{i=0}^{n} \beta_i \partial_y^i,$$

such that $\widetilde{D}$ applied to monomials y^{b+j}, for $j = 0, \ldots, n-1$, vanishes at $y = 1$. By Lemma 2.4 below we may take

$$\beta_i = (-1)^i \frac{(b+n-i-1)!}{(b-1)!} \binom{n}{i}.$$

Now let

$$\alpha_i = (-1)^i \frac{\beta_i}{a \cdot (a+1) \cdots (a+n-i-1)} = \frac{[b+n-i-1]_{n-i}}{[a+n-i-1]_{n-i}} \binom{n}{i},$$

where we used the notation

$$[k]_l = k(k-1) \cdots (k-l+1).$$

With these coefficients α_i, the derivative D applied to the monomials in S_2 vanishes at t_0. We need to prove that D applied to the monomial in S_1 does not vanish at t_0.

We apply D to $x^{-(a+1)} y^{b+n+1}$ and evaluate at t_0 to get

$$\sum_{i=0}^{n} \frac{[b+n-i-1]_{n-i}}{[a+n-i-1]_{n-i}} \binom{n}{i} (-1)^{n-i} [a+n-i]_{n-i} [b+n+1]_i.$$

Now simplify:

$$\frac{[a+n-i]_{n-i}}{[a+n-i-1]_{n-i}} = \frac{a+n-i}{a},$$

$$[b+n-i-1]_{n-i}[b+n+1]_i = \frac{[b+n+1]_{n+2}}{(b+n-i)(b+n-i+1)}.$$

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Replacing i by $n-i$, the sum becomes

$$\frac{[b+n+1]_{n+2}}{a}\sum_{i=0}^{n}(-1)^i\frac{a+i}{(b+i)(b+i+1)}\binom{n}{i}.$$

We can further express

$$\frac{1}{(b+i)(b+i+1)}\binom{n}{i}=\frac{[b+i-1]_{b-1}}{[b+n+1]_{b+1}}\binom{b+n+1}{b+i+1},$$

hence the sum is

$$\frac{[b+n+1]_{n+2}}{a[b+n+1]_{b+1}}\sum_{i=0}^{n}(-1)^i(a+i)[b+i-1]_{b-1}\binom{b+n+1}{b+i+1}.$$

We may ignore the non-zero constant in front of the sum and write the rest as

$$\sum_{i=-(b+1)}^{n}(-1)^i(a+i)[b+i-1]_{b-1}\binom{b+n+1}{b+i+1}-\sum_{i=-(b+1)}^{-1}(-1)^i(a+i)[b+i-1]_{b-1}\binom{b+n+1}{b+i+1}.$$

Since $p(x)=(a+x)[b+x-1]_{b-1}$ is a polynomial of degree b, the first sum vanishes by Lemma 2.3 below. In the second sum, the terms $[b+i-1]_{b-1}$ are zero unless $i=-b-1$ or $i=-b$. Thus, the sum is

$$\begin{aligned}&-\left((-1)^{-b-1}(a-b-1)[-2]_{b-1}\binom{n+b+1}{0}+(-1)^{-b}(a-b)[-1]_{b-1}\binom{n+b+1}{1}\right)\\&\quad=\pm[-1]_{b-1}(bn-a(n+1)).\end{aligned}$$

Now the result follows. □

LEMMA 2.3. *Let $n>0$ be an integer and $p(x)$ a polynomial of degree less than n. Then*

$$\sum_{i=0}^{n}(-1)^i p(i)\binom{n}{i}=0.$$

Proof. We have, for $0\leqslant l<n$,

$$\sum_{i=0}^{n}(-1)^i i(i-1)\cdots(i-l+1)\binom{n}{i}=\partial_x^l(1-x)^n|_{x=1}=0.$$

The polynomials $x(x-1)\cdots(x-l+1)$, for $l=0,\ldots,n-1$, span the space of all polynomials of degree less than n. □

LEMMA 2.4. *Let $n,b>0$ be integers. Then the derivative*

$$\widetilde{D}=\sum_{i=0}^{n}(-1)^i\frac{(b+n-i-1)!}{(b-1)!}\binom{n}{i}\partial_y^i$$

applied to monomials y^{b+j}, $j=0,\ldots,n-1$, vanishes at $y=1$.

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Proof. We apply the derivative $\widetilde{D}$ to the monomial y^{b+j} and evaluate at $y=1$ to get

$$\sum_{i=0}^{n}(-1)^i\frac{(b+n-i-1)!}{(b-1)!}\binom{n}{i}[b+j]_i.$$

Using that $b+j-i+1\leqslant b+n-i-1$, we simplify

$$\begin{aligned}\frac{(b+n-i-1)!}{(b-1)!}[b+j]_i &= (b+j)(b+j-1)\cdots(b+j-i+1)(b+n-i-1)\cdots(b+1)b\\ &= [b+j]_{j+1}[b+n-i-1]_{n-j-1}.\end{aligned}$$

The polynomial $p(x)=[b+n-x-1]_{n-j-1}$ has degree $n-j-1<n$, hence

$$[b+j]_{j+1}\sum_{i=0}^{n}(-1)^i p(i)\binom{n}{i}=0$$

by the previous lemma. □

Remark 2.5. In the case $n=1$, the proof of Theorem 1.2 can be simplified considerably. For this, we first transform the polytope $m\Delta$ by a reflection across the y-axis, shear transformation and translation to get to a $\tilde{\Delta}$ that has its left vertex at $(-1,b)$, right vertex at $(W-1,0)$ and a single lattice point in the second column from the right at $(W-2,0)$. Now the derivative $\partial_x^{W-2}\partial_y$ vanishes on all monomials except the monomial corresponding to the left vertex.

3. Proof of Theorem 1.5

Recall that the weighted projective plane $\mathbb{P}(a,b,c)$ is defined as $\operatorname{Proj} k[x,y,z]$, where the variables x,y,z have degree a,b,c, respectively. A relation $(e,f,-g)$ defines a homogeneous polynomial $x^e y^f - z^g$ of degree $d=gc$.

There is a degree map $\deg:\mathbb{R}^3\to\mathbb{R}$ that maps $(u,v,w)\mapsto au+bv+cw$. The toric variety $\mathbb{P}(a,b,c)$ is then defined by the triangle $\Delta=\deg^{-1}(d)\cap\mathbb{R}^3_{\geqslant 0}$ in the plane $\deg^{-1}(d)\cong\mathbb{R}^2$ and lattice $\deg^{-1}(d)\cap\mathbb{Z}^3\cong\mathbb{Z}^2$. (Indeed, the homogeneous coordinate ring $\bigoplus_{l\geqslant 0}\bigoplus_{m\in l\Delta\cap\mathbb{Z}^3} k\chi^m$ of X_Δ is the dth Veronese subring of $k[x,y,z]$. The Proj of a graded ring and a Veronese subring are isomorphic.) With d coming from the relation, we choose $(0,0,g)$ as the origin of the plane. The unit vector in the 'vertical' direction is then $(e,f,-g)$.

A divisor defined by a homogeneous polynomial of degree d in $k[x,y,z]$ has self-intersection number $d^2/(abc)$, which is the width of Δ:

$$w=\frac{(gc)^2}{abc}=\frac{g^2c}{ab}.$$

This identifies condition (1) of the theorem with condition (1) in Theorem 1.2. To identify conditions (2) in the two theorems, we count lattice points in the columns of $m\Delta$.

Let us construct a linear function h on $\mathbb{R}^3$ that takes value i on the column with index i in $m\Delta$. Since $(0,0,g)$ and $(e,f,0)$ lie in column 0, the function h must be the dot product with

$$\alpha(f,-e,0)$$

for some constant α. We can use h to compute w. The two non-zero vertices of Δ are

$$\left(\frac{cg}{a},0,0\right),\quad \left(0,\frac{cg}{b},0\right).$$

Thus,

$$w = \alpha\left(\frac{cg}{a}f - \frac{cg}{b}(-e)\right) = \alpha\frac{(cg)^2}{ab},$$

from which we solve $\alpha = 1/c$. (Note that we chose the vertex $(cg/a, 0, 0)$ to be on the right-hand side of the plane and the vertex $(0, cg/b, 0)$ on the left-hand side.)

Consider $m\Delta$ and its vertex (on the left-hand side) $P = (0, mcg/b, 0)$. Instead of counting lattice points Q in the second column from the left, we count lattice points $Q - P \in \ker(\deg) \cap \mathbb{Z}^3$, such that $h(Q - P) = 1$. These points are of the form $(u, v, w) \in \mathbb{Z}^3$, $u, w \geqslant 0$, $v \leqslant 0$, satisfying the equations

$$h(u, v, w) = 1 \Leftrightarrow \frac{f}{c}u - \frac{e}{c}v = 1,$$
$$\deg(u, v, w) = 0 \Leftrightarrow au + bv + cw = 0.$$

There is a possibly non-integral point

$$\frac{1}{g}(b, -a, 0)$$

satisfying these equations. Any other point is obtained from this one by subtracting a rational multiple of $(e, f, -g)$:

$$(u, v, w) = \frac{1}{g}(b, -a, 0) + \frac{\delta}{g}(e, f, -g), \quad \delta \leqslant 0.$$

Changing v to $-v$, we get that the number n of lattice points in the second column equals the number of integers $\delta \leqslant 0$, such that

$$(b, a) + \delta(e, -f)$$

has both components non-negative, divisible by g.

By a similar argument we get that the number of lattice points in the nth column from the right is the number of integers $\gamma \geqslant 0$, such that

$$(n - 1)(b, a) + \gamma(e, -f)$$

has both components non-negative and divisible by g.

Finally, if the $(n + 1)$th column from the right has a lattice point on the top edge, then this point corresponds to the solution $\epsilon = 0$, such that

$$n(b, a) + \epsilon(e, -f)$$

has both components non-negative and divisible by g. This happens if and only if $n(b, a) = (0, 0) \pmod g$. □

4. The moduli space $\overline{M}_{0,n}$

We show that the Cox ring of $\overline{M}_{0,n}$ is not finitely generated if the characteristic of k is 0 and $n \geqslant 13$. For this we use the method of Castravet and Tevelev [CT15, Proposition 3.1] to reduce to the case of a weighted projective plane blown up at the identity t_0 of its torus.

Recall that the moduli space $\overline{M}_{0,n}$ of stable n-pointed genus zero curves has been described by Kapranov as the iterated blowup of $\mathbb{P}^{n-3}$ along proper transforms of linear subspaces spanned by $n - 1$ points in linearly general position. The Losev–Manin moduli space $\overline{L}_n$ is constructed similarly by blowing up $\mathbb{P}^{n-3}$ along proper transforms of linear subspaces spanned by $n-2$ points in linearly general position. The space $\overline{L}_n$ is a toric variety and its fan Σ_n is the barycentric subdivision of the fan of $\mathbb{P}^{n-3}$. More precisely, the fan Σ_n has rays generated by all vectors in $\mathbb{R}^{n-3}$ such that each entry is equal to either 0 or 1, and all their negatives.

The main reduction step follows from the result that, given a surjective morphism $X \to Y$ of normal $\mathbb{Q}$-factorial projective varieties, if X has a finitely generated Cox ring, then so does Y (see Okawa [Oka15]). Using [CT15, Proposition 3.1] or its corollary, Theorem 4.1 below, for suitable values of n, a, b, c (e.g. $n = 13$ and $(a, b, c) = (26, 15, 7)$, see § 4.1), one can construct a rational map $\mathrm{Bl}_{t_0}\overline{L}_n \dashrightarrow \mathrm{Bl}_{t_0}\mathbb{P}(a, b, c)$ as a sequence of such surjective morphisms and small modifications (i.e. isomorphisms in codimension one) of normal $\mathbb{Q}$-factorial projective varieties. Moreover, there exist surjective morphisms $\overline{M}_{0,n} \to \mathrm{Bl}_{t_0}\overline{L}_n$ and $\overline{M}_{0,n+1} \to \overline{M}_{0,n}$. Small modifications do not change the Cox ring, hence the non-finite generation of a Cox ring of $\overline{M}_{0,n}$ would follow from the non-finite generation of a Cox ring of $\mathrm{Bl}_{t_0}\mathbb{P}(a, b, c)$.

The following is an immediate corollary of the main reduction result of Castravet and Tevelev [CT15, Proposition 3.1].

THEOREM 4.1 [CT15]. *Let $\overline{L}_n$ be defined by the fan Σ_n in the lattice $N = \mathbb{Z}^{n-3}$, as above. Suppose that there exists a saturated sublattice $N' \subset N$ of rank $n-5$, such that:*

(i) *the vector space $N' \otimes \mathbb{Q}$ is generated by rays of Σ_n;*

(ii) *there exist three rays of Σ_n with primitive generators u, v, w whose images generate N/N' and such that $au + bv + cw = 0 \pmod{N'}$ for some integers $a, b, c > 0$, with $\gcd(a, b, c) = 1$.*

Then there exists a rational map $\mathrm{Bl}_{t_0}\overline{L}_n \dashrightarrow \mathrm{Bl}_{t_0}\mathbb{P}(a, b, c)$ that is a composition of rational maps each of which is either a small modification between normal $\mathbb{Q}$-factorial projective varieties or a surjective morphism between normal $\mathbb{Q}$-factorial projective varieties. In particular, if $\mathrm{Bl}_{t_0}\mathbb{P}(a, b, c)$ does not have a finitely generated Cox ring, then $\mathrm{Bl}_{t_0}\overline{L}_n$ (and $\overline{M}_{0,n}$) does not have it either.

4.1 Proof of Theorem 1.1

We show that Theorem 4.1 applies in the case $n = 13$ and $(a, b, c) = (26, 15, 7)$.

Let $e_1, \dots, e_{10}$ be the canonical basis of $\mathbb{Z}^{10}$. Let

$$\begin{aligned} a_1 &= e_1 + e_5, & a_6 &= e_1 + e_2 + e_3 + e_4 + e_{10},\\ a_2 &= e_1 + e_2 + e_6, & a_7 &= e_5 + e_6 + e_7 + e_8 + e_9 + e_{10},\\ a_3 &= e_1 + e_2 + e_3 + e_7, & a_8 &= e_4 + e_5 + e_7,\\ a_4 &= e_1 + e_2 + e_3 + e_4 + e_8, & a_9 &= e_1,\\ a_5 &= e_1 + e_2 + e_3 + e_4 + e_9, & a_{10} &= e_4. \end{aligned}$$

The matrix A with columns $a_1, \dots, a_{10}$ has determinant 1, so $a_1, \dots, a_{10}$ form a basis of $\mathbb{Z}^{10}$. Let $u = e_1$, $v = e_2$ and $w = -4u - 2v + 2a_1 + a_2 + a_3 - a_8 + a_{10} = e_3 + e_5 + e_6$. So, we have that

$$\begin{aligned} 26u + 15v + 7w &= 11a_1 + 8a_2 + 4a_3 + a_4 + a_5 + a_6 - a_7 - 3a_8,\\ a_9 &= u,\\ a_{10} &= 4u + 2v + w - 2a_1 - a_2 - a_3 + a_8. \end{aligned}$$

Let $N' \subset \mathbb{Z}^{10}$ be the sublattice generated by $a_1, \dots, a_8$. Then $\mathbb{Z}^{10}/N'$ is generated by a_9, a_{10}, both of which can be expressed in terms of u, v, w. The vectors u, v, w satisfy the relation

$$26u + 15v + 7w = 0 \pmod{N'}.$$

We have seen that $\mathrm{Bl}_{t_0}\mathbb{P}(26, 15, 7)$ does not have a finitely generated Cox ring (see Example 1.3), hence by Theorem 4.1, for any $n \geqslant 13$, the moduli space $\overline{M}_{0,n}$ does not have a finitely generated Cox ring either.

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