ON UNIVALENT POLYNOMIALS

by DAVID A. BRANNAN

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1. Introduction. Let P_n be the class of normalised polynomials of the form

$$p_n(z) = z + a_2 z^2 + \ldots + a_n z^n \tag{1.1}$$

of degree n which are univalent in $U = \{|z| < 1\}$. In this note we discuss the coefficients of polynomials in P_n and in some of its subclasses.

Our principal tools will be

LEMMA 1.1. (*Dieudonné criterion*) [6]. *The polynomial $p_n(z)$, of the form* (1.1), *is univalent in U if and only if the associated equation of $p_n(z)$,*

$$\phi(x, \theta) = 1 + \sum_{k=2}^{n} a_k x^{k-1} \sin(k-1)\theta / \sin\theta = 0,$$

has no roots in $|x| < 1$, for any θ with $0 \leqq \theta \leqq \frac{1}{2}\pi$.

LEMMA 1.2. (*Cohn rule*) [9]. *Suppose that*

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n$$

is a polynomial of degree n, and

$$f^*(x) = \bar{a}_n + \bar{a}_{n-1} x + \ldots + \bar{a}_0 x^n.$$

Then, if $|a_0| \geqq |a_n|$, the polynomial

$$f_1(x) = \bar{a}_0 f(x) - \bar{a}_n f^*(x)$$

has the same number of zeros in $|x| < 1$ as has $f(x)$.

Finally we recall the definitions of two classes of univalent functions which will appear later. The analytic function $f(z)$ is said to be *starlike* in U if $f(0) = 0$, and the segment $[0, f(z_0)]$ lies in $f(U)$ for any z_0 in U [7]; analytically this may be expressed by the condition

$$\mathrm{Re}(zf'/f) > 0 \qquad (z \text{ in } U).$$

Further, the analytic function $g(z)$ is said to be *close-to-convex* in U if $g(0) = 0$, and

$$\mathrm{Re}(zg'/f) > 0 \qquad (z \text{ in } U)$$

for some starlike function $f(z)$ [8].

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2. A particular subclass of P_n. If the polynomial $p_n(z)$, of the form (1.1), is univalent in U, $p_n'(z)$ cannot vanish in U; consequently $|a_n| \leq 1/n$. In this section we consider polynomials in P_n where $a_n = 1/n$.

Suppose that

$$p_n(z) = z + a_2 z^2 + \ldots + a_{n-1} z^{n-1} + z^n/n. \tag{2.1}$$

It was shown in [3] that, if $p_n(z)$ belongs to P_n, then

$$(n-k)a_{n-k} = (k+1)\bar{a}_{k+1} \qquad (1 \leqq k \leqq n-2).$$

In the opposite direction we have

THEOREM 2.1. *Suppose that $p_n(z)$ is of the form* (2.1). *Then, if*

$$(n-k)a_{n-k} = (k+1)\bar{a}_{k+1} \qquad (1 \leqq k \leqq n-2) \tag{2.2}$$

and each a_{k+1} is sufficiently small, $p_n(z)$ belongs to P_n.

Proof. The polynomial $p_n(z)$ belongs to P_n if the equation

$$1 + \sum_{k=1}^{n-2} a_{k+1} \frac{\sin(k+1)\theta}{\sin\theta} x^k + \frac{\sin n\theta}{n\sin\theta} x^{n-1} = 0$$

has no roots in $|x| < 1$ for $0 \leqq \theta \leqq \frac{1}{2}\pi$. Applying the Cohn rule to this equation, since $|\sin n\theta/n\sin\theta| \leqq 1$ for $0 \leqq \theta \leqq \frac{1}{2}\pi$, we see that $p_n(z)$ belongs to P_n if

$$0 = 1 - \left(\frac{\sin n\theta}{n\sin\theta}\right)^2 + \sum_{k=1}^{n-2} x^k \left(a_{k+1}\frac{\sin(k+1)\theta}{\sin\theta} - \bar{a}_{n-k}\frac{\sin n\theta \sin(n-k)\theta}{n\sin\theta\sin\theta}\right)$$

$$= 1 - \left(\frac{\sin n\theta}{n\sin\theta}\right)^2 + \sum_{k=1}^{n-2} x^k a_{k+1}\left(\frac{\sin(k+1)\theta}{\sin\theta} - \frac{(k+1)\sin n\theta\sin(n-k)\theta}{(n-k)n\sin\theta\sin\theta}\right)$$

has no roots in $|x| < 1$ for $0 \leqq \theta \leqq \frac{1}{2}\pi$. Now each coefficient of x^r $(0 \leqq r \leqq n-2)$ has a double zero at $\theta = 0$, and the constant term is always positive otherwise. Hence, if all the coefficients of x^r are chosen sufficiently small, this equation has no roots in $|x| < 1$, and $p_n(z)$ belongs to P_n.

In a much underestimated paper [1] Alexander showed that the polynomials $\sum_{k=1}^{n} z^k/k$ and $\sum_{k=0}^{n} z^{2k+1}/(2k+1)$ are univalent in U. We can put this result in a more general setting in

THEOREM 2.2. *Suppose that*

$$p_n(z) = z + \sum_{k=2}^{n} a_k z^k, \quad \text{and} \quad q_n(z) = z + \sum_{k=1}^{n} b_{2k+1} z^{2k+1},$$

where ka_k and $(2k+1)b_{2k+1}$ decrease as k increases. Then $p_n(z)$ and $q_n(z)$ are close-to-convex univalent functions in U.

Proof. We have, for $z \in U$,

$$\operatorname{Re}\left\{\frac{zp_n'(z)}{z/(1-z)}\right\} = \operatorname{Re}\left\{1+(1-2a_2)z+\sum_{k=2}^{n-1}(ka_k-(k+1)a_{k+1})z^k - na_n z^n\right\}$$

$$\geqq 1-(1-2a_2)-\sum_{k=2}^{n-1}(ka_k-(k+1)a_{k+1})-na_n$$

$$= 0.$$

Hence $p_n(z)$ is close-to-convex in U. Similarly for $q_n(z)$.

However, in contrast with Theorems 2.1 and 2.2, we have the following surprising result for starlike polynomials

THEOREM 2.3. *Suppose that $p_n(z)$, of the form* (2.1), *belongs to P_n. Then $p_n(z)$ is starlike in U if and only if $a_k = 0$ for $2 \leqq k \leqq n-1$.*

Proof. If $a_2 = a_3 = \ldots = a_{n-1} = 0$, it is easy to show that $p_n(z)$ is starlike in U. We therefore assume that $p_n(z)$ is starlike in U, and then show that this implies that $a_k = 0$ for $2 \leqq k \leqq n-1$. Then

$$\frac{1+2a_2 z+\ldots+z^{n-1}}{1+a_2 z+\ldots+z^{n-1}/n} = \frac{p_n'(z)}{h(z)}$$

(where $h(z) = p_n(z)/z$) has positive real part in U. Since $p_n(z)$ belongs to P_n, we have that $(k+1)a_{k+1} = (n-k)\bar{a}_{n-k}$ for $1 \leqq k \leqq n-2$. Consequently, on $|z| = 1$, we may define

$$\alpha(\theta) = p_n'(z^2)/z^{n-1} \qquad (z = e^{i\theta})$$

$$= 2[\cos(n-1)\theta + 2|a_2|\cos\{(n-3)\theta-\phi_2\}+\ldots],$$

where $\phi_k = \arg a_k$. Furthermore, we may define $\beta(\theta)$ and $\gamma(\theta)$ by

$$\frac{p_n'(z^2)}{h(z^2)} = \frac{p_n'(z^2)}{z^{n-1}}\bigg/\frac{h(z^2)}{z^{n-1}}$$

$$= \frac{\alpha(\theta)}{\beta(\theta)+i\gamma(\theta)} = \frac{\alpha(\theta)\beta(\theta)-i\alpha(\theta)\gamma(\theta)}{\beta^2(\theta)+\gamma^2(\theta)},$$

where $z = e^{i\theta}$. No difficulty arises from the denominator, since the univalency of $p_n(z)$ ensures that

$$\beta^2(\theta)+\gamma^2(\theta) = |h(e^{2i\theta})|^2 = |p_n(e^{2i\theta})|^2 > 0$$

for $0 \leqq \theta \leqq 2\pi$. We now show that $\alpha(\theta)$ can have only simple zeros for $0 \leqq \theta \leqq 2\pi$. Let ϕ be a zero of $\alpha(\theta)$. Now, with $z = e^{i\theta}$, we have

$$\alpha'(\theta) = \frac{\partial}{\partial\theta}\left[\frac{p_n'(z^2)}{z^{n-1}}\right] = iz\left[\frac{2zp_n''(z^2)}{z^{n-1}} - (n-1)\frac{p_n'(z^2)}{z^n}\right].$$

Now $\alpha(\phi) = 0$, so that $p_n'(z^2) = 0$ when $z = e^{i\phi}$. Hence, if $\alpha'(\phi)$ is also zero, we see that $p_n''(z^2)$ is zero at $z = e^{i\phi}$ as well. But then $p_n''(z)$ is zero at $z = e^{2i\phi}$. This, however, is impossible, since the existence of a double zero of $p_n'(z)$ on $|z| = 1$ is ruled out by the univalency of $p_n(z)$ in U.

Now, the condition $\operatorname{Re}(zp_n'/p_n) \geqq 0$ in U may be written in the form

$$\alpha(\theta)\beta(\theta) \geqq 0 \quad \text{for} \quad 0 \leqq \theta \leqq 2\pi;$$

since the zeros of $\alpha(\theta)$ are simple, this in turn shows that, whenever $\alpha(\theta) = 0$, necessarily $\beta(\theta) = 0$. Now all of its $2(n-1)$ zeros lie in $0 \leqq \theta \leqq 2\pi$ (corresponding to the $n-1$ zeros of $p_n'(z)$ all on $|z| = 1$) in the case of $\alpha(\theta)$, and hence the same must be true of $\beta(\theta)$ since it is also a trigonometric polynomial of degree $n-1$. Since a polynomial which has its maximum number of zeros is determined by these zeros to within a constant factor, it follows that, for some constant C, we have

$$\alpha(\theta) = C\beta(\theta),$$

or

$$p_n'(z^2)/z^{n-1} = C\operatorname{Re}\{p_n(z^2)/z^{n-1}\} \quad \text{on} \quad |z| = 1.$$

Expanding both sides of this equation, and equating the highest terms, with $z = e^{i\theta}$, we find that $C = 2n/(n+1)$. Substituting this value of C, and equating the other terms of the expansion in turn, we find that $a_k = 0$ for $2 \leqq k \leqq n-1$. This completes the proof.

Note. In the case $n = 3$, this result also appears in **[4]**.

3. Some coefficient bounds for P_n. First we give bounds for the central coefficient of particular trinomials in P_{2n+1}.

THEOREM 3.1. *The polynomial*

$$p_{2n+1}(z) = z + az^{n+1} + z^{2n+1}/(2n+1)$$

belongs to P_{2n+1} *if and only if a is real and*

$$|a| \leqq \operatorname*{Min}_{(0,\frac{1}{2}\pi)} \left\{ \frac{1 + [\sin(2n+1)\theta/(2n+1)\sin\theta]}{|\sin(n+1)\theta/\sin\theta|} \right\} = \pi/4n\{1+o(1)\} \quad \textit{for large} \quad n.$$

Note. By (2.2), a must be real, and so we may assume that $a \geqq 0$.

Proof. Applying the Cohn rule to the associated Dieudonné equation for $p_{2n+1}(z)$, and using the fact that $|\sin(2n+1)\theta/(2n+1)\sin\theta| < 1$ for $0 < \theta \leqq \frac{1}{2}\pi$, the first inequality follows. Since $(2/\pi)x \leqq \sin x \leqq x$ for $0 \leqq x \leqq \frac{1}{2}\pi$, we can show that the above minimum occurs in $0 \leqq \theta \leqq 4\pi/(2n+1)$; by elementary differentiation, it must occur at $\pi/(2n+1)\{1+o(1)\}$ for large n. This gives the last inequality.

COROLLARY. *The polynomials* $z + az^2 + \frac{1}{3}z^3$ *and* $z + bz^3 + \frac{1}{5}z^5$ *are univalent in* U *if and only if a and b are real, and* $|a| \leqq 8/9$ *and* $|b| \leqq 3/5$.

We now turn to the estimation of the $(n-1)$th coefficients of polynomials in P_n.

THEOREM 3.2. *Suppose that $p_n(z)$, of the form* (1.1), *belongs to P_n. Then*

$$(n-1)|a_{n-1}| \leqq 1+2n|a_2 a_n| - n^2|a_n|^2 < 4; \tag{3.1}$$

in particular,

$$(n-1)|a_{n-1}| \leqq \begin{cases} 1+|a_2|^2, & \text{if } |a_2| \geqq 1, \\ 2|a_2|, & \text{if } |a_2| < 1. \end{cases}$$

Proof. By the Dieudonné criterion, since $p_n(z)$ belongs to P_n, the equation

$$1+\sum_{k=1}^{n-1} a_{k+1} \frac{\sin(k+1)\theta}{\sin\theta} x^k = 0$$

has no roots in $|x| < 1$ for $0 \leqq \theta \leqq \frac{1}{2}\pi$, and $|a_n| \leqq 1/n$. Applying the Cohn rule, we deduce that the equation

$$1-|a_n|^2\left(\frac{\sin n\theta}{\sin\theta}\right)^2 + \sum_{k=1}^{n-2} x^k \left(a_{k+1}\frac{\sin(k+1)\theta}{\sin\theta} - a_n \bar{a}_{n-k} \frac{\sin n\theta \sin(n-k)\theta}{\sin\theta\sin\theta}\right) = 0$$

has no roots in $|x| < 1$ for $0 \leqq \theta \leqq \frac{1}{2}\pi$. Consequently

$$\begin{aligned} 1-|a_n|^2\left(\frac{\sin n\theta}{\sin\theta}\right)^2 &\geqq \left| a_{n-1}\frac{\sin(n-1)\theta}{\sin\theta} - a_n \bar{a}_2 \frac{\sin n\theta \sin 2\theta}{\sin\theta\sin\theta}\right| \\ &\geqq \left| a_{n-1}\frac{\sin(n-1)\theta}{\sin\theta}\right| - \left| a_n a_2 \frac{\sin n\theta\sin 2\theta}{\sin\theta\sin\theta}\right|; \end{aligned}$$

substituting $\theta = 0$, we obtain

$$1-n^2|a_n|^2 \geqq (n-1)|a_{n-1}| - 2n|a_2 a_n|.$$

This gives the first inequality. The next three follow at once by considering the behaviour of the expression $1+(n|a_n|)(2|a_2|)-(n|a_n|)^2$ where $|a_2| \leqq 2$ and $|a_n| \leqq 1/n$.

Note 1. Suffridge **[10]** has shown that the polynomial

$$\sum_{k=1}^{n} \frac{n-k+1}{n} \cdot \frac{\sin(k\pi/n+1)}{\sin(\pi/n+1)} z^k$$

belongs to P_n. Consequently the constant 4 in (3.1) cannot be improved independently of n.

Note 2. Recent work has determined the coefficient regions for P_3 **[3, 5]**, for starlike polynomials in P_3 **[4]**, and for the subclass of P_n with real Maclaurin coefficients **[10]**. However, much work remains to be done on the general coefficient problem for P_n $(n > 3)$.

Note 3. Results similar to Theorems 2.1, 2.3, 3.1, and 3.2 were obtained in **[2]** for the class of " pseudo-polynomials "

$$\mu_n(z) = z^{-1} + \sum_{k=1}^{n} a_k z^k$$

analytic and univalent in $0 < |z| < 1$.

Note 4. Using the fact that the class of linearly-accessible functions of Biernacki is exactly the class of close-to-convex functions of Kaplan, we may observe that Alexander **[1]** showed that the polynomial

$$p(z) = \int_0^z (1 - e^{i\theta_1} t) \dots (1 - e^{i\theta_{n-1}} t)\, dt,$$

where $0 \leqq \theta_1 < \theta_2 < \dots < \theta_{n-1} < 2\pi \leqq \theta_n = \theta_1 + 2\pi$, is close-to-convex in U if and only if

$$\theta_{j+1} - \theta_j \geqq 2\pi/(n+1) \quad (1 \leqq j \leqq n-1).$$

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University of Glasgow
Glasgow, W.2