REGULAR RANK RINGS

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1. Introduction.

1.1. Throughout this note, $\mathfrak{R}$ will denote an associative ring but we shall *not* require $\mathfrak{R}$ to possess a unit.

If A and B are subsets of $\mathfrak{R}$, then $A + B$ will denote the set $\{x + y \mid x \in A, y \in B\}$. A^r will denote the set $\{u \in \mathfrak{R} \mid au = 0 \text{ for all } a \in A\}$.

Elements a and b will be said to be *orthogonal* if $ab = ba = 0$.

1.2. *We shall assume throughout that $\mathfrak{R}$ is regular in the sense of von Neumann*, i.e., for each a in $\mathfrak{R}$, $axa = a$ for some x in $\mathfrak{R}$. Then obviously the principal right ideal $(a)_r$ (i.e. the smallest right ideal which contains a) coincides with $a\mathfrak{R}$ and with $(e)_r$ where $e = ax$ (idempotent). Similarly, the principal left ideal $(a)_l$ coincides with $\mathfrak{R}a$ and with $(f)_l$ where $f = xa$ (idempotent).

$\bar{R}_{\mathfrak{R}}$ and $\bar{L}_{\mathfrak{R}}$ will denote respectively the set of all principal right ideals and the set of all principal left ideals, each ordered by inclusion. Clearly each of these has a minimum element 0, which consists of the zero element in $\mathfrak{R}$.

1.3. Whenever an ordered set L is under consideration, the symbols $\cup$ and $\cap$ will denote respectively the supremum and infimum (if they exist) in L.

If an ordered set L possesses a minimum element 0, then:

(i) $\mathfrak{C}$ in L will be called a *relative complement* of $\mathfrak{B}$ in $\mathfrak{A}$ if $\mathfrak{B} \cup \mathfrak{C}$ exists and is equal to $\mathfrak{A}$, and $\mathfrak{B} \cap \mathfrak{C}$ exists and is equal to 0;

(ii) $[\mathfrak{A} - \mathfrak{B}]$ will denote any (fixed) relative complement of $\mathfrak{B}$ in $\mathfrak{A}$;

(iii) L will be said to be *relatively complemented* if $[\mathfrak{A} - \mathfrak{B}]$ exists whenever $\mathfrak{B} \leqslant \mathfrak{A}$.

An ordered set L will be called a *lattice* if $\mathfrak{A} \cup \mathfrak{B}$ and $\mathfrak{A} \cap \mathfrak{B}$ exist for all pairs $\mathfrak{A}$ and $\mathfrak{B}$ in L. A lattice will be said to be *modular* if $\mathfrak{A} \geqslant \mathfrak{B}$ implies that $\mathfrak{A} \cap (\mathfrak{B} \cup \mathfrak{C}) \subset \mathfrak{B} \cup (\mathfrak{A} \cap \mathfrak{C})$ (this is equivalent to $=$ since $\supset$ holds always).

If L is a lattice with minimum element 0 and I is any set of indices, then elements $(\mathfrak{A}_i)_{i \in I}$ in L will be said to be *independent* if

$$(\cup_{i \in J} \mathfrak{A}_i) \cap (\cup_{i \in K} \mathfrak{A}_i) = 0$$

whenever J and K are finite disjoint subsets of I. When L is modular, it follows by induction on m that $\mathfrak{A}_1, \ldots, \mathfrak{A}_m$ are independent if $\mathfrak{A}_i \cap (\cup_{j<i} \mathfrak{A}_j) = 0$ for all $1 < i \leqslant m$; see (**4**, Part I, Chapter II). In particular, if $\mathfrak{A}$ and $\mathfrak{B}$ are elements in a relatively complemented modular lattice, then $\mathfrak{A} \cap \mathfrak{B}$, $[\mathfrak{A} - \mathfrak{A} \cap \mathfrak{B}]$, and $[\mathfrak{B} - \mathfrak{A} \cap \mathfrak{B}]$ are independent.

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1.4. A real-valued function $R(a)$ defined for each a in $\mathfrak{R}$ will be called a *rank function on* $\mathfrak{R}$ if:

(1.1) $0 < R(a) < \infty$ for all $a \neq 0$.

(1.2) $R(ab) \leqslant R(a)$, $R(ab) \leqslant R(b)$ for all a, b in $\mathfrak{R}$.

(1.3) $R(e+f) = R(e) + R(f)$ whenever e and f are orthogonal idempotents.

The rank function will be said to be *normalized* if $\mathfrak{R}$ has a unit 1 and $R(1) = 1$.

We note that:

(i) (1.3) implies: $R(0) = R(0+0) = R(0) + R(0)$; hence $R(0) = 0$.

(ii) (1.2) and the regularity of $\mathfrak{R}$ imply that $R(a) \leqslant R(b)$ whenever $(a)_r \subset (b)_r$;

hence $\qquad R(a) = R(b) \qquad$ whenever $\qquad (a)_r = (b)_r$

(if $(a)_r \subset (b)_r$, then $a = by$ for some y in $\mathfrak{R}$; so by (1.2), $R(a) \leqslant R(b)$).

(iii) (1.3) and the regularity of $\mathfrak{R}$ imply (see below) that:

$$R(a+b) \leqslant R(a) + R(b) \qquad \text{for all } a, b \text{ in } \mathfrak{R}.$$

Hence if $R(a)$ is a rank function on a regular ring $\mathfrak{R}$, then the function $\delta(a, b) = R(a - b)$ is a metric (to be called the *rank metric*) on $\mathfrak{R}$. As is well known, this implies that $\mathfrak{R}$ possesses a (unique) metric completion $\mathfrak{R}^\wedge$ which is itself a ring.

The chief purposes of this note are to prove the following:

(1.4) $\mathfrak{R}^\wedge$ is itself a regular ring.

(1.5) The rank function $R(a)$ extends to a rank function on $\mathfrak{R}^\wedge$.

(1.6) $\mathfrak{R}^\wedge$ is complete under its rank metric.

1.5. Von Neumann **(5, 6, 7)** stated this result for the special case of the normalized rank on $\mathfrak{D}_\infty'$ ($\mathfrak{D}_\infty'$ denotes the inductive limit of rings $\mathfrak{D}_m$ with $\mathfrak{D}$ a division ring and $m = 2^n$, $n \geqslant 1$), and gave some indication of his proof.

In this note we present a proof for the general case; this proof generalizes and simplifies a proof found for the case $\mathfrak{D}_\infty'$ by J. W. Alexander **(1)**. I am greatly indebted to Dr. Alexander for the use of his unpublished thesis. In particular, some of the ideas used in the proof of the important Lemma 2.7 below are motivated by his work.

2. Preliminary lemmas for regular rings. Whenever a statement is made about right ideals the corresponding statement about left ideals is to be understood also.

2.1. LEMMA (most of this was given by von Neumann **(4**; Part II, Chapter II; see also **2**, **3)**).

(2.1) *Suppose that* $\mathfrak{A}$, $\mathfrak{B}$ *are in* $\bar{R}_{\mathfrak{R}}$. *If* $\mathfrak{A} + \mathfrak{B}$ *is also in* $\bar{R}_{\mathfrak{R}}$, *then* $\mathfrak{A} \cup \mathfrak{B}$ *exists and coincides with* $\mathfrak{A} + \mathfrak{B}$; *if the set intersection of* $\mathfrak{A}$, $\mathfrak{B}$ *is in* $\bar{R}_{\mathfrak{R}}$, *then* $\mathfrak{A} \cap \mathfrak{B}$ *exists and coincides with this set intersection.*

(2.2) *If e is idempotent, then $u \in (e)_r$ if and only if $eu = u$.*

(2.3) *If e is idempotent and $eb = 0$, then $(e)_r \cap (b)_r$ exists and is* 0.

(2.4) *If e is idempotent and $be = 0$, then*

$$(e)_r + (b)_r = (e + b)_r.$$

(2.5) *If $e_1, \ldots, e_m$ are (pairwise) orthogonal idempotents, then $g = e_1 + \ldots + e_m$ is idempotent, and $e_i g = g e_i = e_i$.*

(2.6) *If e, f are idempotents and $f = ef$, then $(fe)_r = (f)_r$ and $fe, e - fe$ are orthogonal idempotents.*

(2.7) *If e, f are idempotents and $ef = 0$, then $e - fe, f$ are orthogonal idempotents and*

$$(e)_r + (f)_r = (e - fe)_r + (f)_r = (e + f - fe)_r.$$

(2.8) *If $\mathfrak{A}, \mathfrak{B} \in \bar{R}_{\mathfrak{R}}$, then $\mathfrak{A} + \mathfrak{B} \in \bar{R}_{\mathfrak{R}}$.*

(2.9) *If $\mathfrak{A}, \mathfrak{B} \in \bar{R}_{\mathfrak{R}}$, then the set intersection of $\mathfrak{A}, \mathfrak{B}$ is in $\bar{R}_{\mathfrak{R}}$.*

(2.10) *If $\mathfrak{B} \leqslant \mathfrak{A}$ in $\bar{R}_{\mathfrak{R}}$, then $[\mathfrak{A} - \mathfrak{B}]$ exists in $\bar{R}_{\mathfrak{R}}$.*

(2.11) *$\bar{R}_{\mathfrak{R}}$ is a relatively complemented, modular lattice.*

(2.12) *If $\mathfrak{A}_1, \ldots, \mathfrak{A}_m$ are independent in $\bar{R}_{\mathfrak{R}}$ and g is an idempotent with $(g)_r = \bigcup_i \mathfrak{A}_i$, then $g = e_1 + \ldots + e_m$ for pairwise orthogonal idempotents $e_1, \ldots, e_m$ such that $(e_i)_r = \mathfrak{A}_i$.*

(2.13) *For each a in $\mathfrak{R}$, the right ideal $\{u \in \mathfrak{R} \mid au = u\}$ is in $\bar{R}_{\mathfrak{R}}$.*

(2.14) *$\bar{R}_{\mathfrak{R}}$ possesses a maximum element, necessarily $\mathfrak{R}$, if and only if $\mathfrak{R}$ possesses a unit element.*

(2.15) *$(a)_l \subset (b)_l$ if and only if $(a)^r \supset (b)^r$.*

Proof of (2.1). A right ideal in $\mathfrak{R}$ contains each of $\mathfrak{A}, \mathfrak{B}$ if and only if it contains $\mathfrak{A} + \mathfrak{B}$; a right ideal in $\mathfrak{R}$ is contained in each of $\mathfrak{A}, \mathfrak{B}$ if and only if it is contained in the set intersection of $\mathfrak{A}$ and $\mathfrak{B}$.

Proof of (2.2). If $u \in (e)_r$, then $u = ey$ for some y; then since e is idempotent, $eu = e.ey = eu = u$. On the other hand, if $u = eu$, then $u \in (e)_r$.

Proof of (2.3). If u is in both $(e)_r$ and $(b)_r$ and $eb = 0$, then $u = eu = by$ for some y. Then $u = e(by) = 0$.

Proof of (2.4). If $be = 0$, then $(e + b)e = e$; so $e \in (e + b)_r$. Since $e + b \in (e + b)_r$, $b = (e + b) - e \in (e + b)_r$. Hence $(e + b)_r \supset (e)_r + (b)_r$. But for every y in $\mathfrak{R}$, $(e + b)y \in (e)_r + (b)_r$; hence $(e + b)_r \subset (e)_r + (b)_r$, so $(e)_r + (b)_r = (e + b)_r$.

Proof of (2.5). By direct calculation, since $e_i e_i = e_i$ and $e_i e_j = 0$ for $i \neq j$.

Proof of (2.6). Since $(fe)f = f(ef) = ff = f$, so $(f)_r \subset (fe)_r$. Since $fe \in (f)_r$, so $(fe)_r \subset (f)_r$. Hence $(fe)_r = (f)_r$. Finally,

$$fe.fe = f(ef)e = fe; \quad (fe)(e - fe) = fe - fe = 0; \quad (e - fe)fe = fe - fe = 0;$$
$$(e - fe)(e - fe) = (e - fe) - fe(e - fe) = (e - fe) - 0 = e - fe;$$

so fe, $e - fe$ are orthogonal idempotents.

Proof of (2.7).

$$f(e - fe) = fe - fe = 0; \qquad (e - fe)f = ef - f(ef) = 0;$$
$$(e - fe)(e - fe) = (e - fe)e + 0 = e - fe;$$

so $f, e - fe$ are orthogonal idempotents. Then, by (2.4),

$$(e)_r + (f)_r = (e - fe)_r + (f)_r = (e - fe + f)_r.$$

Proof of (2.8). We may suppose that $\mathfrak{A} = (e)_r$, $\mathfrak{B} = (b)_r$ with e idempotent. Then $\mathfrak{A} + \mathfrak{B} = (e)_r + (b - eb)_r = (e)_r + (f)_r$ with idempotent $f = (b - eb)y$ for some y. Since $ef = e(b - eb)y = 0$, it follows from (2.4) that

$$\mathfrak{A} + \mathfrak{B} = (f + e)_r \in \bar{R}_{\mathfrak{R}}.$$

Proof of (2.9). We may suppose that $\mathfrak{A} = (e)_r$, $\mathfrak{B} = (f)_r$ with e, f idempotents. Then for $u \in \mathfrak{R}$ the conditions $u \in (e)_r$ *and* $u \in (f)_r$ are equivalent (successively) to each of the conditions:

(i) $u = eu = fu$.

(ii) $u = eu$ and $(e - f)u = 0$.

(iii) $u = eu$ and $(e - fe)u = 0$.

(iv) $u = eu$ and $gu = 0$ (where g is an idempotent such that $(g)_l = (e - fe)_l$, which implies that $g = ge$).

(v) $u = eu$ and $(eg)u = 0$ (since $geg = gg = g$).

(vi) $u = (e - eg)u$ (since $u = (e - eg)u$ implies $u = eu$).

(vii) $u \in (e - eg)_r$ (since $e - eg$ is idempotent).

Proof of (2.10). We may suppose that $\mathfrak{A} = (e)_r$ and $\mathfrak{B} = (f)_r$ with e, f idempotents. Then $f = ef$. Hence by (2.6), $(e - fe)_r$ satisfies the requirements for $[\mathfrak{A} - \mathfrak{B}]$.

Proof of (2.11). From (2.1), (2.8), (2.9), and (2.10), it follows that $\bar{R}_{\mathfrak{R}}$ is a relatively complemented lattice. To show that $\bar{R}_{\mathfrak{R}}$ is modular, we may suppose that $(a)_r \supset (b)_r$ and that $u \in (a)_r \cap ((b)_r \cup (c)_r)$, and we need only show that

(2.16) $u \in (b)_r \cup ((a)_r \cap (c)_r)$.

We have that $u = ax = by + cz$ for suitable $x, y, z \in \mathfrak{R}$. Then

$$cz = ax - by \in (a)_r;$$

so cz is in the set intersection of $(a)_r$ and $(c)_r$. Now (2.16) follows.

Proof of (2.12). We may suppose that $\mathfrak{A}_i = (a_i)_r$. Then for suitable x_j in $\mathfrak{R}$, $g = \sum_j a_j x_j$ and $a_i = ga_i = \sum_j a_j x_j a_i$. Since the $(a_i)_r$ are independent, it follows that $a_i x_i a_i = a_i$ and for $j \neq i$, $a_j x_j a_i = 0$. Set $e_i = a_i x_i$. These e_i satisfy the requirements of (2.12).

Proof of (2.13). Let e, f be idempotents such that $(e)_r = (a)_r$ and

$(f)_l = (e - ae)_l$. Then for $u \in \mathfrak{R}$ the condition $au = u$ is equivalent (successively) to each of the conditions:

(i) $au = eu = u$.
(ii) $eu = u$, $(e - ae)u = 0$.
(iii) $eu = u, fu = 0$.
(iv) $eu = u$, $efu = 0$ (since $f.ef = ff = f$).
(v) $(e - ef)u = u$ (since this condition implies that $eu = u$).
(vi) $u \in (e - ef)_r$ (since $e - ef$ is idempotent).

Proof of (2.14). If $\mathfrak{A} \supset (a)_r$ for all $a \in \mathfrak{R}$, then necessarily $\mathfrak{A} \supset \mathfrak{R}$ so $\mathfrak{A} = \mathfrak{R}$. If $\mathfrak{R} = (e)_r$ with e idempotent, then $a = ea$ for all a in $\mathfrak{R}$, and for some x in $\mathfrak{R}$,

$$\begin{aligned} a - ae = (a - ae)x(a - ae) &= (a - ae)(ex)(a - ae) \\ &= (ae - ae)x(a - ae) = 0; \end{aligned}$$

thus e must be a unit in $\mathfrak{R}$. On the other hand, if e is a unit in $\mathfrak{R}$, then $\mathfrak{R} = (e)_r$.

Proof of (2.15). If $(a)_l \subset (b)_l$, then $a = yb$ for some y in $\mathfrak{R}$ and hence $(b)^r \subset (a)^r$. On the other hand, if $(b)^r \subset (a)^r$, let e be an idempotent such that $(e)_l = (b)_l$. Then $b = be$; so $b(z - ez) = 0$ for all z in $\mathfrak{R}$; hence $az = aez$, $(a - ae)z = 0$ for all z in $\mathfrak{R}$. Since $a - ae = (a - ae)z(a - ae)$ for some z in $\mathfrak{R}$, so $a - ae = 0$; $a \in (e)_l$; $(a)_l \subset (b)_l$.

2.2. Corollary to (2.12). *If $\mathfrak{A}, \mathfrak{B} \in \bar{R}_{\mathfrak{R}}$, then there exist orthogonal idempotents e, f, g such that*

$$(e)_r = \mathfrak{A} \cap \mathfrak{B}, \qquad (e + f)_r = \mathfrak{A}, \qquad (e + g)_r = \mathfrak{B},$$
$$(e + f + g)_r = \mathfrak{A} \cup \mathfrak{B}.$$

In particular, if $\mathfrak{A} \cap \mathfrak{B} = 0$, then

$$\mathfrak{A} = (f)_r, \qquad \mathfrak{B} = (g)_r, \qquad \mathfrak{A} \cup \mathfrak{B} = (f + g)_r.$$

Proof. Since $\mathfrak{A} \cap \mathfrak{B}$, $[\mathfrak{A} - \mathfrak{A} \cap \mathfrak{B}]$, and $[\mathfrak{B} - \mathfrak{A} \cap \mathfrak{B}]$ are independent, they can be represented as $(e)_r$, $(f)_r$, $(g)_r$ for suitable orthogonal idempotents. From this and (2.4) the Corollary follows.

2.3. Lemma. *If $a \in \mathfrak{R}$, there exists an idempotent e with $(e)_r \subset (a)_r$ and $a - e = ay(a^2 - a)$ for suitable y in $\mathfrak{R}$.*

Proof. For some idempotent f, $(a^2 - a)_l = (f)_l$. This means that $(a^2 - a)f = a^2 - a$ (hence $a^2 - a^2f = a - af$), and $f = y(a^2 - a)$ for some y in $\mathfrak{R}$.

Set $e = a - af$. Then

$$e = ae;$$
$$fe = f(a - af) = y(a^2 - a)(a - af) = ya((a^2 - a) - (a^2 - a)f) = 0.$$

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Hence $ee = (a - af)e = ae = e$, so e is idempotent and satisfies the requirements of Lemma 2.3.

2.4. LEMMA. *If e, f are idempotents, there exist orthogonal idempotents e_1, f_1 such that $(e_1)_r \subset (e)_r$, $(f_1)_r \subset (f)_r$ and $e - e_1 \in (ef)_r, f - f_1 \in (fe)_r$.*

Proof. Since $(ef)_r \subset (e)_r$, (2.12) implies that $e = e_1 + e_2$ with orthogonal idempotents e_1, e_2 such that:

$$(e_2)_r = (ef)_r \quad \text{and} \quad (e_1)_r = [(e)_r - (ef)_r].$$

Similarly, $f = f_1 + f_2$ with orthogonal idempotents f_1, f_2 such that $(f_2)_r = (fe)_r$. Then

$$e_1 f_1 = e_1 eff_1 = e_1 e_2 eff_1 = 0 \quad \text{and} \quad f_1 e_1 = f_1 fee_1 = f_1 f_2 fee_1 = 0.$$

2.5. LEMMA. *If e is an idempotent and $(e)_r \cap (b)^r = 0$, then $(be)_l = (e)_l$.*

Proof. Since $(be)_l \subset (e)_l$, (2.12) shows that for some idempotent e_1 with $ee_1 = e_1 e = e_1$ we have $[(e)_l - (be)_l] = (e_1)_l$. Then

$$e_1 = ee_1 \in (e)_r \quad \text{and} \quad be_1 = b(e_1 e) \in (e_1)_l \cap (be)_l,$$

so $be_1 = 0$. Since $(e)_r \cap (b)^r = 0$ by hypothesis, therefore $e_1 = 0$. Hence $(e)_l = (be)_l$, as stated.

2.6. COROLLARY TO LEMMA 2.5. *If $(a)_r \cap (b)^r = 0$, then $a = uba$ for some u in $\mathfrak{R}$.*

Proof. Let e be an idempotent with $(e)_r = (a)_r$. Then by Lemma 2.5, $e = ube$ for some u in $\mathfrak{R}$. Since $a = ea$, therefore $a = uba$.

2.7. LEMMA. *If $a, b, x \in \mathfrak{R}$ and $axa = a$, then there exists $y \in \mathfrak{R}$ such that $byb = b$ and $x - y = w_1 + w_2$ with each w_i of the form $u(a - b)v$ for suitable u, v in $\mathfrak{R}$.*

Proof. By (2.13) and (2.12), applied to $\{z \in \mathfrak{R} | bxz = z\} \subset (b)_r$, it follows that $(b)_r = (e_1)_r \cup (e_2)_r$, with orthogonal idempotents e_1, e_2 such that

$$(e_1)_r = \{z \in \mathfrak{R} |\ bxz = z\}.$$

This implies that $bxe_1 = e_1$; $(e_1 + e_2)b = b$; $e_2 = bq$ for some q in $\mathfrak{R}$.

Set $y = x - xe_2 + qe_2$. Then

$$byb = bye_1 b + bye_2 b = bxe_1 b + bqe_2 b = e_1 b + e_2 b = b.$$

Next, $y - x = (q - x)e_2$. Since $(e_2)_r \cup (e_1)_r = (b)_r$ it follows that

$$(e_2)_r \cup [(e_1)_r - (e_1)_r \cap (a)_r] \cup ((e_1)_r \cap (a)_r) = (b)_r,$$
$$e_2 \in [(b)_r - (e_1)_r \cap (a)_r]$$

for a suitable relative complement. Thus, for suitable relative complements

$$e_2 \in [(b)_r - (b)_r \cap (a)_r] \cup [(b)_r \cap (a)_r - (e_1)_r \cap (a)_r], \qquad e_2 = w_1 + w_2$$

with $w_1 \in [(b)_r - (b)_r \cap (a)_r]$ and $w_2 \in [(b)_r \cap (a)_r - (e_1)_r \cap (a)_r]$.

Now $(w_2)_r \cap ((a-b)x)^r = 0$; in fact, $w_2 \in (a)_r$; so if $((a-b)x)(w_2 z) = 0$, it follows that

$$axw_2 z = bxw_2 z; \quad w_2 z = bxw_2 z\ ; \quad e_1 w_2 z = w_2 z \in (e_1)_r \cap (a)_r; \quad w_2 z = 0.$$

Then by Corollary 2.6, $w_2 = u(a-b)xw_2 = u(a-b)v$ for some u, v in $\mathfrak{R}$.

Next, $w_1 = bv$ for some v in $\mathfrak{R}$ and $bvz \in (a)_r$ implies that $w_1 z = 0$ for each z in $\mathfrak{R}$.

In particular, $bvz = avz$ implies that $w_1 z = 0$. Thus $((a-b)v)^r \subset (w_1)^r$. Hence, by (2.15), $w_1 \in ((a-b)v)_l$ so $w_1 = u(a-b)v$ for suitable u, v in $\mathfrak{R}$.

Thus each of w_1, w_2 is of the form $u(a-b)v$. Since $y - x = (q-x)(w_1 + w_2)$, Lemma 2.6 follows.

3. Completion of regular rank ring.

3.1. Throughout this section $R(a)$ will denote a rank function assumed given on a regular ring $\mathfrak{R}$.

3.2. LEMMA.

$$R(-a) = R(a), \quad R(a-b) \leqslant R(a) + R(b), \quad R(a+b) \leqslant R(a) + R(b).$$

Proof. $axa = a$ implies that $-a = ax(-a)$, so $R(-a) \leqslant R(a)$. Hence $R(a) = R(-(-a)) \leqslant R(-a)$, so $R(-a) = R(a)$.

Now by Corollary 2.2, for certain orthogonal idempotents e, f, g we have

$$(a)_r = (e+f)_r, \qquad (b)_r = (e+g)_r,$$
$$(a)_r \cup (b)_r = (e+f+g)_r, \qquad a + b \in (a)_r + (b)_r.$$

Hence

$$a + b = (e+f+g)(a+b);$$
$$R(a+b) \leqslant R(e+f+g) = R(e+f) + R(g)$$
$$\leqslant R(e+f) + R(e+g) = R(a) + R(b).$$

Finally, $R(a-b) \leqslant R(a) + R(-b) = R(a) + R(b)$.

3.3. *Definition.* A sequence $(a_n)_{n\geqslant 1}$ with all $a_n \in \mathfrak{R}$ will be said to be *fundamental* if $R(a_n - a_m) \to 0$ as $n, m \to \infty$.

If (a_n), (b_n) are fundamental sequences, we write $(a_n) \equiv (b_n)$ if $R(a_n - b_n) \to 0$ as $n \to \infty$.

$\mathfrak{R}$ will be said to be *complete* with respect to its rank metric R if for every fundamental sequence (a_n) there exists an element a in $\mathfrak{R}$ such that $R(a_n - a) \to 0$ as $n \to \infty$.

3.4. LEMMA. (i) *The relation* $\equiv$ *is an equivalence relation.*

(ii) *If* (a_n) *is a fundamental sequence, then* $\lim_{n\to\infty} R(a_n)$ *exists;* $(a_n) \equiv (b_n)$ *implies that* $\lim_{n\to\infty} R(a_n) = \lim_{n\to\infty} R(b_n)$.

(iii) *If* (a_n), (b_n) *are fundamental sequences, then so are* $(a_n + b_n)$, $(a_n - b_n)$,

$(a_n b_n)$, and these are changed into equivalent sequences if (a_n), (b_n) are replaced by equivalent sequences.

Proof. The lemma follows from the relations:

$$\begin{aligned} &R(a+b) \leqslant R(a) + R(b), \\ &|R(a) - R(b)| \leqslant R(a-b), \\ &R(ab - a'b') \leqslant R(ab - ab') + R(ab' - a'b') \\ &\qquad \leqslant R(b-b') + R(a-a'). \end{aligned}$$

3.5. *Definition.* $\mathfrak{R}^\wedge$ denotes the set of equivalence classes of fundamental sequences, with addition, multiplication, and rank defined by the rules:

$$(a_n) + (b_n) = (a_n + b_n), \quad (a_n)(b_n) = (a_n b_n), \quad R((a_n)) = \lim_{n\to\infty} R(a_n).$$

3.6. LEMMA. *$\mathfrak{R}^\wedge$ is a ring. The map:*

$$\begin{aligned} &i_{\mathfrak{R}}: \mathfrak{R} \to \mathfrak{R}^\wedge \\ &\qquad a \to (a_n) \text{ with } a_n = a \text{ for all } n \end{aligned}$$

is a ring-isomorphic imbedding of $\mathfrak{R}$ into $\mathfrak{R}^\wedge$ preserving rank, $i_{\mathfrak{R}}(\mathfrak{R}) = \mathfrak{R}^\wedge$ if and only if $\mathfrak{R}$ is complete.

Proof. The usual proof.

3.7. THEOREM.

(i) *$\mathfrak{R}^\wedge$ is a regular ring.*

(ii) *R is a rank function on $\mathfrak{R}^\wedge$.*

(iii) *$\mathfrak{R}^\wedge$ is complete with respect to its rank metric R.*

(iv) *$\mathfrak{R}^\wedge$ has a unit if and only if* $\sup(R(a) | a \in \mathfrak{R}) < \infty$.

(v) *A fundamental sequence (a_n) is in the centre of $\mathfrak{R}^\wedge$ if and only if $R(a_n x - x a_n) \to 0$ as $n \to \infty$ for each $x \in \mathfrak{R}$.*

Proof of (i). Suppose that (a_n) is a fundamental sequence. We need to show that $(a_n)(x_n)(a_n) = (a_n)$ for some fundamental sequence (x_n).

By replacing (a_n) by a suitable subsequence we may suppose that $\sum_n R(a_{n+1} - a_n) < \infty$.

We choose x_1 to be any element in $\mathfrak{R}$ such that $a_1 x_1 a_1 = a_1$. Then, using Lemma 2.7, we choose elements x_n, $n > 1$, by induction on n so that for $n \geqslant 1$, $a_n x_n a_n = a_n$ and

$$x_{n+1} - x_n = u_n(a_{n+1} - a_n)v_n + u_n'(a_{n+1} - a_n)v_n'$$

for some u_n, v_n, u_n', v_n' in $\mathfrak{R}$. This implies that

$$R(x_{n+1} - x_n) \leqslant 2R(a_{n+1} - a_n);$$

$$R(x_m - x_n) \leqslant \sum_{i=\min(m,n)}^{\infty} R(x_{i+1} - x_i) \to 0 \quad \text{as } m, n \to \infty;$$

hence (x_n) is a fundamental sequence.

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Since $(a_n)(x_n)(a_n) = (a_n)$, this proves that $\mathfrak{R}^\wedge$ is regular.

Proof of (ii). If (a_n) is a fundamental sequence,

$$R((a_n)) = \lim_{n\to\infty} R(a_n);$$

since $0 \leqslant R(a_n) < \infty$ for all n, it follows that $0 \leqslant R((a_n)) < \infty$. From the definition of equivalence, $(a_n) \equiv 0$ if and only if $R((a_n)) = 0$.

Next, $R((a_n)(b_n)) = \lim_{n\to\infty} R(a_n b_n)$; since $R(a_n b_n) \leqslant R(a_n)$ and $\leqslant R(b_n)$ for all n, it follows that $R((a_n)(b_n)) \leqslant R((a_n))$ and $\leqslant R((b_n))$.

Finally, suppose that $e = (a_n)$ and $f = (b_n)$ are orthogonal idempotents in $\mathfrak{R}^\wedge$. Then

$$0 = e - ee = (a_n) - (a_n)(a_n) = (a_n - a_n a_n),$$

so

$$R(a_n - a_n a_n) \to 0 \quad \text{as } n \to \infty.$$

Then by Lemma 2.3 there exist idempotents e_n in $\mathfrak{R}$ such that $R(a_n - e_n) \leqslant R(a_n - a_n a_n)$. This means that $e = (e_n)$ with all e_n idempotent.

Similarly, $f = (f_n)$ with all f_n idempotent.

Then $0 = ef = (e_n)(f_n) = (e_n f_n)$, so $R(e_n f_n) \to 0$ as $n \to \infty$. Similarly, $R(f_n e_n) \to 0$ as $n \to \infty$. Now by Lemma 2.4 there exist, for each n, orthogonal idempotents e_n', f_n' such that

$$R(e_n - e_n') \leqslant R(e_n f_n), \qquad R(f_n - f_n') \leqslant R(f_n e_n).$$

This means that

$$e = (e_n'), \qquad f = (f_n'), \qquad e + f = (e_n' + f_n'),$$

$$R(e + f) = \lim_{n\to\infty} R(e_n' + f_n') = \lim_{n\to\infty} (R(e_n') + R(f_n')) = R(e) + R(f).$$

This proves (ii).

Proof of (iii). The usual proof.

Proof of (iv). Suppose that the condition $\sup (R(a) | a \in \mathfrak{R}) = k < \infty$ holds (a sufficient but not necessary condition for this to hold is: $\mathfrak{R}$ possesses a unit e_0; then $R(a) = R(ae_0) \leqslant R(e_0)$ for all $a \in \mathfrak{R}$). Then there exists a sequence a_n, $n \geqslant 1$ in $\mathfrak{R}$ with $R(a_n) \geqslant k - 1/n$. For each $n \geqslant 1$ let e_n be an idempotent in $\mathfrak{R}$ with

$$(e_n)_r = \bigcup_{i=1}^{n} (a_i)_r.$$

Then

$$(e_1)_r \subset (e_2)_r \subset \ldots; \quad R(e_1) \leqslant R(e_2) \leqslant \ldots; \quad R(e_n) \to k \text{ as } n \to \infty;$$
$$R(e_m - e_n) = R(e_m) - R(e_n) \qquad \text{if } n \leqslant m$$

since

$$(e_m)_r = (e_n)_r \cup (e_m - e_n)_r \text{ and } (e_n)_r \cap (e_m - e_n)_r = 0$$

(use Corollary 2.2).

Since $R(e_n) - R(e_m) \to 0$ as $n, m \to \infty$, the sequence (e_n) is fundamental.

For each fundamental sequence (x_n) we have

$$(x_n) - (e_n)(x_n) = (x_n - e_n x_n).$$

In $\mathfrak{R}$, for each $n \geqslant 1$: $(x_n - e_n x_n)_r \cap (e_n)_r = 0$ since $e_n(x_n - e_n x_n) = 0$ (use (2.3)). Hence, by Corollary 2.2, there exist orthogonal idempotents f, g such that

$$(f)_r = (x_n - e_n x_n)_r \quad \text{and} \quad (g)_r = (e_n)_r.$$

Hence

$$k \geqslant R(f + g) = R(f) + R(g) = R(x_n - e_n x_n) + R(e_n).$$

Since $R(e_n) \to k$, hence $R(x_n - e_n x_n) \to 0$ as $n \to \infty$. This proves that

$$(x_n) = (e_n)(x_n).$$

Since $\mathfrak{R}^\wedge$ is now known to be regular, it follows (with the argument used in the proof of (2.14)) that $(x_n)(e_n) = (x_n)$; hence (e_n) is a unit in $\mathfrak{R}^\wedge$.

On the other hand, if $\mathfrak{R}^\wedge$ possesses a unit e', then for every fundamental sequence (a_n):

$$R((a_n)) = R((a_n)e') \leqslant R(e').$$

In particular, if $a_n = a$ (fixed element in $\mathfrak{R}$) for all $n \geqslant 1$, then

$$R((a_n)) = \lim_{n\to\infty} R(a_n) = \lim_{n\to\infty} R(a) = R(a),$$

so $R(a) \leqslant R(e')$. Thus $\sup(R(a) | a \in \mathfrak{R}) \leqslant R(e') < \infty$.

Proof of (v). If (a_n) is in the centre of $\mathfrak{R}$, then $(a_n)(x_n) - (x_n)(a_n) = 0$ for every fundamental sequence (x_n) with $x_n = x$ (fixed in $\mathfrak{R}$) for all n. Hence it is *necessary* that $R(a_n x - xa_n) \to 0$ as $n \to \infty$.

On the other hand, if $R(a_n x - xa_n) \to 0$ as $n \to \infty$, for each $x \in \mathfrak{R}$, and (b_n) is any fundamental sequence, then

$$\begin{aligned} R(a_n b_n - b_n a_n) &\leqslant R(a_n b_p - b_p a_n) + R(a_n(b_n - b_p)) + R((b_n - b_p)a_n) \\ &\leqslant R(a_n b_p - b_p a_n) + 2R(b_n - b_p). \end{aligned}$$

For given $\epsilon > 0$ we can choose p (fixed) so that $R(b_m - b_p) < \epsilon/2$ for all $m \geqslant p$, then n large enough so that $R(a_n b_p - b_p a_n) < \epsilon/2$. This shows that $(a_n)(b_n) = (b_n)(a_n)$.

Note. Alexander's unpublished thesis shows that the centre of $(\mathfrak{D}_\infty')^\wedge$ is ring-isomorphic to the centre $\mathfrak{D}_\infty'$, and hence to the centre of $\mathfrak{D}$.

3.8. *Examples* ($\mathfrak{D}$ denotes any fixed division ring).

(i) Let $\mathfrak{R}$ denote the ring of all matrices $a = (a_{ij})_{i,j\geqslant 1}$ with all $a_{ij} \in \mathfrak{D}$ and $a_{ij} = 0$ with a finite number of exceptions. Let $R(a)$ denote the usual (right column, left row) rank of a (thus $R(a) = 0, 1, 2, \ldots$). This $\mathfrak{R}$ is a regular rank ring, and for it $\sup(R(a) | a \in \mathfrak{R})$ is not finite.

(ii) Let $\mathfrak{R}$ denote the ring of all sequences $a = (a_n)_{n \geqslant 1}$ with all a_n in $\mathfrak{D}$ and $a_n = 0$ with a finite number of exceptions (with componentwise addition and multiplication in $\mathfrak{R}$). Let $R(a)$ denote

$$\sum_{n=1}^{\infty} \frac{\bar{a}_n}{n^2},$$

where $\bar{a}_n = 1$ if $a_n \neq 0$ and $\bar{a}_n = 0$ if $a_n = 0$. Then $\mathfrak{R}$ is a regular rank ring without unit, but

$$\sup(R(a) \mid a \in \mathfrak{R}) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty .$$

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