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F-ABUNDANT SEMIGROUPS*

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Abstract. The investigation of general F-abundant semigroups is initiated. After obtaining some properties of such semigroups, the structure of a class of F-abundant semigroups is established. In addition, a problem raised in [**2**] is positively answered.

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1. Introduction and preliminaries. A semigroup S is called *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent. An abundant semigroup is called *quasi-adequate* if its idempotents form a subsemigroup. Moreover, a quasi-adequate semigroup is called *adequate* if the idempotent subsemigroup is a semilattice. Also an adequate semigroup S is called of *type A* if for all $a \in S$ and for all idempotent e, $eS \cap aS = eaS$ and $Se \cap Sa = Sae$. Abundant semigroups are a generalization of regular semigroups while quasi-adequate [adequate] semigroups generalize orthodox [inverse] semigroups. As a class of semi-groups intermediate between that of abundant semigroups and that of regular ones, El-Qallali and Fountain [**2**] defined and studied idempotent-connected abundant semigroups. An *idempotent-connected (IC) abundant semigroup* is an abundant semigroup in which for each $a \in S$ and for some $a^+ \in R^*_a \cap E(S)$, $a^* \in L^*_a \cap E(S)$, there is a bijection $\theta : \langle a^+ \rangle \to \langle a^* \rangle$ such that $xa = a(x\theta)$, for all $x \in \langle a^+ \rangle$, where $\langle a^+ \rangle$ is the subsemigroup of S generated by $eE(S)e$. Indeed, θ is an isomorphism; (see [**2**]). Various kinds of abundant semigroups have been investigated by many authors; (see [**2–7,9**] and their references). It is worth mentioning that Lawson [**9**] considered the natural partial order on an abundant semigroup.

An *F-inverse semigroup* is an inverse semigroup whose congruence classes modulo the least group congruence contain greatest elements with respect to the natural partial order. McFadden and O'Carroll [**10**] determined the structure of such semigroups. After that Edwards [**1**] studied regular semigroups satisfying the same condition, called *F-regular semigroups*. She established the construction of F-regular semigroups. In this paper, we shall be concerned with F-abundant semigroups, a generalization of F-regular semigroups in the class of abundant semigroups.

In Section 2, we introduce (strongly) F-abundant semigroups and their properties. Section 3 is concerned with the construction of strongly F-abundant semigroups.

Throughout this paper we shall use the terminology and notations of [**5,9**]. The following Lemma is repeatedly used in the sequel.

LEMMA 1.1. *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent*:

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(1) $a\mathcal{R}^*b$;
(2) *for all* $x, y \in S^1$, $xa = ya \Longleftrightarrow xb = yb$.
As an easy but useful consequence, we have the following result.

COROLLARY 1.2. *Let a be an element of S and e an idempotent. Then the following statements are equivalent:*
(1) $a\mathcal{R}^*e$;
(2) $ea = a$ *and for all* $x, y \in S^1$, $xa = ya \Rightarrow xe = ye$.

For an abundant semigroup S, $E(S)$ (or E) denotes the set of idempotents of S. For the sake of simplicity, a typical idempotent in the $\mathcal{L}^*$-class [resp. $\mathcal{R}^*$-class] of an element a of S will be denoted by a^* [resp. a^+]. If $e \in E(S)$, $\omega(e)$ indicates the set $\{f \in E(S) : f = fe = ef\}$. The next lemma gives an alternative description of *IC* abundant semigroups.

LEMMA 1.3. *Let S be abundant. Then the following statements are equivalent.*
(1) *S is IC.*
(2) *For each* $a \in S$, *two conditions hold*:
(i) for some [for all] a^* *[and* a^+*] and for all* $e \in \omega(a^*)$, *there exists* $b \in S[b \in \omega(a^+)]$ *such that* $ae = ba$;
(ii) for some [for all] a^+ *[and* a^**] and for all* $h \in \omega(a^+)$, *there exists* $c \in S[c \in \omega(a*)]$ *such that* $ha = ac$.

Throughout this paper, the natural partial order on an abundant semigroup is in the sense of [**9**]. Equivalently, for an abundant semigroup S and $a, b \in S$, $a \leq b$ if and only if, for some $e, f \in E(S)$, $a = eb = bf$. Moreover, we have the following result.

LEMMA 1.4. (from [**9**, Proposition 2.5 and its dual]). *Let S be an abundant semigroup and* $a, b \in S$. *Then the following statements are equivalent*:
(1) $a \leq b$;
(2) *for each* b^+ *and* b^*, *there exists* $a^+ \in \omega(b^+)$, $a^* \in \omega(b^*)$ *such that* $a = a^+b = ba^*$.

LEMMA 1.5. *Let S be an abundant semigroup. If* $a, b \in S$ *with* $a\mathcal{R}^*b$ $(a\mathcal{L}^*b)$ *and* $a \leq b$, *then* $a = b$.

2. Strongly F-abundant semigroups. A congruence ρ on a semigroup S is called *cancellative* if S/ρ is cancellative. Since the intersection of any non-empty set of cancellative congruences on a semigroup is itself cancellative, every semigroup S has a minimum cancellative congruence which we denote by σ_S or simply by σ if there is no danger of ambiguity. The σ-class of an element a of S is denoted by σ_a. If S is abundant and if σ_a contains a greatest element under the natural partial order, then this element is uniquely determined and we denote it by m_a.

DEFINITION 2.1. An abundant semigroup is called *F-abundant* if each σ-class of S has a greatest element with respect to the natural partial order.
We remark that, using Lemma 1.4, it is easy to see that if ρ is a cancellative congruence on an abundant semigroup and if every ρ-class has a greatest element,

then $\rho = \sigma$ and so S is F-abundant. We give some basic properties of F-abundant semigroups in the next proposition.

PROPOSITION 2.2. *Let S be an F-abundant semigroup. Then the following statements are true*:

(1) *S is an IC quasi-adequate semigroup*;
(2) $\mathcal{H} \cap \sigma = \iota_S$;
(3) *for all $a \in S$, $Em_a^+ \subseteq m_a^+E$ and $Em_a^* \supseteq m_a^*E$*;
(4) *S is monoid.*

Proof. (1) Let $a \in S$. Since S is F-abundant, σ_a has a greatest element. This element is uniquely determined and, as before we denote it by m_a. By Lemma 1.4, $a = a^+m_a$, for some $a^+ \in \omega(m_a^+)$. If $e \in \omega(a^+)$, then $ea \in \sigma_a$. Consider

$$ea = ea^+m_a = em_a = a^+em_a.$$

As $em_a \in \sigma_a$, from Lemma 1.4 we deduce that $em_a = m_af$, for some $f \in E(S)$. Now

$$ea = a^+m_af = af.$$

From this, together with its dual argument, it follows from Lemma 1.3 that S is *IC*.

We shall next verify that S is quasi-adequate. Now let $e, f \in E(S)$. Clearly, $ef \in \sigma_e$. Notice that $a \leq e$ implies $a \in E(S)$. It suffices to verify that $m_e \in E(S)$. But $m_e^+ \in \sigma_e$ and further $m_e^+ \leq m_e$. In virtue of Lemma 1.5, $m_e^+ = m_e$, as required.

(2) Assume that $a, b \in S$ with $(a, b) \in \mathcal{H}^* \cap \sigma$. Then for some a^+ and b^*, $a = a^+m_a$ and $b = m_ab^*$. Hence, $ab^* = a^+b$. As $a\mathcal{H}^*b$, $a\mathcal{L}^*b^*$ and $b\mathcal{R}^*a^+$, it follows that $a = b$ and so (2) holds.

(3) Here we prove only that $Em_a^+ \subseteq m_a^+E$.

Let $a \in S$ and $e \in E(S)$. Obviously $em_a \in \sigma_a$. By Lemma 1.4, for all m_a^+ and for some $f \in \omega(m_a^+)$, $em_a = fm_a$. Then

$$em_a^+ = fm_a^+ = f = m_a^+f.$$

Now $Em_a^+ \subseteq m_a^+E$. The other statement is dual.

(4) Let $e \in E(S)$. Then $e\sigma$ is an idempotent in the cancellative semigroup S/σ. It follows that $E(S) \subseteq e\sigma$. Let x be the greatest element in $e\sigma$ and let x^+ be any idempotent in R_x^*. Then $x^+ \in e\sigma$, so that $x^+ \leq x$. Hence, by Lemma 1.5, $x^+ = x$ and so x is idempotent.

For any idempotent e we have $e \leq x$ so that $ex = e = xe$, since $\leq$ is the natural partial order on S. Now, if $s \in S$, then

$$s = s^+s = xs^+s = x(s^+s) = xs$$

and similarly, $s = sx$. Thus x is the identity of S and S is a monoid.

In general, we do not know whether $Em_a^+ = m_a^+E$ and $Em_a^* = m_a^*E$ in an F-abundant semigroup. But in F-regular (F-orthodox) semigroups, this holds. To see this, from [**1**], $Ee = eE$ for some $e \in R_{m_a} \cap E(S)$. It suffices to verify that, for all $f \in R_{m_a} \cap E(S)$, $e = f$. Indeed $f = ef = efe = e$, as required. Similarly, one can show that the other equality holds.

DEFINITION 2.3. An F-abundant semigroup S is called *strong* if for all $a \in S$, $Em_a^+ = m_a^+E$ and $Em_a^* = m_a^*E$.

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As stated above, the following is immediate.

PROPOSITION 2.4. *Let S be a strongly F-abundant semigroup. Then for all* $a \in S$, $|L^+_{m_a} \cap E| = 1 = |R^*_{m_a} \cap E|$.

It is worth recording the following here. For an F-abundant semigroup S, M denotes the set of all the elements m_a. Under the multiplication of S, M need not constitute a subsemigroup. But with respect to the multiplication given by

$$m * n = m_{mn} \, (m \in M,\ n \in M)$$

M is a semigroup. Moreover, we have the following result.

PROPOSITION 2.5. $(M, *)$ *is a semigroup and isomorphic to* S/σ.

Concluding this section, we consider *IC* quasi-adequate semigroups. These results are used in a sequence of corresponding papers. The next Theorem shows that all *IC* quasi-adequate semigroups are type *W*, which answers an open problem raised by El-Qallali and Fountain. Following [**3**], on a quasi-adequate semigroup S we define a relation δ as follows:

$$a\delta b \Leftrightarrow E(a^+)aE(a^*) = E(b^+)bE(b^*), \text{ for some } a^+,\ a^* \text{ and } b^+,\ b^*,$$

where $E(e)$ is a $\mathcal{D}$-class of E containing $e(\in E)$. In fact, $a\delta b$ if and only if $a = ebf$, for some $e \in E(b^+)$, $f \in E(b^*)$. In the remainder of the section, $E(e) \leq E(f)$ means that $E(e)E(f) \subseteq E(e)$.

THEOREM 2.6. *Let S be an IC quasi-adequate semigroup. Then* δ *is a good congruence.*

Proof. We verify first the assertion: if $e, f \in E(S)$ with $a = ebf$, then $E(a^+) \leq E(e)$ and $E(a^*) \leq E(f)$. To see this, as $a = ebf$, we have $ea = a$ and $bf = b$. Now $ea^+ = a^+$ and $b^*f = b^*$. It follows that $E(a^+) \leq E(e)$ and $E(b^*) \leq E(f)$.

From [**3**, Proposition 2.6], it suffices to check that δ is left and right compatible. Let $a, b, c \in S$ and $a\delta b$. Then for some $e \in E(b^+)$ and $f \in E(b^*)$, $a = ebf$. Thus

$$\begin{aligned} ca &= cebf = cc^*eb^+bf \\ &= cc^*eb^+c^*b^+c^*eb^+bf \, (\text{since } c^*eb^+ \in E(c^*b^+)) \\ &= cc^*eb^+c^*b^+c^*eb^+bf \\ &= gcbhf \, (\text{for some } g, h \in E(S)) \, (\text{by Lemma 1.3}) \\ &= g(cb)^+cb(cb)^*hf. \end{aligned}$$

By the assertion above, $E((ca)^+) \leq E(g(cb)^+)$ and $E((ca)^*) \leq E((cb)^*hf)$. Hence $E((ca)^+) \leq E((cb)^+)$ and $E((ca)^*) \leq E((cb)^*)$. Again, because $a = ebf$, we obtain $b^+ab^* = b$. Applying the dual discussion to $b = b^+ab^*$, one can obtain that $E((cb)^+) \leq E((ca)^+)$ and $E((cb)^*) \leq E((ca)^*)$. Thus $E((ca)^+) = E((cb)^+)$ and $E((ca)^*) = E((cb)^*)$. Now $E((cb)^+) = E(g(cb)^+)$ and $E((cb)^*) = E((cb)^*hf)$. Therefore $ca\delta cb$; that is, δ is left compatible.

Dually, we can verify that δ is right compatible.

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Corollary 2.7. *Let S be an IC quasi-adequate semigroup. Then*

$$\sigma = \{(a, b) \in S \times S : eae = ebe, \text{ for some } e \in E(S)\}.$$

Proof. Let $a, b \in S$ with $a\sigma b$. By Theorem 2.6 and [**3**, Proposition 2.6], S/δ is type A. Then

$$\begin{aligned} a\sigma b &\Rightarrow a\delta\sigma b\delta; \\ &\Rightarrow \text{for some } e \in E, e\delta \bullet a\delta = e\delta \bullet b\delta; \\ &\Rightarrow \text{for some } e, f, g \in E, ea = febg; \\ &\Rightarrow gfe \bullet a \bullet gfe = gfe \bullet b \bullet gfe. \end{aligned}$$

Thus $\sigma \subseteq \{(a, b) \in S \times S\text{: for some } e \in E, eae = ebe\}$. The reverse inclusion is obvious. Now we have completed the proof.

3. Structure of strongly F-abundant semigroups. In this section we show first how to construct a class of strongly F-abundant semigroups in terms of specific ingredients. After obtaining some properties of such semigroups, we shall verify that any strongly F-abundant semigroup is isomorphic to some F-abundant semigroup constructed in this manner.

For a set X let f be a mapping of X to itself. We identify f with the set $\{(x, f(x)) \in X \times X : x \in X\}$. Denote by ε_X the identity mapping on X. $r(f)$ denotes the image set of f. Sometime we write also this set as $f(X)$.

Definition 3.1. Let S be a semigroup and ϕ an endomorphism of S (on the left). ϕ is called an *r-isomorphism* on S if there exists an endomorphism ψ of S, such that $\varepsilon_{r(\psi)} \subseteq \psi\phi$ and $\varepsilon_{r(\phi)} \subseteq \phi\psi$. In this case ψ is called an *r-inverse* of ϕ with respect to the set $r(\psi)$.

The following fact is easily checked and we omit the proof.

Proposition 3.2. *Let ϕ be an endomorphism of a semigroup S. Then the following statements are equivalent*:

(1) *ϕ is r-isomorphic on S*;

(2) *for some endomorphism ψ of S, $\phi\psi\phi = \phi$ and $\psi\phi\psi = \psi$*;

(3) *for some endomorphism ψ of S, $\psi|_{r(\phi)}$ and $\phi|_{r(\psi)}$ are mutually inverse isomorphisms.*

The following observation is useful in the proofs of this section.

Lemma 3.3. *Let x be an element of a band E. Then $xE = Ex$ if and only if x is central in E.*

Proof. Clearly, if x is central, we have $xE = Ex$. Conversely, if $xE = Ex$, then for any element $y \in E$ we have $xy = zx$ and $yx = xt$, for some $z, t \in E$. Now $xyx = zx^2 = zx = xy$ and $xyx = x^2t = xt = yx$, so that $xy = yx$ and x is central.

Definition 3.4. Let M be a cancellative monoid with identity 1 and E a band with identity e. Let $\Phi = \{\varphi_t : t \in M\}$, $\Psi = \{\psi_t : t \in M\}$ be two families of

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r-isomorphisms of E, such that φ_t and ψ_t are mutually r-inverse for all $t \in M$. (M, E; Φ, Ψ) is called an *SF-system* if the following conditions are satisfied:

(SF1) φ_1 is the identity mapping on E;
(SF2) for all $t \in M$, $E\varphi_t(e) = \varphi_t(e)E$ and $E\psi_t\varphi_t(e) = \psi_t\varphi_t(e)E$;
(SF3) for all $s, t \in M$ and $x \in E$, $\varphi_s\varphi_t(x) = \varphi_s\varphi_t(e)\varphi_{st}(x)$;
(SF4) for all $s \in M$, $r(\varphi_s) = E\varphi_s(e)$ and $r(\psi_s) = E\psi_s\varphi_s(e)$.

Given an *SF*-system (M, B; Φ, Ψ), put

$$SF(M, E; \Phi, \Psi) = SF = \{(m, x) \in M \times E : x \in \omega(\varphi_m(e))\}$$

with the multiplication

$$(m, x)(n, y) = (mn, x(\varphi_m y)).$$

LEMMA 3.5. *With the multiplication above, SF is a monoid.*

Proof. Let $(m, x), (n, y), (p, z) \in SF$. Since

$$\begin{aligned} x(\varphi_m y) &= x \bullet \varphi_m(\varphi_n(e)y) = x \bullet \varphi_m\varphi_n(e) \bullet \varphi_m(y) \\ &= x \bullet \varphi_m\varphi_n(e) \bullet \varphi_{mn}(e)\varphi_m(y) \; (by\ (SF3)) \\ &= x \bullet \varphi_m\varphi_n(e) \bullet \varphi_m(y)\dot{\varphi}_{mn}(e) \; (by\ (SF2)) \\ &= x\varphi_m y) \bullet \varphi_{mn}(e) = \varphi_{mn}(e) \bullet x\varphi_m(y) \; (by\ (SF2)), \end{aligned}$$

$x(\varphi_m y) \in \omega(\varphi_{mn}(e))$. This means that $(mn, x(\varphi_m y)) \in SF$; that is, $(m, x) \bullet (n, y) \in SF$. Thus *SF* is closed with respect to the multiplication above.

With notation as above, we have

$$\begin{aligned} (m, x)((n, y)(p, z)) &= (m, x)(np, y(\varphi_n z)) \\ &= (m(np), x \bullet \varphi_m m(y(\varphi_n z))) \\ &= ((mn)p, x \bullet \varphi_m(y) \bullet \varphi_m\varphi_n(z)) \\ &= ((mn)p, x \bullet \varphi_m(y) \bullet \varphi_m\varphi_n(e) \bullet \varphi_{mn}(z)) \\ &= ((mn)p, x \bullet \varphi_m(y\varphi_n(e)) \bullet \varphi_{mn}(z)) \\ &= ((mn)p, x \bullet \varphi_m(y) \bullet \varphi_{mn}(z)) \\ &= (mn, x(\varphi_m y))(p, z) \\ &= ((m, x)(n, y))(p, z), \end{aligned}$$

which shows that the multiplication is associative. Thus *SF* is a semigoup. In addition, by (SF4), it is easy to check that $(1, e)$ is the identity of *SF*. Therefore SF is a monoid.

The next lemma follows from (SF1).

LEMMA 3.6. $E(SF) = \{(1, x) : x \in E\}$ *and isomorphic to E. Moreover, E(SF) has (1, e) as its identity.*

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THEOREM 3.7 *Let* $(M, E; \Phi, \Psi)$ *be an SF-system. Then the following statements are true.*

(1) *For all* $(m, x), (n, y) \in SF$, $(m, x)\mathcal{R}^*(n, y)$ *if and only if* $x\mathcal{R}y$.
(2) *For all* $(m, x), (n, y) \in SF$, $(m, x)\mathcal{L}^*(n, y)$ *if and only if* $\psi_m(x)\mathcal{L}\psi_n(y)$.
(3) *For all* $(m, x), (n, y) \in SF$, $(m, x) \leq (n, y)$ *if and only if* $m = n$ *and* $x \leq y$.
(4) *SF is an IC quasi-adequate monoid.*
(5) *For all* $(m, x), (n, y) \in SF$, $(m, x)\sigma(m, y)$ *if and only if* $m = n$.
(6) *SF is strongly F-abundant.*

Proof. (1) We verify first that $(m, x)\mathcal{R}^*(1, x)$. Now let $(p, u), (q, v) \in SF$ with $(p, u)(m, x) = (q, v)(m, x)$. Then

$$(pm, u(\varphi_p x)) = (qm, v(\varphi_q x)),$$

so that $pm = qm$ and $u(\varphi_p x) = v(\varphi_q x)$. The prior equality implies that $p = q$ since M is cancellative. Hence

$$(p, u)(1, x) = (p, u(\varphi_p x)) = (q, v(\varphi_q x)) = (q, v)(1, x).$$

From this, together with $(1, x)(m, x) = (m, x)$, we have $(1, x)\mathcal{R}^*(m, x)$.

By the proof above, we have

$$\begin{aligned}(m, x)\mathcal{R}^*(n, y) &\Leftrightarrow (1, x)\mathcal{R}(1, y);\\ &\Leftrightarrow x = yx, y = xy;\\ &\Leftrightarrow x\mathcal{R}y.\end{aligned}$$

(2) We verify first that $(m, x)\mathcal{L}^*(1, \psi_m(x))$. Since $x \in E\varphi_m(e)$,

$$\begin{aligned}(m, x)(1, \psi_m(x)) &= (m, x \bullet \varphi_m\psi_m(x))\\ &= (m, x \bullet x) = (m, x).\end{aligned}$$

Assume that $(p, u), (q, v) \in SF$ with $(m, x)(p, u) = (m, x)(q, v)$. Then

$$(mp, x \bullet \varphi_m(u)) = (mq, x \bullet \varphi_m(v)),$$

so that $mp = mq$ and $x \bullet \varphi_m(u) = x \bullet \varphi_m(v)$. The prior equality implies that $p = q$. Consider

$$\begin{aligned}\psi_m(x)u = \psi_m(\varphi_m(e) \bullet x)u &= \psi_m\varphi_m(e) \bullet \psi_m(x)u\\ &= \psi_m(x)u \bullet \psi_m\varphi_m(e) \in r(\psi_m)\end{aligned}$$

and similarly $\psi_m(x)v \in r(\psi_m)$. Since $x \in \omega(\varphi_m(e))$, $x \in r(\varphi_m)$. Thus

$$\begin{aligned}\varphi_m(\psi_m(x)u) &= \varphi_m\psi_m(x) \bullet \varphi_m(u) = x \bullet \varphi_m(u)\\ &= x \bullet \varphi_m(v) = \varphi_m(\psi_m(x) \bullet v).\end{aligned}$$

By Proposition 3.2, $\psi_m(x)u = \psi_m(x)v$. Now

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$$(1, \psi_m(x))(p, u) = (p\psi_m(x)u) = q, \psi_m(x)v)$$
$$= (1, \psi_m(x))(q, v).$$

From these equations, by the dual of Corollary 1.2, $(m, x)\mathcal{L}^*(1, \psi_m(x))$.

The rest of the proof is similar to that in (1).

(3) Suppose that $(m, x), (n, y) \in SF$ with $(m, x) \leq (n, y)$. Then, for some $(1, u), (1, v) \in SF$, we have

$$(m, x) = (1, u)(n, y) = (n, y)(1, v);$$

that is,

$$(m, x) = (n, uy) = (m, y(\varphi_n v)),$$

so that $m = n$ and $x = uy = y \bullet \varphi_n(v)$. The latter equality yields $x \leq y$. Thus the direct part holds.

Conversely, let $(m, x), (n, y) \in SF$ and $m = n, x = uy = yv(u, v \in E)$. Then $y \in \omega(\varphi_n(e))$. Clearly, $\varphi_n(e)v \in \omega(\varphi_n(e)) = r(\varphi_n)$. We have $\varphi_n(e)v = \varphi_n(z)$, for some $z \in E$. Hence

$$(m, x) = (1, u)(m, x) = (m, ux) = (m, yv)$$
$$= (n, y \bullet \varphi_n(z)) = (m, y)(1, z);$$

that is, $(m, x) \leq (n, y)$.

(4) By virtue of (1) and (2), it suffices to prove that SF is IC. Now let $(m, x) \in SF$ and $(1, y) \leq (1, x)$. Then $y \leq x \leq \varphi_m(e)$ and so $x, y \in r(\varphi_m)$. Hence, for some $u \in E, \varphi_m(u) = y$. Thus, using (3), we obtain

$$(1, y)(m, x) = (m, y) = (m, xyx)$$
$$= (m, x\varphi_m(u)x)$$
$$= (m, x)(1, u\psi_m(x)) \ \ (\text{since } x \in r(\varphi_m)).$$

If $(1, v) \leq (1, \psi_m(x))$, then

$$(m, x)(1, v) = (m, x \bullet \varphi_m(v)) = (m, x \bullet \varphi_m(v \bullet \psi_m(x)))$$
$$= (m, x \bullet \varphi_m(v) \bullet \varphi_m\psi_m(x)) = (m, x \bullet \varphi_m(v) \bullet x)$$
$$= (1, x \bullet \varphi_m(v))(m, x).$$

Thus, from Lemma 1.3, SF is IC.

(5) Let $(m, x), (n, y) \in SF$. Then

$$(m, x)\sigma(n, y) \Leftrightarrow \text{for some } (1, u) \text{ we have } (1, u)(m, x)(1, u) = (1, u)(n, y)1, u)$$
$$\Leftrightarrow \exists u \in E \text{ such that } u \bullet x \bullet \varphi_m(u) = u \bullet y \bullet \varphi_n(u)$$
$$\Leftrightarrow m = n.$$

The reason why the last $\Leftrightarrow$ holds is that

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$$(1, \psi_m(x, y) \bullet y)(m, x)(1, \psi_m(xy) \bullet y) = (m, \psi_m(xy) \bullet yxy \bullet \varphi_m(y))$$
$$= (1, \psi_m(xy) \bullet y)(m, y)(1, \psi_n(xy) \bullet y).$$

(6) This follows from (3), (5) and the definition of SF.

In the remainder of this section, we shall prove that any strongly F-abundant semigroup is isomorphic to some $SF(M, E; \Phi, \Psi)$. For the sake of simplicity, we always assume that S is a strongly F-abundant semigroup with idempotent band E in the next part. $(M, *)$ denotes the cancellative monoid with identity 1 consisting of the greatest elements in all σ-classes of S (in the sense of Section 2). In addition, e denotes the identity of E.

For $m \in M$, by the fact that E is a band, we have $\langle m^+ \rangle = \omega(m^+)$. Notice that there exists an isomorphism $\theta_m : \omega(m^*) \to \omega(m^+)$ such that $mx = \theta_m(x)m$, for all $x \in \omega(m^*)$. Here we fix θ_m, for all $m \in M$. On E, define mappings ϕ_m and ψ_m as follows: for all $y \in E$, set

$$\phi_m(y) = \theta_m(m^* y), \psi_m(y) = \theta_m^{-1}(ym^+).$$

If $x, y \in E$, then

$$\phi_m(xy) = \theta_m(m^* xy) = \theta_m(m^* xm^* y) \text{ (by Proposition 2.2)}$$
$$= \theta_m(m^* x)\theta_m(m^* y) = \phi_m(x)\phi_m(y).$$

Thus ϕ_m is an endomorphism of E. Similarly, ψ_m is an endomorphism of E. Clearly, ϕ_m and ψ_m are mutually r-inverse. It is easy to see that $\phi_m(e) = m^+$, $\psi_m\phi_m(e) = m^*$, so that $r(\phi_m) = E\phi_m(e)$ and $r(\psi_m) = E\psi_m\phi_m(e) = \psi_m\phi_m(e)E$.

Take $\Phi = \{\phi_m : m \in M\}$, $\Psi = \{\psi_m : m \in M\}$. From the definition of ϕ_m, ϕ_1 is the identity. Moreover, we can prove that $(M, E; \Phi, \Psi)$ is an SF-system. We still need a lemma.

Lemma 3.8. *Let $m, n \in M$. Then $mn = \phi_m\phi_n(e) \bullet (m * n)$.*

Proof. By Lemma 1.4, for some $f \in \omega((m * n)^+)$ with $f\mathcal{R}^* mn$, $mn = f(m * n)$ and clearly $mn = fmn$. Thus $mn^+ = fmn^+$. Since $m^+mn = mn$, we have $m^+f = f$. It follows that $f \in \omega(m^+)$. With the notation above, we have $fm = m(\theta_m^{-1}(f))$ and further

$$m \bullet \theta_m^{-1}(f)n^+ = fmn^+ = m \bullet n^+,$$

so that $m^*\theta_m^{-1}(f)n^+ = m^*n^+$. We have, since S is strongly F-abundant,

$$m^*n^+\theta_m^{-1}(f) = m^*\theta_m^{-1}(f)n^+ = \theta_m^{-1}(f)m^*n^+ = m^*n^+;$$

that is, $\theta_m^{-1}(f) \geq m^*n^+$. Thus, since θ_m is isomorphic,

$$f = \theta_m(\theta_m^{-1}(f)) \geq \theta_m(m^*n^+) = \theta_m(m^*\phi_n(e)) = \phi_m\phi_n(e).$$

From this and the fact that

$$\begin{aligned} f \ \mathcal{R}^* \ & mn\mathcal{R}^*mn^+ = m\phi_n(e) \\ = \ & m \bullet m^*\phi_n(e) = \phi_m\phi_n(e) \bullet m \\ \mathcal{R}^* \ & \phi_m\phi_n(e) \bullet m^+ = \phi_m\phi_n(e), \end{aligned}$$

it follows from Lemma 1.5 that $f = \phi_m\phi_n(e)$. Thus $mn = \phi_m\phi_n(e) \bullet (m * n)$.

Lemma 3.9. *$(M, E; \Phi, \Psi)$ is an SF-system.*

Proof. From the statement above, all that remains to be proved is that $(SF3)$ holds. To verify $(SF3)$, suppose that $s, t \in M$. Then, by Lemma 1.4, $st = (s * t)f$, for some $f \in \omega((s * t)^*)$. Since

$$\begin{aligned} \phi_{s*t}(e)st &= \phi_{s*t}(e)(s * t)f = (s * t)ef \\ &= (s * t)f = st \end{aligned}$$

and, by the proof of Lemma 3.8, $st\mathcal{R}^*\phi_s\phi_t(e)$, we have $\phi_{s*t}(e) \bullet \phi_s\phi_t(e) = \phi_s\phi_t(e)$. Let $x \in E$. Computing

$$\begin{aligned} \phi_s\phi_t(e)\phi_{s*t}(x) \bullet (s * t) &= \phi_s\phi_t(e) \bullet (s * t)x \\ &= st \bullet x = \phi_s\phi_t(x)st \\ &= \phi_s\phi_t(x) \bullet \phi_s\phi_t(e) \bullet s * t. \end{aligned}$$

From this and the fact that $\phi_{s*t}(e)\mathcal{R}^*(s * t)$, we obtain that, since S is strongly F-abundant,

$$\begin{aligned} \phi_s\phi_t(e) \bullet \phi_{s*t}(x) &= \phi_s\phi_t(e) \bullet \phi_{s*t}(x) \bullet \phi_{s*t}(e) \\ &= \phi_s\phi_t(x) \bullet \phi_s\phi_t(e) \bullet \phi_{s*t}(e) \\ &= \phi_s\phi_t(x) \bullet \phi_{s*t}(e) \bullet \phi_s\phi_t(e) \\ &= \phi_s\phi_t(x)\phi_s\phi_t(e) = \phi_s\phi_t(x), \end{aligned}$$

as required.

Theorem 3.10. $S \cong SF(M, E; \Phi, \Psi)$.

Proof. Define $\tau : S \to SF(M, E; \Phi, \Psi)$ as follows:

$$a \to \tau(a) = (m_a, x_a),$$

where, $x_a \in \omega(m_a^+)$ with $x\mathcal{R}^*a$, $a = x_a m_a$. It is sufficient to check that τ is an isomorphism.

Let $a \in S$. Then, from Lemma 1.4, $a = x_a \bullet m_a$, for some $x_a \in \omega(m_a^+)$ with $x_a\mathcal{R}^*a$. Now let another element $y \in \omega(m_a^+)$ satisfy the same condition as x_a. Then $x_a m_a = y m_a$, so that

$$x_a = x_a m_a^+ = y m_a^+ = y.$$

Thus τ is well defined. By the proof above, we easily see that for all $(m, x) \in SF$, $\tau(xm) = (m, x)$. Accordingly, τ is surjective.

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Now let $a, b \in S$ and $\tau(a) = \tau(b)$. That is, $(m_a, x_a) = (m_b, x_b)$. Then $m_a = m_b$, $x_a = x_b$. It follows that $a = b$. Thus τ is injective.

Finally, suppose that $a, b \in S$. Using the above notation,

$$\begin{aligned}\tau(a)\tau(b) &= (m_a, x_a)(m_b, x_b) = (m_a * m_b, x_a(\phi_{m_a} x_b)) \\ &= \tau(x_a(\phi_{m_a} x_b)(m_a \bullet m_b)) \\ &= \tau(x_a(\phi_{m_a} x_b) \bullet (\phi_{m_a}\phi_{m_b}(e))(m_a * m_b)) \\ &= \tau(x_a(\phi_{m_a} x_b) \bullet m_a m_b) = \tau(x_a m_a(m_a^* x_b) m_b) \\ &= \tau(x_a m_a \bullet x_b m_b) = \tau(ab).\end{aligned}$$

Thus τ is homomorphism.

Up to now we have proved that τ is an isomorphism.

Summing up Theorem 3.7 and Theorem 3.10 in one theorem, we have our final result.

THEOREM 3.11. *Let* $(M, E; \Phi, \Psi)$ *be an SF-system. Then* $SF(M, E; \Phi, \Psi)$ *is a strongly F-abundant semigroup whose idempotent band is isomorphic to E. Conversely, any strongly F-abundant semigroup can be constructed in this manner.*

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