Bull. Aust. Math. Soc. **90** (2014), 257–263
doi:10.1017/S0004972714000215

BORNOLOGIES AND LOCALLY LIPSCHITZ FUNCTIONS

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(Received 6 February 2014; accepted 13 February 2014; first published online 15 May 2014)

Abstract

Let $\langle X, d\rangle$ be a metric space. We characterise the family of subsets of X on which each locally Lipschitz function defined on X is bounded, as well as the family of subsets on which each member of two different subfamilies consisting of uniformly locally Lipschitz functions is bounded. It suffices in each case to consider real-valued functions.

2010 *Mathematics subject classification*: primary 26A16, 46A17; secondary 54E35.

Keywords and phrases: Lipschitz function, locally Lipschitz function, uniformly locally Lipschitz function, Lipschitz in the small function, function preserving Cauchy sequences, bornology, totally bounded set, Bourbaki bounded set.

1. Introduction

Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces with at least two points. A function $f : X \to Y$ is called *α-Lipschitz* for $\alpha \in (0, \infty)$ if for all $\{x, w\} \subseteq X$, we have $\rho(f(x), f(w)) \le \alpha d(x, w)$. Less specifically, f is called Lipschitz if it is α-Lipschitz for some positive α. Whenever B is a d-bounded subset of X and f is α-Lipschitz, its image $f(B)$ is a ρ-bounded subset of Y, as $\operatorname{diam}(f(B)) \le \alpha \operatorname{diam}(B)$. Conversely, if B fails to be d-bounded, we can find a real-valued 1-Lipschitz function f on X such that $f(B)$ fails to be a bounded subset of $\mathbb{R}$. In fact, the same function works for all unbounded subsets of X: $x \mapsto d(x, x_0)$ where x_0 is a fixed but arbitrary point of X.

In general, if $f : X \to Y$, the family of subsets $\mathcal{B}_f$ of X on which f is bounded contains the singletons, is hereditary, and is stable under finite unions. A family of subsets of X with these properties is called a *bornology* on X [3, 4, 6, 13, 16]. The smallest bornology on X consists of its finite subsets and the largest is its power set $\mathcal{P}(X)$. Between the extremes are these bornologies: (1) the bornology of subsets that have compact closure; (2) the bornology $\mathcal{B}_d(X)$ of d-bounded subsets; (3) the bornology of d-totally bounded subsets.

Since the intersection of a nonempty family of bornologies is again a bornology, given a family of functions $\mathcal{C}$ defined on X with values in one or more metric spaces, $\bigcap_{f\in\mathcal{C}} \mathcal{B}_f$ is again a bornology on X. This is the bornology of subsets on which

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https://doi.org/10.1017/S0004972714000215 Published online by Cambridge University Press

each function in the family is bounded. It is of interest to identify this bornology for standard classes of continuous functions on X. For example, if X is a complex normed linear space, then $\{B \subseteq X : \forall f \in X^*, f(B) \text{ is a bounded subset of } \mathbb{C}\}$ consists by the Uniform Boundedness Principle [15, Theorem 9.2] of the norm bounded subsets of X. It is the purpose of this note to identify $\bigcap_{f \in \mathcal{C}} \mathcal{B}_f$ for an arbitrary metric space $\langle X, d\rangle$ where $\mathcal{C}$ is one of three classes of locally Lipschitz functions.

Let $B_d(x, \delta)$ denote the open ball in X with centre x and radius $\delta > 0$. Recall that $f : X \to Y$ is called *locally Lipschitz* [8, 9, 14] provided for each $x \in X$, there exists $\delta_x > 0$ such that the restriction of f to $B_d(x, \delta_x)$ is Lipschitz. We call f *uniformly locally Lipschitz* if δ_x can be chosen independent of x. For example, $f : (0, \infty) \to (0, \infty)$ defined by $f(x) = 1/x$ is locally Lipschitz but not uniformly locally Lipschitz.

A stronger requirement is that the local Lipschitz constant can be chosen independent of the point x as well as the size of the neighbourhood. Following Garrido and Jaramillo [9], we call such a function *Lipschitz in the small*. Put differently, f is Lipschitz in the small if there exist $\delta > 0$ and $\alpha > 0$ such that $d(x, w) < \delta$ implies $\rho(f(x), f(w)) \leq \alpha d(x, w)$. Clearly, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly locally Lipschitz but not Lipschitz in the small. On the other hand, if we equip $X = \bigcup_{n=1}^{\infty} [n - \frac{1}{4}, n + \frac{1}{4}]$ with the usual metric inherited from $\mathbb{R}$, then the locally constant function defined by $f(x) = n^2$ if $n - \frac{1}{4} \leq x \leq n + \frac{1}{4}$ is Lipschitz in the small but fails to be Lipschitz on X. Obviously, each Lipschitz in the small function is uniformly continuous whereas a uniformly locally Lipschitz function need not be.

Each class of locally Lipschitz functions gives rise to a standard bornology. In the case of the locally Lipschitz functions, we get the subsets of X with compact closure. For the uniformly locally Lipschitz functions, we get the totally bounded subsets of X. Finally, for the Lipschitz in the small functions, we get the subsets of X that are Bourbaki bounded, also called finitely chainable relative to X [1].

2. Preliminaries

For $x \in X$ and A a nonempty subset of $\langle X, d\rangle$, we put $d(x, A) := \inf\{d(x, A) : a \in A\}$. If A and B are nonempty subsets of X, we define the *gap* $D_d(A, B)$ between them [2, page 28] by the formula

$$D_d(A, B) := \inf\{d(a, b) : a \in A, b \in B\} = \inf\{d(a, B) : a \in A\}.$$

We denote the *ε-enlargement* of a nonempty subset A of X by $B_d(A, \varepsilon)$, that is, $B_d(A, \varepsilon) := \{x \in X : d(x, A) < \varepsilon\} = \bigcup_{a \in A} B_d(a, \varepsilon)$. Taking successive enlargements is not in general additive, that is, $B_d(B_d(A, \varepsilon), \delta)$ can be a proper subset of $B_d(A, \varepsilon + \delta)$; equality is achieved in those metric spaces for which the metric d is almost convex [2, page 108]. With this in mind, given $x \in X$, we define the sets $B_d^n(x, \varepsilon)$ for $n = 0, 1, 2, \ldots$ recursively by

$$B_d^0(x, \varepsilon) = \{x\} \quad \text{and} \quad B_d^{n+1}(x, \varepsilon) = B_d(B_d^n(x, \varepsilon), \varepsilon).$$

Clearly, $B_d^n(x, \varepsilon) \subseteq B_d^{n+1}(x, \varepsilon)$.

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A subset A of X is called *totally bounded* if for each $\varepsilon > 0$ there exists a finite subset F_ε of X such $A \subseteq B_d(F_\varepsilon, \varepsilon)$. Each subset of X with compact closure is totally bounded while each totally bounded set is bounded, and the family of totally bounded subsets forms a bornology. If A is totally bounded, then the sets F_ε can be chosen inside A, which means that total boundedness is an intrinsic property of A and does not depend on the enveloping space X in which A is isometrically embedded. This is not the case for example for the bornology of subsets with compact closure: $(0, 1)$ fails to have compact closure in $(0, \infty)$ while it has compact closure in $\mathbb{R}$. As is well known, A is totally bounded if and only if each sequence in A has a Cauchy subsequence [11, page 155].

By an *ε-chain of length n from* x to y in $\langle X, d\rangle$ we mean a finite sequence of points (not necessarily distinct) $x_0, x_1, x_2, x_3, \ldots, x_n$ in X such that $x = x_0, y = x_n$ and for each $j \in \{1, 2, \ldots, n\}, d(x_{j-1}, x_j) < \varepsilon$. Clearly, there is an ε-chain of length n from x to y if and only if $y \in B_d^n(x, \varepsilon)$.

Put $x \simeq_\varepsilon y$ on X if there is an ε-chain of some length from x to y. The relation $\simeq_\varepsilon$ is an equivalence relation, and if $x/\simeq_\varepsilon$ and $y/\simeq_\varepsilon$ are distinct equivalence classes, we have $D_d(x \simeq_\varepsilon, y \simeq_\varepsilon) \geq \varepsilon$. We will need the formula $x/\simeq_\varepsilon = \bigcup_{n=0}^\infty B_d^n(x, \varepsilon)$.

We call a subset A of X *Bourbaki bounded* if for each $\varepsilon > 0$ there is a finite subset $\{x_1, x_2, \ldots, x_k\}$ of X and $n \in \mathbb{N}$ such that $A \subseteq \bigcup_{j=1}^k B_d^n(x_j, \varepsilon)$ [5, 12, 16]. The Bourbaki bounded subsets form a bornology lying between the d-totally bounded subsets and $\mathcal{B}_d(X)$. The unit ball in each infinite dimensional normed linear space is Bourbaki bounded but not totally bounded. In any infinite set X equipped with the 0–1 metric, each infinite subset is bounded but not Bourbaki bounded. The bornology of Bourbaki bounded sets fails to be an intrinsic bornology: in the Hilbert space ℓ_2 the standard orthonormal base $A = \{e_n : n \in \mathbb{N}\}$ is Bourbaki bounded, whereas it fails to be Bourbaki bounded within itself. Note that each bounded subset of a metric space $\langle X, d\rangle$ is Bourbaki bounded in some larger metric space, as we can isometrically embed $\langle X, d\rangle$ in a Banach space [7, page 338]. Recently, Garrido and Meroño [10] characterised Bourbaki boundedness of a subset A in terms of a type of subsequence that each sequence in A must have.

3. Results

In what follows, we freely use the following fact: if $f_i : \langle X, d\rangle \to \mathbb{R}$ is Lipchitz restricted to $A \subseteq X$ for $i = 1, 2, \ldots, n$, then $\max\{f_1, f_2, \ldots, f_n\}$ is Lipschitz restricted to A.

Theorem 3.1. *Let $\langle X, d\rangle$ be a metric space and let B be a nonempty subset. The following conditions are equivalent:*

(1) $\mathrm{cl}(B)$ *is compact;*

(2) *whenever $\langle Y, \rho\rangle$ is a metric space and $f : X \to Y$ is continuous, then $f(B) \in \mathcal{B}_\rho(Y)$;*

(3) *whenever $\langle Y, \rho\rangle$ is a metric space and $f : X \to Y$ is locally Lipschitz, then $f(B) \in \mathcal{B}_\rho(Y)$;*

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(4) *whenever* $f : X \to \mathbb{R}$ *is locally Lipschitz, then* $f(B)$ *is a bounded set of real numbers.*

PROOF. The implication (1) $\Rightarrow$ (2) follows from the fact that $f(\mathrm{cl}(B))$ must be compact if f is continuous, while (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.

For (4) $\Rightarrow$ (1), we prove the contrapositive. Suppose $\mathrm{cl}(B)$ fails to be compact; then we can find a sequence $\langle b_n \rangle$ in B with distinct terms that has no cluster point in X. For each $n \in \mathbb{N}$, put $\mu_n = d(b_n, \{b_j : j \neq n\}) > 0$ and then put $\varepsilon_n := \min\{1/n, \mu_n/3\}$. The family of open balls $\{B_d(b_n, \varepsilon_n) : n \in \mathbb{N}\}$ is a pairwise disjoint family as, whenever $n \neq j$, $\varepsilon_n + \varepsilon_j < \max\{\mu_n, \mu_j\}$. For each $n \in \mathbb{N}$, let $g_n : X \to \mathbb{R}$ be the Lipschitz function defined by

$$g_n(x) = n - \frac{n}{\varepsilon_n} d(x, b_n).$$

Notice that $g_n(x) > 0$ if and only if $d(x, b_n) < \varepsilon_n$.

We are now ready to describe our globally defined badly behaved locally Lipschitz function f:

$$f(x) = \begin{cases} g_n(x) & \text{if } x \in B_d(b_n, \varepsilon_n), \\ 0 & \text{otherwise.} \end{cases}$$

Since $f(b_n) = n$, $f(B)$ is unbounded. To see that f is locally Lipschitz, let $x_0 \in X$ be arbitrary. Since $\varepsilon_n \leq 1/n$ for each n and x_0 is not a cluster point of $\langle b_n \rangle$, there exists $\delta > 0$ such that $B_d(x_0, \delta)$ either fails to hit any $B_d(b_n, \varepsilon_n)$ or it hits $B_d(b_n, \varepsilon_n)$ for at most finitely many n, say $n_1, n_2, \ldots, n_k$. In the first case, f restricted to $B_d(x_0, \delta)$ is the zero function, while in the second, whenever $d(x, x_0) < \delta$, we have $f(x) = \max\{0, g_{n_1}(x), g_{n_2}(x), \ldots, g_{n_k}(x)\}$. Either way, f restricted to $B_d(x_0, \delta)$ is Lipschitz. □

The equivalence of conditions (1) and (2) is of course well known, and (2) $\Rightarrow$ (1) is most easily proved using the Tietze extension theorem. We next turn to uniformly locally Lipschitz functions.

THEOREM 3.2. *Let* $\langle X, d \rangle$ *be a metric space and let* B *be a nonempty subset. The following conditions are equivalent:*

(1) B *is totally bounded;*

(2) *whenever* $\langle Y, \rho \rangle$ *is a metric space and* $f : X \to Y$ *maps Cauchy sequences to Cauchy sequences, then* $f(B) \in \mathcal{B}_\rho(Y)$*;*

(3) *whenever* $\langle Y, \rho \rangle$ *is a metric space and* $f : X \to Y$ *is uniformly locally Lipschitz, then* $f(B) \in \mathcal{B}_\rho(Y)$*;*

(4) *whenever* $f : X \to \mathbb{R}$ *is uniformly locally Lipschitz, then* $f(B)$ *is a bounded set of real numbers.*

PROOF. (1) $\Rightarrow$ (2). Suppose f maps Cauchy sequences to Cauchy sequences. Using the sequential characterisation of total boundedness, it is easy to see that $f(B)$ is actually totally bounded, else one could find a sequence $\langle b_n \rangle$ in B such that $\inf\{\rho(f(b_j), f(b_n)) : j \neq n\} > 0$ while $\langle b_n \rangle$ has a Cauchy subsequence.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4). These are trivial.

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(4) $\Rightarrow$ (1). Suppose B fails to be totally bounded. Then for some $\varepsilon > 0$, the set B fails to be contained in the ε-enlargement of any finite subset of X. Inductively we can construct a sequence $\langle b_n \rangle$ in B such that, for each n, $b_{n+1} \notin B_d(\{b_1, b_2, \ldots, b_n\}, \varepsilon)$. The family of balls $\{B_d(b_n, \frac{1}{4}\varepsilon) : n \in \mathbb{N}\}$ is uniformly discrete: for all $x \in X$, $B_d(x, \frac{1}{4}\varepsilon)$ intersects at most one of the balls $B_d(b_n, \frac{1}{4}\varepsilon)$. As a result, $f : X \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} n - \dfrac{4n}{\varepsilon} d(x, b_n) & \text{if } x \in B_d\Big(b_n, \dfrac{1}{4}\varepsilon\Big), \\ 0 & \text{otherwise,} \end{cases}$$

is a uniformly locally Lipschitz function that is unbounded on B. □

In the proof of our final theorem, we modify a construction of Atsuji [1, pages 14–15] used to show that if each uniformly continuous real-valued function on $\langle X, d \rangle$ is bounded, then X is a Bourbaki bounded subset of itself.

Theorem 3.3. *Let $\langle X, d \rangle$ be a metric space and let B be a nonempty subset. The following conditions are equivalent:*

(1) *B is Bourbaki bounded;*
(2) *whenever $\langle Y, \rho \rangle$ is a metric space and $f : X \to Y$ is uniformly continuous, then $f(B) \in \mathcal{B}_\rho(Y)$;*
(3) *whenever $\langle Y, \rho \rangle$ is a metric space and $f : X \to Y$ is Lipschitz in the small, then $f(B) \in \mathcal{B}_\rho(Y)$;*
(4) *whenever $f : X \to \mathbb{R}$ is Lipschitz in the small, then $f(B)$ is a bounded set of real numbers.*

Proof. For the string (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1), only (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1) require proof. For (1) $\Rightarrow$ (2), choose $\delta > 0$ such that

$$\forall x \in X \ \forall w \in X, \ d(x, w) < \delta \Rightarrow \rho(f(x), f(w)) < 1.$$

By Bourbaki boundedness, choose $\{x_1, x_2, \ldots, x_k\} \subseteq X$ and $n \in \mathbb{N}$ such that $B \subseteq \bigcup_{i=1}^k B_d^n(x_i, \delta)$; then $f(B) \subseteq B_\rho(\{f(x_1), f(x_2), \ldots, f(x_k)\}, n)$ and so $f(B) \in \mathcal{B}_\rho(Y)$.

(4) $\Rightarrow$ (1). Suppose B fails to be Bourbaki bounded. Choose $\varepsilon > 0$ such that whenever F is a finite subset of X and $n \in \mathbb{N}$ we have $B \not\subseteq \bigcup_{x \in F} B_d^n(x, \varepsilon)$. We consider two mutually exclusive and exhaustive cases for the structure of B:

(a) for some $b \in B$, for all $n \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $B \cap B_d^n(b, \varepsilon)$ is a proper subset of $B \cap B_d^{n+j}(b, \varepsilon)$;
(b) for all $b \in B$, there exists $n \in \mathbb{N}$ such that, for all $j \in \mathbb{N}, B \cap B_d^n(b, \varepsilon) = B \cap B_d^{n+j}(b, \varepsilon)$, that is, $B \cap B_d^n(b, \varepsilon) = B \cap b/\simeq_\varepsilon$.

In the first case, for $x \simeq_\varepsilon b$, let $n(x)$ be the smallest n such that $x \in B \cap B_d^n(b, \varepsilon)$. We define our real-valued function f by

$$f(x) = \begin{cases} (n(x) - 1)\varepsilon + d(x, B_d^{n(x)-1}(b, \varepsilon)) & \text{if } x \neq b \text{ and } x \simeq_\varepsilon b, \\ 0 & \text{otherwise.} \end{cases}$$

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By assumption (a), f is unbounded on B. We intend to show that if $x \neq w$ and $d(x,w) < \varepsilon$ then $|f(x) - f(w)| \leq 2d(x,w)$. Now if either x or w is not related to b, then the same is true for the other because $d(x,w) < \varepsilon$. Thus $|f(x) - f(w)| = 0 < 2d(x,w)$. We now pass to the situation where both $x \simeq_\varepsilon b$ and $w \simeq_\varepsilon b$.

Without loss of generality we may assume $n(x) \geq n(w)$. If $n(w) = 0$, that is, $w = b$, then $0 < d(x,w) < \varepsilon$ implies $n(x) = 1$ and

$$|f(x) - f(w)| = f(x) = (1-1)\varepsilon + d(x, B_d^0(b,\varepsilon)) = d(x,b) = d(x,w).$$

Otherwise, $n(w) \geq 1$ and either $n(x) = n(w)$ or $n(x) = n(w) + 1$. When $n(x) = n(w)$ we easily compute

$$|f(x) - f(w)| = |d(x, B_d^{n(x)-1}(b,\varepsilon)) - d(w, B_d^{n(x)-1}(b,\varepsilon))| \leq d(x,w).$$

We are left with the possibility that $n(x) = n(w) + 1$. This gives $f(x) > f(w)$, because

$$(n(x) - 1)\varepsilon - (n(w) - 1)\varepsilon = \varepsilon$$

and

$$d(x, B_d^{n(x)-1}(b,\varepsilon)) - d(w, B_d^{n(w)-1}(b,\varepsilon)) > 0 - \varepsilon.$$

We now claim that $d(w, B_d^{n(w)-1}(b,\varepsilon)) \geq \varepsilon - d(x,w)$; if this fails, then

$$d(x, B_d^{n(w)-1}(b,\varepsilon)) \leq d(x,w) + d(w, B_d^{n(w)-1}(b,\varepsilon)) < d(x,w) + \varepsilon - d(x,w),$$

making $n(x) \leq n(w) - 1 + 1$, which is impossible. The claim having been established,

$$\begin{aligned}|f(x) - f(w)| = f(x) - f(w) &= \varepsilon + d(x, B_d^{n(x)-1}(b,\varepsilon)) - d(w, B_d^{n(w)-1}(b,\varepsilon)) \\ &\leq \varepsilon + d(x,w) - (\varepsilon - d(x,w)) \leq 2d(x,w).\end{aligned}$$

Case (b) is easier. Let $b_1 \in B$ be arbitrary. Choose n_1 such that $B \cap B_d^{n_1}(b_1,\varepsilon) = B \cap b_1/\simeq_\varepsilon$. Since B is not Bourbaki bounded, there exists $b_2 \in B$ with $b_2 \notin B_d^{n_1}(b_1,\varepsilon)$. By the choice of n_1, $b_1/\simeq_\varepsilon \neq b_2/\simeq_\varepsilon$. Choose $n_2 > n_1$ such that $B \cap B_d^{n_2}(b_2,\varepsilon) = B \cap b_2/\simeq_\varepsilon$. Since $B \not\subseteq \bigcup_{j=1}^{2} B_d^{n_2}(b_j,\varepsilon)$, we can find $b_3 \in B \backslash (b_1/\simeq_\varepsilon \cup b_2/\simeq_\varepsilon)$. Thus, we can find a sequence $\langle b_j \rangle$ with distinct terms in B such that whenever $j \neq k$, we have $b_j/\simeq_\varepsilon \neq b_k/\simeq_\varepsilon$. We now define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} j & \text{if } x \simeq_\varepsilon b_j \text{ for some } j, \\ 0 & \text{otherwise.} \end{cases}$$

Since $f(b_j) = j$, f is unbounded on B; further, for each $x \in X$, f is constant on $B_d(x,\varepsilon)$ and so f is Lipschitz in the small. □

There is a third family of uniformly locally Lipschitz functions that one might consider: the family $\mathcal{C}$ of locally Lipschitz functions f where the local Lipschitz constant but not the size of the neighbourhood can be chosen independent of the point. We have no idea what $\bigcap_{f \in \mathcal{C}} \mathcal{B}_f$ comprises and leave this as an open problem to the interested reader.

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There is another approach to obtaining Theorems 3.1 and 3.3: one can use the known uniform density of the locally Lipschitz functions in the continuous real-valued functions on $\langle X, d\rangle$ [8, Corollary 2.8] and the uniform density of the Lipschitz in the small functions within the uniformly continuous real-valued functions [9, Theorem 1]. We feel that it is valuable to give simple direct constructions rather than appeal to these deeper results.

Results such as we have given can be used to show that an unbounded function with a particular continuity property defined on some subset of $\langle X, d\rangle$ cannot be extended to a globally defined function with the same continuity property. For example, by our last result, there is no globally defined uniformly continuous function on ℓ_2 mapping each e_n to n because $\{e_n : n \in \mathbb{N}\}$ is a Bourbaki bounded subset of the Hilbert space.

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https://doi.org/10.1017/S0004972714000215 Published online by Cambridge University Press