BULL. AUSTRAL. MATH. SOC. 34B18
VOL. 66 (2002) [95-104]

POSITIVE SOLUTIONS OF FOURTH-ORDER SUPERLINEAR SINGULAR BOUNDARY VALUE PROBLEMS

GUOLIANG SHI AND SHAOZHU CHEN

This paper investigates fourth-order superlinear singular two-point boundary value problems and obtains necessary and sufficient conditions for existence of C^2 or C^3 positive solutions on the closed interval.

1. INTRODUCTION

In this paper, we are concerned with the fourth-order singular two-point boundary value problem

$$\begin{cases} u^{(4)}(t) = f\big(t, u(t)\big), & 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1}$$

where $f \in C\big((0,1) \times [0,+\infty), [0,+\infty)\big)$ and is *quasi-homogeneous* with respect to the second variable, namely, there are constants λ, μ, N, M with $1 < \lambda \leqslant \mu < \infty$ and $0 < N \leqslant 1 \leqslant M$ such that for all $0 < t < 1$, $u \geqslant 0$,

$$c^{\mu} f(t,u) \leqslant f(t,cu) \leq c^{\lambda} f(t,u), \quad \text{if} \quad 0 < c \leqslant N, \tag{2}$$

$$c^{\lambda} f(t,u) \leqslant f(t,cu) \leq c^{\mu} f(t,u), \quad \text{if} \quad c \geqslant M. \tag{3}$$

A typical quasi-homogeneous function is $f = f_1(t)u^{\lambda_1} + \cdots + f_m(t)u^{\lambda_m}$, where $\lambda \leqslant \lambda_i \leqslant \mu$, $i = 1, \ldots, m$.

Singular or nonsingular fourth-order boundary value problems have been extensively studied by many authors (see [**1, 2, 3, 4, 5, 6, 7**] for nonsingular cases and [**8, 9**] for singular cases). In [**3, 4, 5**] the right hand side function in the equation of (1) has separated variables, namely, $f(t,u) = \lambda a(t) g(u)$, and in [**1, 6, 7, 8**] the function f involves the second derivative u''. O'Regan considered the singular case where $f(t,u,u'')$ is singular at $u = 0$ or $u'' = 0$, while in [**9**] singularity occurs at $t = 0$ or $t = 1$. Using a modified upper and lower solution method, Chen and Zhang [**10**] established necessary and sufficient conditions for existence of positive solutions to second-order sublinear

Received 8th January, 2002
This research was supported by the Chinese NSF under Grant 10071043

boundary value problems on a half-line. Using a similar method, Wei [9] obtained necessary and sufficient conditions for existence of positive solutions to the fourth-order problem (1) in the sublinear case. The results in [9, 10] involve integrability conditions in terms of the function f and the Green's function. To this connection, however, the upper and lower solution method can hardly be used to treat the superlinear case.

In this paper, based on a careful analysis of the Green's function, we shall apply a fixed point theorem in cones to the superlinear problem (1) and obtain necessary and sufficient conditions for existence of a positive solution with different smoothness on the closed interval.

2. Main Results

Our main results are the two following theorems.

Theorem 1. *The boundary value problem* (1) *has a positive solution* $u \in C^2[0,1] \cap C^4(0,1)$, *if and only if,*

$$\int_0^1 t(1-t) f\big(t, t(1-t)\big)\, dt < \infty. \tag{4}$$

Theorem 2. *The boundary value problem* (1) *has a positive solution* $u \in C^3[0,1] \cap C^4(0,1)$, *if and only if,*

$$\int_0^1 f\big(t, t(1-t)\big)\, dt < \infty. \tag{5}$$

We note that (5) implies (4). To prove Theorems 1 and 2, we shall prepare some lemmas. First, we state a fixed point theorem in a cone as follows:

Lemma 1. ([11, Theorem 2.3.4].) *Let E be a Banach space and P a cone in E. Suppose that Ω_1 and Ω_2 are two bounded open subsets of E with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. If $T : P \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to P$ is a completely continuous operator satisfying*

$$\|Tx\| \leqslant \|x\| \text{ for } x \in P \cap \partial\Omega_1 \text{ and } \|Tx\| \geqslant \|x\| \text{ for } x \in P \cap \partial\Omega_2,$$

then T has a fixed point in $P \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

Let $E = \{u \in C^2[0,1] : u(0) = u(1) = 0,\ u''(0) = u''(1) = 0\}$. Define the norm $\|u\|$ for every $u \in E$ by $\|u\| = |u|_0 + |u''|_0$, where $|\cdot|_0$ is the usual sup-norm for continuous functions over $[0,1]$. It is seen that E equipped with the norm $\|\cdot\|$ is a Banach space.

Let $G(t,s)$ be the Green's function of the second-order boundary value problem

$$\begin{cases} -u''(t) = 0, \\ u(0) = u(1) = 0, \end{cases}$$

https://doi.org/10.1017/S0004972700020712 Published online by Cambridge University Press

that is,

$$G(t,s)=\begin{cases} s(1-t), & 0\leqslant s\leqslant t\leqslant 1,\\ t(1-s), & 0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

Let

$$h(t,s)=\int_0^1 G(t,\tau)G(\tau,s)d\tau.$$

Then $h(t,s)$ is the Green's function of the homogeneous fourth-order boundary value problem corresponding to (1). It is easily seen that

$$G(t,s)\leqslant G(s,s),\quad 0\leqslant t,\ s\leqslant 1 \tag{6}$$

and for $1/4\leqslant t\leqslant 3/4$,

$$G(t,s)\geqslant \frac{1}{4}G(s,s),\qquad 0\leqslant s\leqslant 1. \tag{7}$$

Denote

$$P=\Big\{u\in E\ \big|\ u(t)\geqslant 0,\ u''(t)\leqslant 0,\ 0\leqslant t\leqslant 1;\\ u(t)\geqslant \frac{1}{4}|u|_0,\ -u''(t)\geqslant \frac{1}{4}|u''|_0,\ \frac{1}{4}\leqslant t\leqslant \frac{3}{4}\Big\}.$$

It can be easily seen that P is a cone in E.

Next, we define an operator $T:P\to E$ by

$$(Tu)(t)=\int_0^1 h(t,s)f\big(s,u(s)\big)\,ds,\quad u\in P. \tag{8}$$

We observe that a fixed point of T in E is indeed a positive solution of the boundary value problem (1).

Using the Green's function, for every $u\in P$, we shall have an estimate for $u(t)$ in terms of the magnitude of its second derivative, namely, for $t\in[0,1]$,

$$u(t)=\int_0^1 G(t,s)\big(-u''(s)\big)\,ds\leqslant\bigg(\int_0^t s(1-t)\,ds+\int_t^1 t(1-s)\,ds\bigg)|u''|_0$$
$$=\frac{1}{2}t(1-t)|u''|_0. \tag{9}$$

Let $u\in P$ and let c be a positive number such that $c\geqslant M$ and $|u''|_0/(2c)\leqslant N$. From (9), $u(s)/\big(cs(1-s)\big)\leqslant |u''|_0/(2c)\leqslant M$. Then, from (2) and (3),

$$\begin{aligned}\big|Tu(t)\big| &\leqslant \int_0^1 G(s,s)c^\mu f\big(s,u(s)/c\big)\,ds\\ &= c^\mu\int_0^1 G(s,s)f\Big(s,\frac{u(s)}{cs(1-s)}s(1-s)\Big)\,ds\\ &\leqslant c^\mu\Big(\frac{|u''|_0}{2c}\Big)^\lambda\int_0^1 s(1-s)f\big(s,s(1-s)\big)\,ds.\end{aligned}$$

Hence, T is well defined on P provided that (4) or (5) holds.

https://doi.org/10.1017/S0004972700020712 Published online by Cambridge University Press

Lemma 2. *If (4) holds, then $T(P) \subset P$.*

Proof: Let $u \in P$. Obviously, $Tu(t) \geqslant 0$ and $-(Tu)''(t) \geqslant 0$. For $1/4 \leqslant t \leqslant 3/4$, we claim that

$$Tu(t) \geqslant \frac{1}{4}|Tu|_0.$$

Indeed, from (4), by Fubini's theorem, (8) can be rewritten as

$$(Tu)(t) = \int_0^1 G(t,\tau) \int_0^1 G(\tau,s) f(s,u(s))\, ds\, d\tau. \tag{10}$$

It follows from (6) that

$$|Tu|_0 \leqslant \int_0^1 G(\tau,\tau) \int_0^1 G(\tau,s) f(s,u(s))\, ds\, d\tau. \tag{11}$$

On the other hand, for $1/4 \leqslant t \leqslant 3/4$, (7) together with (11) gives

$$(Tu)(t) \geqslant \frac{1}{4} \int_0^1 G(\tau,\tau) \int_0^1 G(\tau,s) f(s,u(s))\, ds\, d\tau \geq \frac{1}{4}|Tu|_0. \tag{12}$$

Next, we claim that $-(Tu)''(t) \geqslant (1/4)\big|(Tu)''\big|_0$ for $t \in [1/4, 3/4]$. In fact, from

$$-(Tu)''(t) = \int_0^1 G(t,s) f(s,u(s))\, ds,$$

it follows from (6) and (7) that

$$\big|(Tu)''\big|_0 \leq \int_0^1 G(s,s) f(s,u(s))\, ds$$

and, for $1/4 \leqslant t \leqslant 3/4$,

$$-(Tu)''(t) \geqslant \frac{1}{4} \int_0^1 G(s,s) f(s,u(s))\, ds \geqslant \frac{1}{4}\big|(Tu)''\big|_0. \tag{13}$$

We now conclude that $T : P \to P$ from (12) and (13) and complete the proof. ∎

Lemma 3. *If (4) holds, then T is a completely continuous operator on P.*

Proof: If $u_n \in P$ and $u_n \to u_0$ in E as $n \to \infty$, then we have that $u_0 \in P$ by the definition of the cone P and that $\{\|u_n\|\}$ is bounded, say, $\|u_n\| \leqslant C_0$, $n \geqslant 1$. As a result, from (9), we have

$$u_n(t) \leqslant \frac{C_0}{2} t(1-t). \tag{14}$$

https://doi.org/10.1017/S0004972700020712 Published online by Cambridge University Press

Let c be a positive number such that $c \geqslant M$ and $C_0/2c \leqslant N$. From (2) and (3),

$$\begin{aligned}
\left|(Tu_n)(t)\right| &\leqslant \int_0^1 s(1-s) f\big(s, u_n(s)\big)\, ds \\
&\leqslant \int_0^1 s(1-s) c^{\mu} f(s, u_n(s)/c)\, ds \\
&\leqslant \int_0^1 s(1-s) c^{\mu} \Big(\frac{u_n(s)}{s(1-s)c}\Big)^{\lambda} f\big(s, s(1-s)\big)\, ds \\
&\leqslant c^{\mu-\lambda} \Big(\frac{C_0}{2}\Big)^{\lambda} \int_0^1 s(1-s) f\big(s, s(1-s)\big)\, ds.
\end{aligned}$$

Now, from (4), an application of Lebesgue's dominant convergence theorem gives the continuity of T on P.

To prove T is a compact operator, we shall show that for every bounded sequence $\{u_n\}$ in P, the sequence $\{Tu_n\} \subset P$ has a convergent subsequence in E. Since $\{Tu_n\}$ is bounded in E, $\big\{\left|(Tu_n)''\right|_0\big\}$ is bounded and hence $\big\{Tu_n(t)\big\}$ is equicontinuous. By Ascoli—Arzela's lemma, it suffices to show that $\big\{(Tu_n)''(t)\big\}$ is equicontinuous. Let C_0 be a positive number such that $\|u_n\| \leqslant C_0$, $n = 1, 2, \ldots$. Then (14) holds from (9). Again, choose a $c \geqslant \max\big\{M, C_0/(2N)\big\}$. Then

$$\begin{aligned}
(Tu)'''(t) &= \int_0^t s f\big(s, u(s)\big)\, ds - \int_t^1 (1-s) f\big(s, u(s)\big)\, ds \\
&\leqslant \int_0^t s f\big(s, u(s)\big)\, ds + \int_t^1 (1-s) f\big(s, u(s)\big)\, ds \\
&\leqslant C_1 \Big(\int_0^t s f\big(s, s(1-s)\big)\, ds + \int_t^1 (1-s) f\big(s, s(1-s)\big)\, ds \Big) \\
&=: F(t),
\end{aligned}$$

where $C_1 = c^{\mu-\lambda}(C_0/2)^{\lambda}$. Since, in view of (4),

$$\begin{aligned}
\int_0^1 F(t)\, dt &= C_1 \int_0^1 \int_0^t s f\big(s, u(s)\big)\, ds\, dt + C_1 \int_0^1 \int_t^1 (1-s) f\big(s, u(s)\big)\, ds \\
&= 2C_1 \int_0^1 s(1-s) f\big(s, s(1-s)\big)\, ds < \infty,
\end{aligned}$$

we have the equicontinuity of the sequence $\big\{(Tu_n)''(t)\big\}$ from the uniform continuity of the convergent integral of $F(t)$ with respect to the Lebesgue measure over $[0,1]$.

Therefore, T is a compact operator on P and the proof of Lemma 3 is complete. ∎

We are now in a position to prove our main results.

PROOF OF THEOREM 1: Necessity. Let $u \in C^2[0,1] \cap C^4(0,1)$ be a positive solution of (1). Obviously, $u''(t) \leqslant 0$ for $0 \leqslant t \leqslant 1$ and hence $u(t)$ is concave. It follows from $u(0) = u(1) = 0$ that $u'(0) > 0$ and $u'(1) < 0$. Consequently, there must be a positive number k such that $u(t) \geqslant kt(1-t)$. Let $c \geqslant \max\{M, 1/(kN)\}$. Then, for $0 < t < 1$, $t(1-t)/(cu(t)) < N$, and we get

$$
\begin{aligned}
f(t, t(1-t)) &\leqslant c^{\mu} f\Big(t, t(1-t)u(t)/\big(cu(t)\big)\Big) \\
&\leqslant c^{\mu-\lambda}k^{-\lambda} f\big(t, u(t)\big) = c^{\mu-\lambda}k^{-\lambda}u^{(4)}(t).
\end{aligned} \tag{15}
$$

Since $u''(0) = u''(1) = 0$, there is a $t_0 \in (0,1)$ such that $u'''(t_0) = 0$. Then

$$
u''(t_0) = \int_0^{t_0} u'''(s)\,ds = -\int_0^{t_0}\int_s^{t_0} u^{(4)}(\tau)d\tau\,ds = -\int_0^{t_0} \tau u^{(4)}(\tau)\,d\tau. \tag{16}
$$

On the other hand,

$$
\begin{aligned}
u''(t_0) &= -\int_{t_0}^1 u'''(s)\,ds = -\int_{t_0}^1\int_{t_0}^s u^{(4)}(\tau)\,d\tau\,ds \\
&= -\int_{t_0}^1 (1-\tau)u^{(4)}(\tau)\,d\tau.
\end{aligned} \tag{17}
$$

Therefore,

$$
\begin{aligned}
\int_0^1 t(1-t)u^{(4)}(t)\,dt &= \left(\int_0^{t_0} + \int_{t_0}^1\right) t(1-t)u^{(4)}(t)\,dt \\
&\leqslant \int_0^{t_0} tu^{(4)}(t)\,dt + \int_{t_0}^1 (1-t)u^4(t)\,dt \\
&= 2\big(-u''(t_0)\big) < \infty.
\end{aligned} \tag{18}
$$

We now obtain (4) from (15) and (18), and complete the proof of the necessity.

Sufficiency. Let $\Omega_1 = \{u \in E \mid \|u\| < r\}$, where

$$
r \leqslant \min\left\{2N, 2\left(\int_0^1 s(1-s)f\big(s, s(1-s)\big)\,ds\right)^{1/(1-\lambda)}\right\}. \tag{19}
$$

Let $u \in \partial\Omega_1 \cap P$. Then $\|u\| = |u|_0 + |u''|_0 = r$, and $|u|_0 \leqslant r$, $|u''|_0 \leqslant r$. It follows from (9) that

$$
u(t) \leqslant \frac{1}{2}t(1-t)|u''|_0 \leqslant \frac{r}{2}t(1-t) \leqslant Nt(1-t). \tag{20}
$$

In view of (2), (3), and (20), we have

$$\begin{aligned} Tu(t) &= \int_0^1 h(t,s) f\big(s,u(s)\big)\,ds \\ &\le \int_0^1 h(t,s)\Big(\frac{u(s)}{s(1-s)}\Big)^{\lambda} f\big(s,s(1-s)\big)\,ds \\ &\le 2^{-\lambda} r^{\lambda} \int_0^1 s(1-s) f\big(s,s(1-s)\big)\,ds \end{aligned}$$

and

$$|Tu|_0 \leqslant 2^{-\lambda} r^{\lambda} \int_0^1 s(1-s) f\big(s,s(1-s)\big)\,ds, \quad u \in \partial\Omega_1 \cap P. \tag{21}$$

On the other hand,

$$\begin{aligned} -(Tu)''(t) &= \int_0^1 G(t,s) f\big(s,u(s)\big)\,ds \\ &\le \int_0^1 G(t,s)\Big(\frac{u(s)}{s(1-s)}\Big)^{\lambda} f\big(s,s(1-s)\big)\,ds \\ &\leqslant 2^{-\lambda} r^{\lambda} \int_0^1 s(1-s) f\big(s,s(1-s)\big)\,ds, \end{aligned}$$

and so

$$\big|(Tu)''\big|_0 \leqslant 2^{-\lambda} r^{\lambda} \int_0^1 s(1-s) f\big(s,s(1-s)\big)\,ds. \tag{22}$$

Thus, from (21), (22), and (19),

$$\begin{aligned} \|Tu\| = |Tu|_0 + \big|(Tu)''\big|_0 &\leqslant 2^{1-\lambda} r^{\lambda} \int_0^1 s(1-s) f\big(s,s(1-s)\big)\,ds \\ &\leqslant r = \|u\|, \quad u \in \partial\Omega_1 \cap P. \end{aligned} \tag{23}$$

Next, set $\Omega_2 = \big\{u \in E \bigm| \|u\| < R\big\}$, where

$$R = \max\Big\{288M,\ 2^{(9\lambda+1)/(\lambda-1)}\Big(\int_{1/4}^{3/4} s(1-s) f\big(s,s(1-s)\big)\,ds\Big)^{1/(1-\lambda)}\Big\}. \tag{24}$$

Let $u \in \partial\Omega_2 \cap P$. Then $\|u\| = |u|_0 + |u''|_0 = R$, $|u|_0 \le R, |u''|_0 \le R$. From (9), we have

$$|u|_0 \leqslant \frac{1}{8}|u''|_0, \quad |u''|_0 \geqslant \frac{8}{9}R. \tag{25}$$

https://doi.org/10.1017/S0004972700020712 Published online by Cambridge University Press

Also, by the definition of the cone P, we have that for $1/4 \leqslant t \leqslant 3/4$,

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\big(-u''(s)\big)\,ds \geq \int_{1/4}^{3/4} G(t,s)\big(-u''(s)\big)\,ds \\ &\geqslant \frac{1}{4^2}|u''|_0 \int_{1/4}^{3/4} G(s,s)\,ds \geqslant \frac{1}{2^8}|u''|_0, \end{aligned}$$

and hence,

$$|u|_0 \geqslant \frac{1}{2^8}|u''|_0. \tag{26}$$

Since $u \in P$, from (26), we have

$$\frac{u(s)}{s(1-s)} \geqslant 4u(s) \geqslant |u|_0 \geqslant \frac{1}{2^8}|u''|_0, \tag{27}$$

and so, from (24) and (25), for $1/4 \leqslant s \leqslant 3/4$,

$$\frac{u(s)}{s(1-s)} \geqslant \frac{1}{2^8}\frac{8}{9}R \geqslant M. \tag{28}$$

For $1/4 \leqslant t \leqslant 3/4$, from (27) and (28), we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,\tau)\int_0^1 G(\tau,s)f\big(s,u(s)\big)\,dsd\tau \\ &\geqslant \int_{1/4}^{3/4} G(t,\tau)\int_{1/4}^{3/4} G(\tau,s)f\big(s,u(s)\big)\,dsd\tau \\ &\geqslant \frac{1}{4^2}\int_{1/4}^{3/4}\tau(1-\tau)d\tau\int_{1/4}^{3/4} s(1-s)\Big(\frac{u(s)}{s(1-s)}\Big)^{\lambda} f\big(s,s(1-s)\big)\,ds \\ &\geqslant \frac{1}{2^8}|u|_0^{\lambda}\int_{1/4}^{3/4} s(1-s)f\big(s,s(1-s)\big)\,ds. \end{aligned} \tag{29}$$

On the other hand, from (27),

$$\begin{aligned} -(Tu)''(t) &= \int_0^1 G(t,s)f\big(s,u(s)\big)\,ds \\ &\geqslant \int_{1/4}^{3/4} G(t,s)\Big(\frac{u(s)}{s(1-s)}\Big)^{\lambda} f\big(s,s(1-s)\big)\,ds \\ &\geqslant 2^{-8\lambda-2}|u''|_0^{\lambda}\int_{1/4}^{3/4} s(1-s)f\big(s,s(1-s)\big)\,ds, \end{aligned}$$

https://doi.org/10.1017/S0004972700020712 Published online by Cambridge University Press

and hence,

$$|(Tu)''|_0 \geqslant 2^{-8\lambda-2}|u''|_0^\lambda \int_{1/4}^{3/4} s(1-s)f\big(s, s(1-s)\big)\, ds. \tag{30}$$

Now, from (29), (30), and the fact that $a^\lambda + b^\lambda \geqslant 2^{1-\lambda}(a+b)^\lambda$ for $\lambda \geqslant 1$ and a, $b > 0$, we arrive at

$$\begin{aligned} \|Tu\| &\geqslant \big(2^{-8}|u|_0^\lambda + 2^{-8\lambda-2}|u''|_0^\lambda\big) \int_{1/4}^{3/4} s(1-s)f\big(s, s(1-s)\big)\, ds \\ &\geqslant 2^{-8\lambda-2}\big(|u|_0^\lambda + |u''|_0^\lambda\big) \int_{1/4}^{3/4} s(1-s)f\big(s, s(1-s)\big)\, ds \\ &\geqslant 2^{-9\lambda-1}\big(|u|_0 + |u''|_0\big)^\lambda \int_{1/4}^{3/4} s(1-s)f\big(s, s(1-s)\big)\, ds. \end{aligned}$$

Consequently, by the definition of R, we have

$$\begin{aligned} \|Tu\| = |Tu|_0 + |(Tu)''|_0 &\geqslant 2^{-9\lambda-1}R^\lambda \int_{1/4}^{3/4} s(1-s)f\big(s, s(1-s)\big)\, ds \\ &\geqslant R = \|u\|, \quad u \in \partial\Omega_2 \cap P. \end{aligned} \tag{31}$$

Finally, from (23) and (31), by Lemma 1, the operator T has at least one fixed point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ which is a positive $C^2[0,1]$ solution to the boundary value problem (1).

The proof of Theorem 1 is complete. □

PROOF OF THEOREM 2: We prove the sufficiency first. Since (5) implies (4), Theorem 1 provides a $C^2[0,1]$ solution $u \in P$. From (9), $u(t) \leqslant (1/2)t(1-t)|u''|_0$.

To prove that $u \in C^3[0,1]$, choose a positive number $c \geqslant \max\{M, |u''|_0/(2N)\}$. Then, from (2) and (3), we have

$$\begin{aligned} \int_0^1 |u^{(4)}(s)|\, ds = \int_0^1 f\big(s, u(s)\big)\, ds &\leqslant c^\mu \int_0^1 f\Big(s, \frac{u(s)}{c}\Big)\, ds \\ &\leqslant c^\mu \Big(\frac{|u''|_0}{2c}\Big)^\lambda \int_0^1 f\big(s, s(1-s)\big)\, ds. \end{aligned}$$

Now, $u^{(4)}(t)$ is absolute integrable over $[0,1]$ from (5), and hence, $u \in C^3[0,1]$.

To prove the necessity, let there be a positive solution $u \in C^3[0,1]$ of (1). The same reasoning at the beginning of the proof of Theorem 1 asserts that $u(t) \geqslant k_1 t(1-t)$ for

https://doi.org/10.1017/S0004972700020712 Published online by Cambridge University Press

all $t \in [0,1]$ and for some constant $k_1 > 0$. Let $c \geqslant \max\{M, 1/(k_1 N)\}$. Then, from (2) and (3),

$$f\big(t, t(1-t)\big) \leqslant c^{\mu} f\Big(t, \big(t(1-t)u(t)\big)/\big(cu(t)\big)\Big) \leqslant c^{\mu-\lambda} k_1^{-\lambda} f\big(t, u(t)\big),$$

and hence,

$$\int_0^1 f\big(t, t(1-t)\big)\,dt \leqslant c^{\mu-\lambda} k_1^{-\lambda} \int_0^1 f\big(t, u(t)\big)\,dt = c^{\mu-\lambda} k_1^{-\lambda} \int_0^1 u^{(4)}(t)\,dt$$
$$= c^{\mu-\lambda} k_1^{-\lambda} \big[u'''(1) - u'''(0)\big] < +\infty.$$

Thus, (5) holds and the proof of Theorem 2 is complete. ∎

References

[1] A.R. Aftabizadeh, 'Existence and uniqueness theorems for fourth-order boundary value problems', *J. Math. Anal. Appl.* **116** (1986), 415–426.

[2] R.P. Agarwal, 'On fourth-order boundary value problems arising in beam analysis', *Differential Integral Equations* **2** (1989), 91–110.

[3] J.R. Graef and B. Yang, 'Existence and nonexistence of positive solutions of fourth order nonlinear boundary value problem', *Appl. Anal.* **74** (2000), 201–214.

[4] J.R. Graef and B. Yang, 'On a nonlinear boundary value problems for fourth order equations', *Appl. Anal.* **72** (1999), 439–451.

[5] R. Ma and H. Wang, 'On the existence of positive solutions of fourth-order ordinary differential equations', *Appl. Anal.* **59** (1995), 225–231.

[6] R. Ma, J. Zhang, and S. Fu, 'The method of lower and upper solutions for fourth-order two-point boundary value problems', *J. Math. Anal. Appl.* **215** (1997), 415–422.

[7] Y. Yang, 'Fourth-order two-point boundary value problems', *Proc. Amer. Math. Soc.* **104** (1988), 175–180.

[8] D. O'Regan, 'Solvability of some fourth (and higher) order singular boundary value problems', *J. Math. Anal. Appl.* **161** (1991), 78–116.

[9] Z. Wei, 'Positive solutions of singular boundary value problems of fourth order differential equations', *Acta Math. Sinica* **42** (1999), 715–722.

[10] S. Chen and Y. Zhang, 'Singular boundary value problem on a half-line', *J. Math. Anal. Appl.* **195** (1995), 449–468.

[11] D. Guo and V. Lakshmikantham, *Nonlinear problems in abstract cones*, Notes and Reports in Mathematics in Science and Engineering **5** (Academic Press, Inc., Boston, New York, 1988).

Department of Mathematics
Shandong University
Jinan
Shandong 250100
People's Republic of China
e-mail: szchen@ sdu.edu.cn

https://doi.org/10.1017/S0004972700020712 Published online by Cambridge University Press