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On the Steinberg Map and Steinberg Cross-Section for a Symmetrizable Indefinite Kac-Moody Group

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Abstract. Let G be a symmetrizable indefinite Kac-Moody group over $\mathbb{C}$. Let $\mathrm{Tr}_{\Lambda_1}, \ldots, \mathrm{Tr}_{\Lambda_{2n-l}}$ be the characters of the fundamental irreducible representations of G, defined as convergent series on a certain part $G^{\text{tr-alg}} \subseteq G$. Following Steinberg in the classical case and Brüchert in the affine case, we define the Steinberg map $\chi := (\mathrm{Tr}_{\Lambda_1}, \ldots, \mathrm{Tr}_{\Lambda_{2n-l}})$ as well as the Steinberg cross section C, together with a natural parametrisation $\omega \colon \mathbb{C}^n \times (\mathbb{C}^\times)^{n-l} \to C$. We investigate the local behaviour of χ on C near $\omega\Big((0,\ldots,0) \times (1,\ldots,1)\Big)$, and we show that there exists a neighborhood of $(0,\ldots,0) \times (1,\ldots,1)$, on which $\chi \circ \omega$ is a regular analytical map, satisfying a certain functional identity. This identity has its origin in an action of the center of G on C.

Introduction

Let G be a semisimple, simply connected linear algebraic group over $\mathbb{C}$ of rank r. Let T be a maximal torus of G, N the normalizer of T, and $\mathcal{W} = N/T$ the corresponding Weyl group. Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_r$ be the fundamental dominant characters with respect to some Borel subgroup containing T. Denote by Tr_{Λ_i} the character of the rational irreducible G-representation $\big(L(\Lambda_i), \pi_{\Lambda_i}\big)$ belonging to Λ_i.

G acts on itself by the conjugation action. The adjoint quotient, which is the categorical quotient in the category of affine algebraic varieties, can be realized by the Steinberg map:

$$\chi := (\mathrm{Tr}_{\Lambda_1}, \mathrm{Tr}_{\Lambda_2}, \ldots, \mathrm{Tr}_{\Lambda_r}) \colon G \to \mathbb{C}^r.$$

The regular elements of G, which are contained in a fiber of χ, form a single conjugacy class. This class is open and dense in the fiber.

Denote by U_i the root group belonging to the simple root α_i, and let $æ_i \colon \mathbb{C} \to U_i$ be an isomorphism such that $t æ_i(s) t^{-1} = æ_i\big(\alpha(t)s\big)$, $t \in T$, $s \in \mathbb{C}$. Let $n_i \in N$ be an element which represents the simple reflection σ_i. Then

$$C := U_1 n_1 \cdots U_r n_r$$

is a closed affine subvariety of G, isomorphic to $\mathbb{C}^r$ by:

$$\omega \colon \mathbb{C}^r \to C$$
$$(s_1, \ldots, s_r) \mapsto æ_1(s_1) n_1 \cdots æ_r(s_r) n_r$$

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C consists only of regular elements of G. It is a cross section of χ, which means that $\chi|_C : C \to \mathbb{C}^r$, or equivalently $\chi \circ \omega : \mathbb{C}^r \to \mathbb{C}^r$, is an isomorphism of affine varieties. C is called the Steinberg cross section.

Semisimple Lie algebras are generalized by symmetrizable Kac-Moody algebras. The indecomposable, nonclassical ones can be divided into affine and indefinite Kac-Moody algebras. For a given Kac-Moody algebra there exists an analogue of a semi-simple, simply connected algebraic group, the Kac-Moody group.

In [B], G. Brüchert gave the following approach to the adjoint quotient and the Steinberg cross section for an affine Kac-Moody group: Every G-module $L(\Lambda_i)$, $i = 1, \ldots, n+1$, carries a contravariant, positive definite hermitian form, uniquely determined up to a nonzero scalar factor. (Here $n \times n$ is the size of the generalized Cartan matrix, and $n + 1$ the dimension of the torus T.) Denote by G^{tr} the semigroup of elements $g \in G$, such that for all i, the linear map $\pi_{\Lambda_i}(g)$ can be extended to a trace class operator on the Hilbert space completion of $L(\Lambda_i)$. G. Brüchert determined a conjugation invariant subsemigroup $G^{>1}$ of G^{tr}. (As shown in [M], $G^{>1}$ equals G^{tr}.) Using the trace functions on the semigroups of trace class operators, the Steinberg map $\chi : G^{>1} \to \mathbb{C}^{n+1}$ is defined in the obvious way. He also gave an obvious generalization of the Steinberg cross section C and its parametrization $\omega : \mathbb{C}^n \times \mathbb{C}^\times \to C$. Due to his results, $\omega^{-1}(C \cap G^{>1}) = \mathbb{C}^n \times \mathbb{C}^{>1}$, and $\chi \circ \omega : \mathbb{C}^n \times \mathbb{C}^{>1} \to \mathbb{C}^{n+1}$ is a regular analytic map. (Here $\mathbb{C}^{>1} := \{z \in \mathbb{C} \mid |z| > 1\}$.) He defined an action of the identity component $Z(G)^\circ \cong \mathbb{C}^\times$ of the center $Z(G)$ on the Steinberg cross section, which leads to a functional identity for $\chi \circ \omega$.

In this paper, we define for a symmetrizable indefinite Kac-Moody group the Steinberg map $\chi := (\mathrm{Tr}_{\Lambda_1}, \ldots, \mathrm{Tr}_{\Lambda_{2n-l}}) : G^{\text{tr-alg}} \to \mathbb{C}^{2n-l}$, and the Steinberg cross section C, together with a natural parametrization. ($n \times n$ is the size of the generalized Cartan matrix, and l its rank.) Here $G^{\text{tr-alg}}$ is a certain part of G, on which the traces Tr_{Λ_i} can be realized as convergent series. We investigate the local behaviour of χ on C near $n_1 \cdots n_n = \omega\big((0, \ldots, 0) \times (1, \ldots, 1)\big)$, and we show that there exists a neighborhood of $(0, \ldots, 0) \times (1, \ldots, 1)$, on which the map $\chi \circ \omega$ is regular analytic. Furthermore it obeys a certain functional identity, which has its origin in an action of the center $Z(G)$ of G on C.

The local nature of our result is technically related to the following fact: We show that for an indefinite Kac-Moody group there is no reasonable part of G^{tr}, which is invariant under G-conjugation. To look for a conjugation invariant part of G, on which the characters are defined, we should look at subdomains in $G^{\text{tr-alg}}$. However here, general convergence considerations are not easily manageable.

For the proof of the functional identity in the affine case, G. Brüchert used a case by case inspection of the affine Kac-Moody groups. The proof given in this paper for the indefinite case is of general nature. It can be modified also to work in the affine case.

There is a well known relation between simple singularities and simple algebraic groups, [Sl 1]. This relation has been extended to simple elliptic singularities and certain completions of affine Kac-Moody groups, which correspond to holomorphic loops, [H,Sl]. For the proof, the regularity of χ on C has been used in an essential way. Hopefully, our result for indefinite Kac-Moody groups will help to extend this relation further.

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1 Preliminaries

In this section we recall some basic facts about Kac-Moody algebras and Kac-Moody groups, which are used later, merely to introduce our notation.

The Kac-Moody group given in [K,P1], [K,P2] corresponds to the derived Kac-Moody algebra. We work with a slightly enlarged group, corresponding to the full Kac-Moody algebra, as in [Ti], [Mo,Pi].

All the material stated in this subsection about Kac-Moody algebras can be found in the books [K], [Mo,Pi], and about Kac-Moody groups in [K,P1], [K,P2], [Mo,Pi].

We denote by $\mathbb{N} = \mathbb{Z}^+$, $\mathbb{Q}^+$ resp. $\mathbb{R}^+$ the sets of strictly positive numbers of $\mathbb{Z}$, $\mathbb{Q}$ resp. $\mathbb{R}$, and the sets $\mathbb{N}_0 = \mathbb{Z}_0^+$, $\mathbb{Q}_0^+$, $\mathbb{R}_0^+$ contain, in addition, the zero. We denote by $\mathbb{R}^\times$, $\mathbb{C}^\times$ the unit groups of $\mathbb{R}$, $\mathbb{C}$.

Generalized Cartan Matrices Starting point for the construction of a Kac-Moody algebra and its associated simply connected Kac-Moody group is a *generalized Cartan matrix*, which is a matrix $A = (a_{ij}) \in M_n(\mathbb{Z})$ with $a_{ii} = 2$, $a_{ij} \leq 0$ for all $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$. Denote by l the rank of A and set $I := \{1, 2, \dots, n\}$.

For the properties of the generalized Cartan matrices, in particular their classification, we refer to the book [K]. In this paper we assume A to be symmetrizable.

Realizations A *simply connected minimal free realization* of A consists of dual free $\mathbb{Z}$-modules H, P of rank $2n-l$, and linear independent sets $\Pi^\vee = \{h_1, \dots, h_n\} \subseteq H$, $\Pi = \{\alpha_1, \dots, \alpha_n\} \subseteq P$, such that $\alpha_i(h_j) = a_{ji}$, $i, j = 1, \dots, n$. Furthermore $H_I := Q^\vee := \mathbb{Z}\text{-span}\,\{h_1, \dots, h_n\}$ is saturated in H, which means that for all $n \in \mathbb{N}$, $x \in H$ we have: $nx \in H \Rightarrow x \in H$.

P is called the *weight lattice*, $Q := \mathbb{Z}\text{-span}\,\{\alpha_i \mid i \in I\}$ the *root lattice*, and $Q^\vee$ the *coroot lattice*.

Set $Q_0^{\pm} := \mathbb{Z}_0^{\pm}\text{-span}\,\{\alpha_i \mid i \in I\}$, set $Q^{\pm} := Q_0^{\pm} \setminus \{0\}$, and order the elements of $\mathbf{h}^*$ by $\lambda \leq \lambda'$ if and only if $\lambda' - \lambda \in Q_0^+$. For an element $q = \sum_i k_i \alpha_i \in Q$ set $\operatorname{supp}(q) := \{i \in I \mid k_i \neq 0\}$.

We fix a complement H_{rest} of H_I in H. This complement determines fundamental dominant weights $\Lambda_1, \dots, \Lambda_n$ by:

$$\Lambda_i(h_j) := \delta_{ij} \quad (j = 1, \dots, n), \qquad \Lambda_i(h) := 0 \quad (h \in H_{\text{rest}}).$$

We extend $h_1, \dots, h_n \in H_I$ with elements $h_{n+1}, \dots, h_{2n-l} \in H_{\text{rest}}$ to a base of H, and extend $\Lambda_1, \dots, \Lambda_n$ to the corresponding dual base $\Lambda_1, \dots, \Lambda_{2n-l}$, which we call a *system of fundamental dominant weights*.

The Weyl Group and the Tits Cone Define the following vector spaces over $\mathbb{C}$:

$$\mathbf{h} := H \otimes_{\mathbb{Z}} \mathbb{C}, \quad \mathbf{h}^* := P \otimes_{\mathbb{Z}} \mathbb{C}.$$

H and P are identified with $H \otimes 1$, $P \otimes 1$, and $\mathbf{h}^*$ is interpreted as the dual of $\mathbf{h}$. Because A is symmetrizable, we can choose a symmetric matrix $B \in M_n(\mathbb{Q})$ and a

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diagonal matrix $D = \operatorname{diag}(\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_1, \ldots, \epsilon_n \in \mathbb{Q}^+$, such that $A = DB$. Define a nondegenerate symmetric bilinear form on $\mathbf{h}$ by:

$$(h_i \mid h) = (h \mid h_i) = \alpha_i(h)\epsilon_i \quad i \in I, \quad h \in \mathbf{h}$$
$$(h' \mid h'') = 0 \quad h', h'' \in \mathbf{h}_{\text{rest}} := H_{\text{rest}} \otimes \mathbb{C}.$$

The induced nondegenerate symmetric form on $\mathbf{h}^*$ is also denoted by $(\ \mid\ )$.

The *Weyl group* $\mathcal{W} = \mathcal{W}(A)$ is the Coxeter group with generators σ_i, $i \in I$, and relations:

$$\sigma_i^2 = 1 \quad (i \in I), \quad (\sigma_i\sigma_j)^{m_{ij}} = 1 \quad (i, j \in I, i \neq j).$$

The m_{ij} given by:

$a_{ij}a_{ji}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	no relation between σ_i and σ_j

.

The Weyl group $\mathcal{W}$ acts faithfully (resp. contragrediently) by

$$\sigma_i h := h - \alpha_i(h)h_i \quad i \in I, \quad h \in \mathbf{h}$$
$$\sigma_i \lambda := \lambda - \lambda(h_i)\alpha_i \quad i \in I, \quad \lambda \in \mathbf{h}^*$$

on $\mathbf{h}$ and $\mathbf{h}^*$, leaving the lattices $Q^\vee$, H, Q, P and the forms invariant.

$\Delta_{\text{re}} := \mathcal{W}\{\alpha_i \mid i \in I\} \subseteq Q$ is called the set of *real roots* and $\Delta_{\text{re}}^\vee := \mathcal{W}\{h_i \mid i \in I\} \subseteq Q^\vee$ the set of *real coroots*. The map $\alpha_i \mapsto h_i$, $i \in I$, can be extended to a $\mathcal{W}$-equivariant bijection $\alpha \mapsto h_\alpha$.

Define the real subspaces $\mathbf{h}_\mathbb{R} := H \otimes \mathbb{R} \subseteq \mathbf{h}$, $\mathbf{h}_\mathbb{R}^* := P \otimes \mathbb{R} \subseteq \mathbf{h}^*$. To illustrate the action of $\mathcal{W}$ on $\mathbf{h}_\mathbb{R}^*$ geometrically, for $J \subseteq I$ set:

$$F_J = \{\lambda \in \mathbf{h}_\mathbb{R}^* \mid \lambda(h_i) = 0 \text{ for } i \in J,\ \lambda(h_i) > 0 \text{ for } i \in I \setminus J\}$$
$$\overline{F}_J = \{\lambda \in \mathbf{h}_\mathbb{R}^* \mid \lambda(h_i) = 0 \text{ for } i \in J,\ \lambda(h_i) \geq 0 \text{ for } i \in I \setminus J\}.$$

Call $C := F_\varnothing = \{\lambda \in \mathbf{h}_\mathbb{R}^* \mid \lambda(h_i) > 0 \text{ for } i \in I\}$ the *open fundamental chamber*, and call $\overline{C} := \overline{F}_\varnothing = \{\lambda \in \mathbf{h}_\mathbb{R}^* \mid \lambda(h_i) \geq 0 \text{ for } i \in I\}$ the *fundamental chamber*. The *Tits cone* $X := \mathcal{W}\overline{C}$ is a convex $\mathcal{W}$-invariant cone, a $\mathcal{W}$-invariant partition into facets given by $\{\sigma F_J \mid \sigma \in \mathcal{W}, J \subseteq I\}$.

The chamber $\overline{C} = \dot{\bigcup}_{J \subseteq I} F_J$ is a fundamental region of X, and the parabolic subgroup $\mathcal{W}_J$ of $\mathcal{W}$ is the stabilizer of every element $\lambda \in F_J$.

The action of $\mathcal{W}$ on $\mathbf{h}_\mathbb{R}$ can be illustrated in a similar way, by using the dual Tits cone $X^\vee := \mathcal{W}\overline{C^\vee}$, where $\overline{C^\vee} := \{h \in \mathbf{h}_\mathbb{R} \mid \alpha_i(h) \geq 0 \text{ for } i \in I\}$ is the dual fundamental chamber.

Note that an element $h \in \mathbf{h}$ can be written uniquely in the form $h = h_{\text{re}} + ih_{\text{im}}$ with $h_{\text{re}}, h_{\text{im}} \in \mathbf{h}_\mathbb{R}$.

The Kac-Moody Algebra The *Kac-Moody algebra* $\mathbf{g} = \mathbf{g}(A)$ is the Lie algebra over $\mathbb{F}$ generated by the abelian Lie algebra $\mathbf{h}$ and $2n$ elements e_i, f_i $(i \in I)$ with the following relations, which hold for any $i, j \in I$, $h \in \mathbf{h}$:

$$[e_i, f_j] = \delta_{ij}h_i, \quad [h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i$$
$$(\operatorname{ad} e_i)^{1-a_{ij}}e_j = (\operatorname{ad} f_i)^{1-a_{ij}}f_j = 0 \quad (i \neq j).$$

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The *compact involution* $*$ of $\mathbf{g}$ is the involutive antilinear antiautomorphism determined by:

$$e_i^* = f_i, \quad f_i^* = e_i \quad (i \in I), \quad h^* = h, \quad (h \in H).$$

The space $\mathbf{h}$ and the elements e_i, f_i $(i \in I)$ can be identified with their images in $\mathbf{g}$. The nondegenerate symmetric bilinear form $(\ \mid\ )$ on $\mathbf{h}$ can be uniquely extended to a nondegenerate symmetric invariant bilinear form $(\ \mid\ )$ on $\mathbf{g}$. We have the *root space decomposition*:

$$\mathbf{g} = \bigoplus_{\alpha \in \mathbf{h}^*} \mathbf{g}_\alpha \quad \text{where} \quad \mathbf{g}_\alpha := \{x \in \mathbf{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathbf{h}\}.$$

In particular $\mathbf{g}_0 = \mathbf{h}$, $\mathbf{g}_{\alpha_i} = \mathbb{C}e_i$, $\mathbf{g}_{-\alpha_i} = \mathbb{C}f_i$ $(i \in I)$.

The set of roots $\Delta := \{\alpha \in \mathbf{h}^* \setminus \{0\} \mid \mathbf{g}_\alpha \neq \{0\}\}$ is invariant under the Weyl group and spans the root lattice Q. We have $\Delta_{\mathrm{re}} \subseteq \Delta$, and $\Delta_{\mathrm{im}} := \Delta \setminus \Delta_{\mathrm{re}}$ is called the set of *imaginary roots*.

Δ, Δ_{re} and Δ_{im} decompose into the disjoint union of the sets of *positive* and *negative* roots $\Delta^\pm := \Delta \cap Q^\pm$, $\Delta_{\mathrm{re}}^\pm := \Delta_{\mathrm{re}} \cap Q^\pm$, $\Delta_{\mathrm{im}}^\pm := \Delta_{\mathrm{im}} \cap Q^\pm$, and we have $\Delta^\pm = -\Delta^\mp$, $\Delta_{\mathrm{re}}^\pm = -\Delta_{\mathrm{re}}^\mp$, $\Delta_{\mathrm{im}}^\pm = -\Delta_{\mathrm{im}}^\mp$.

Corresponding to the decomposition into positive and negative roots there is a *triangular decomposition* $\mathbf{g} = \mathbf{n}^- \oplus \mathbf{h} \oplus \mathbf{n}^+$ where $\mathbf{n}^\pm := \bigoplus_{\alpha \in \Delta^\pm} \mathbf{g}_\alpha$.

For a real root α, the subalgebra $\mathbf{g}_\alpha \oplus [\mathbf{g}_\alpha, \mathbf{g}_{-\alpha}] \oplus \mathbf{g}_{-\alpha}$ of $\mathbf{g}$ is isomorphic to $\mathrm{sl}(2, \mathbb{C})$.

The derived Lie algebra of $\mathbf{g}$ is given by $\mathbf{g}' = \bigoplus_{\alpha \in \Delta} \mathbf{g}_\alpha \oplus \mathbf{h}'$, where $\mathbf{h}'$ is spanned by the elements h_i, $i \in I$. The center of $\mathbf{g}$ is given by $z(\mathbf{g}) = \{h \in \mathbf{h} \mid \alpha_i(h) = 0$ for all $i\} = \{a_1 h_1 + \cdots + a_n h_n \in \mathbf{h} \mid (a_1, \ldots, a_n)A = 0\}$.

The Kac-Moody Group To construct the Kac-Moody group, call a representation (V, π) of $\mathbf{g}$ *admissible* if:

(1) V is $\mathbf{h}$-diagonalizable with set of weights $P(V) \subseteq P$.
(2) $\pi(x)$ is locally nilpotent for all $x \in \mathbf{g}_\alpha$, $\alpha \in \Delta_{\mathrm{re}}$.

Examples are the the adjoint representation $(\mathbf{g}, \mathrm{ad})$, and for $\Lambda \in P^+ := P \cap \overline{C}$ the irreducible highest weight representation $(L(\Lambda), \pi_\Lambda)$, which is (up to isomorphy) the unique irreducible representation with a nonzero element v_Λ satisfying

$$\pi_\Lambda(\mathbf{n}^+)v_\Lambda = 0, \quad \pi_\Lambda(h)v_\Lambda = \Lambda(h)v_\Lambda \quad (h \in \mathbf{h}).$$

Let $\tilde{G}$ be the free product of the additive groups $\mathbf{g}_\alpha$, $\alpha \in \Delta_{\mathrm{re}}$, and the torus $H \otimes_{\mathbb{Z}} \mathbb{C}^\times$. For any admissible representation (V, π) we get, due to (1), (2), and the universal property of $\tilde{G}$, a homomorphism $\tilde{\pi} \colon \tilde{G} \to \mathrm{GL}(V)$, mapping $x_\alpha \in \mathbf{g}_\alpha$ to $\exp\big(\pi(x_\alpha)\big)$ and $h \otimes s \in H \otimes_{\mathbb{Z}} \mathbb{C}^\times$ to an element $t_h(s)$ defined by

$$t_h(s)v_\lambda := s^{\lambda(h)}v_\lambda \quad v_\lambda \in V_\lambda, \quad \lambda \in P(V).$$

Let $\tilde{N}$ be the intersection of all kernels of homomorphisms $\tilde{\pi}'$ corresponding to admissible representations (V, π'). The *Kac-Moody group* is defined as $G := G(A) :=$

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$\tilde{G}/\tilde{N}$ and, due to its definition, $\tilde{\pi}\colon \tilde{G} \to \mathrm{GL}(V)$ factors to a representation $\Pi\colon G \to \mathrm{GL}(V)$, often also denoted by π.

For $\alpha \in \Delta_{\mathrm{re}}$ we get, by composing the injection of $\mathbf{g}_\alpha$ into $\tilde{G}$ with the projection onto G, an injective homomorphism $\exp\colon \mathbf{g}_\alpha \to G$. Its image U_α is called the *root group* belonging to α. Similarly, we get an injective homomorphism $t\colon H \otimes_{\mathbb{Z}} \mathbb{C}^\times \to G$, its image being denoted by T.

The torus T can be described by the exponential map $\exp\colon \mathbf{h} \to T$, in explicit terms: $\exp(\sum_{i=1}^{2n-l} c_i h_i) := \prod_{i=1}^{2n-l} t_{h_i}(e^{c_i})$, $c_i \in \mathbb{C}^\times$.

The derived group G', which is identical with the Kac-Moody group as defined in [K,P1], is generated by the root groups U_α, $\alpha \in \Delta_{\mathrm{re}}$, and we have $G = G' \rtimes T_{\mathrm{rest}}$, where T_{rest} is the subtorus of T generated by the elements $t_{h_i}(s)$, $i = n+1, \ldots, 2n-l$, $s \in \mathbb{C}^\times$.

The *compact involution* $*\colon G \to G$ is the involutive antiisomorphism determined by:

$$\exp(x_\alpha)^* := \exp(x_\alpha^*) \quad (x_\alpha \in \mathbf{g}_\alpha, \alpha \in \Delta_{\mathrm{re}})$$

$$t_h(s)^* := t_{h^*}(s) \quad (h \in H, s \in \mathbb{C}^\times).$$

The Kac-Moody group has the following important structural properties:

Let $\alpha \in \Delta_{\mathrm{re}}^+$ and $x_\alpha \in \mathbf{g}_\alpha$, $x_{-\alpha} \in \mathbf{g}_{-\alpha}$ such that $[x_\alpha, x_{-\alpha}] = h_\alpha$. There exists an injective homomorphism of groups $\phi_\alpha\colon\ \mathrm{SL}(2,\mathbb{C}) \to G$ with

$$\phi_\alpha \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} := \exp(sx_\alpha), \quad \phi_\alpha \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} := \exp(sx_{-\alpha}), \quad (s \in \mathbb{C}^\times).$$

Denote by N the subgroup generated by T and $n_\alpha := \phi_\alpha \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\alpha \in \Delta_{\mathrm{re}}$. Let $B^\pm$ be the subgroups generated by T and U_α, $\alpha \in \Delta_{\mathrm{re}}^\pm$, and let $U^\pm$ be the subgroups generated by U_α, $\alpha \in \Delta_{\mathrm{re}}^\pm$.

Then $\left(G, (U_\alpha)_{\alpha \in \Delta_{\mathrm{re}}}, T\right)$ is a root groups data system, leading to the twinned BN-pairs $B^\pm$, N, which have the property $B^+ \cap B^- = B^+ \cap N = B^- \cap N = T$. The common Coxeter group N/T can be identified with the Weyl group $\mathcal{W}$, the isomorphism $\kappa\colon N/T \to \mathcal{W}$ being given by $\kappa(n_\alpha T) := \sigma_\alpha$, $\alpha \in \Delta_{\mathrm{re}}$.

In particular, the twinned BN-pairs lead to the *Bruhat-* and *Birkhoff decompositions* $G = \dot{\bigcup}_{\sigma \in \mathcal{W}} B^\pm \sigma B^\pm$.

Denote an arbitrary element $n \in N$ with $\kappa(nT) = \sigma \in \mathcal{W}$ by n_σ. If (V, π) is an admissible $\mathbf{g}$-module, then its set of weights $P(V)$ is $\mathcal{W}$-invariant, and $n_\sigma V_\lambda = V_{\sigma\lambda}$, $\lambda \in P(V)$.

Define $T^+ := \exp(\mathbf{h}_{\mathbb{R}})$, and the *compact form* $K := \{g \in G \mid g^* = g^{-1}\}$. We have the *Iwasawa decompositions* $G = KT^+U^\pm = U^\pm T^+ K$.

Properties of the Admissible Irreducible Highest Weight Representations For $\Lambda \in P^+ := \overline{C} \cap P$ there exists a positive definite hermitian form $\langle\langle\ \mid\ \rangle\rangle : L(\Lambda) \times L(\Lambda) \to \mathbb{C}$ which is contravariant, *i.e.*, $\langle\langle v \mid xw \rangle\rangle = \langle\langle x^* v \mid w \rangle\rangle$ for all $v, w \in L(\Lambda)$, $x \in \mathbf{g}$ resp. $x \in G$. (We assume $\langle\langle\ \mid\ \rangle\rangle$ to be antilinear in the first entry.) This form is uniquely determined up to a nonzero positive multiplicative scalar.

For the properties of the set of weights $P(\Lambda)$ of $L(\Lambda)$ we refer to the book [K, Sections 11.1, 11.2 and 11.3].

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2 Bounded Elements and Trace Class Elements in Kac-Moody Groups

To define the Steinberg map in the classical case, the characters of the fundamental irreducible highest weight representations are needed. For a Kac-Moody group the irreducible highest weight representations are in general infinite dimensional. The corresponding characters can't be realized as functions on the whole Kac-Moody group, but nevertheless on a part of the Kac-Moody group:

Call an element $g \in G$ bounded, if for all $\Lambda \in P^+$ the linear map $\pi_\Lambda(g)$ is bounded with respect to a contravariant, positive definite hermitian form on $L(\Lambda)$. For $M \subseteq G$ denote by M^b the set of bounded elements of M.

Call an element $g \in G$ of trace class, if for all $\Lambda \in P^+$ the linear map $\pi_\Lambda(g)$ can be extended to a trace class operator, also denoted by $\pi_\Lambda(g)$, on the Hilbert space completion of $L(\Lambda)$. For $M \subseteq G$ denote by M^{tr} the set of all trace class elements of M.

G^{tr} is a semigroup, G^b is a monoid, both related by $G^b G^{\mathrm{tr}} \subseteq G^{\mathrm{tr}} \supseteq G^{\mathrm{tr}} G^b$.

Fix an element $\Lambda \in P^+$. The G-character Tr_Λ of the irreducible highest weight representation $L(\Lambda)$ can be defined on G^{tr} by:

$$\mathrm{Tr}_\Lambda(g) := \mathrm{Tr}\big(\pi_\Lambda(g)\big), \quad g \in G^{\mathrm{tr}}.$$

Here Tr denotes the trace function of the trace class operators on the Hilbert space completion of $L(\Lambda)$. Obviously G^{tr} and the characters $\mathrm{Tr}_\Lambda \colon G^{\mathrm{tr}} \to \mathbb{C}$, $\Lambda \in P^+$, are $*$-invariant, and invariant under conjugation by elements of the compact form K.

There is a reasonable way to extend these characters to a larger subset $G^{\text{tr-alg}}$ of G. Fix $\Lambda \in P^+$. For $\lambda \in P(\Lambda)$ denote by $i_\lambda \colon L(\Lambda)_\lambda \to L(\Lambda)$ the injection, and by $p_\lambda \colon L(\Lambda) \to L(\Lambda)_\lambda$ the projection corresponding to the weight space decomposition of $L(\Lambda)$. For $g \in G$ let $(\mathrm{Tr}_\Lambda)_\lambda(g)$ be the trace of the linear map $p_\lambda \circ \pi_\lambda(g) \circ i_\lambda \colon L(\Lambda)_\lambda \to L(\Lambda)_\lambda$. Denote by $G^{\text{tr-alg}}$ the set of elements $g \in G$, such that for all $\Lambda \in P^+$ the series

$$\sum_{\lambda \in P(\Lambda)} (\mathrm{Tr}_\Lambda)_\lambda(g) \tag{1}$$

is absolutely convergent. Obviously, $G^{\text{tr-alg}}$ and the G-characters $\mathrm{Tr}_\Lambda \colon G^{\text{tr-alg}} \to \mathbb{C}$ defined by the series (1) are $*$-invariant, and invariant under conjugation by elements of N.

In [Sl 2], [Sl 3], P. Slodowy posed the question: Find a conjugation invariant part of $G^{\text{tr-alg}}$, on which the characters are realized as conjugation invariant convergent functions. He also pointed out, that parts of G^{tr} may be good candidates.

In the affine case, G. Brüchert found a conjugation invariant part of G^{tr}, on which the characters are conjugation invariant functions, compare [B, Theorem 1, Theorem 2, Theorem 3 a)]. In [M, Proposition 4.1], it was shown that this part equals G^{tr}:

$$G^{\mathrm{tr}} = G'(T_{\mathrm{rest}})^{\mathrm{tr}} = U^{\pm} N^{\mathrm{tr}} U^{\pm} = K (T^+)^{\mathrm{tr}} U^{\pm}.$$

In this section we give some results about bounded elements and trace class elements for a symmetrizable, indecomposable Kac-Moody group of general type. They

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are used for our investigation of the Steinberg cross section. We also show that, in contrast to the affine case, in the indefinite case there is no reasonable part of G^{tr}, which is invariant under G-conjugation.

Let c be the edge of the Tits cone X, *i.e.*, $c = \{\lambda \in \mathbf{h}_{\mathbb{R}}^* \mid \lambda(h_i) = 0$ for all $i \in I\}$. Our first aim is to determine the norms of the elements of the torus T on the irreducible highest weight representations $L(\Lambda)$, $\Lambda \in P^+$. This is trivial for the 1-dimensional irreducible highest weight representations $L(\Lambda)$, $\Lambda \in c \cap P^+$, and we may restrict to $\Lambda \in P^+ \setminus c$.

Proposition 2.1 *Let A be indecomposable and $h \in \mathbf{h}$. For $\Lambda \in P^+ \setminus c$ we have:*

$$\big\| \pi_\Lambda\big(\exp(h)\big) \big\| = \begin{cases} e^{\Lambda(h_{\mathrm{re}})} & \text{for } h_{\mathrm{re}} \in X^\vee \\ \infty & \text{for } h_{\mathrm{re}} \notin \overline{X^\vee}. \end{cases}$$

In particular $\{\exp(h) \mid h_{\mathrm{re}} \in X^\vee\} \subseteq T^b \subseteq \{\exp(h) \mid h_{\mathrm{re}} \in \overline{X^\vee}\}$.

Proof $L(\Lambda)$ is the orthogonal sum of the weight spaces, on which the elements of the torus act diagonally. Therefore:

$$\big\| \pi_\Lambda\big(\exp(h)\big) \big\| = \sup\{e^{\lambda(h_{\mathrm{re}})} \mid \lambda \in P(\Lambda)\}.$$

Let $h_{\mathrm{re}} \in X^\vee$. Since $P(\Lambda)$ is $\mathcal{W}$-invariant, the value of this norm doesn't depend on the $\mathcal{W}$-orbit of h, and we may assume $h_{\mathrm{re}} \in \overline{C^\vee}$. Because of $\Lambda \in P(\Lambda) \subseteq \Lambda - Q_0^+$ we get $\big\| \pi_\Lambda\big(\exp(h)\big) \big\| = e^{\Lambda(h_{\mathrm{re}})}$.

Let $h_{\mathrm{re}} \notin \overline{X^\vee}$. Due to [K, Proposition 5.8], there exists a root $\alpha \in \Delta_{\mathrm{im}}^+$ such that $\alpha(h_{\mathrm{re}}) < 0$. First we show that there exists an element $\lambda_0 \in P(\Lambda)$ such that $\big(\lambda_0 \mid \alpha\big) > 0$: In the affine case, due to the description of the positive imaginary roots, we have $\mathrm{supp}(\alpha) = I$. We can take $\lambda_0 = \Lambda$, because there exists an element $i \in I$ such that $(\Lambda \mid \alpha_i) > 0$. In the indefinite case, there exists a root $\beta \in \Delta_{\mathrm{im}}^+ \cap (-C)$ such that $\mathrm{supp}(\beta) = I$, compare [K, Theorem 5.6 c)]. We can take $\lambda_0 = \Lambda - \beta$, because we have $\Lambda - \beta \in C$, and due to [K, Proposition 11.2 a)], $\Lambda - \beta \in P(\Lambda)$.

Let $k \in \mathbb{N}_0$. Due to [K, Corollary 11.9], we get $\lambda_0 - k\alpha \in P(\Lambda)$, and therefore

$$\big\| \pi_\Lambda\big(\exp(h)\big) \big\| \geq e^{\lambda_0(h_{\mathrm{re}})} (\underbrace{e^{-\alpha(h_{\mathrm{re}})}}_{>1})^k. \qquad \blacksquare$$

Remark Let $\Lambda \in P^+ \setminus c$.

1) If A is of affine type, the due to [B, Lemma 3], we have:

$$\big\| \pi_\Lambda\big(\exp(h)\big) \big\| < \infty \iff h_{\mathrm{re}} \in X^\vee.$$

2) If A is of hyperbolic type, then:

$$\big\| \pi_\Lambda\big(\exp(h)\big) \big\| < \infty \iff h_{\mathrm{re}} \in \overline{X^\vee}.$$

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The direction "$\Leftarrow$" can be proved as follows: Using [K, exercise 5.15], we find that $\overline{X^\vee}$ and $-\overline{X}$ are dual convex cones. Due to $P(\Lambda) \subseteq X$ we have $\left\| \pi_\Lambda\big(\exp(h)\big) \right\| \le e^0 = 1$. ■

The domain of absolute convergence of the series (1) on the torus $T = \exp(\mathbf{h})$ has been studied by Kac and Peterson. Their results are presented in [K, Sections 10.6 and 11.10]: For A indecomposable and $\Lambda \in P^+ \setminus c$ the set

$$\Big\{ h \in \mathbf{h} \;\Big|\; \sum_{\lambda \in P(\Lambda)} \dim\big(L(\Lambda)_\lambda\big)\, |e^{\alpha(h)}| < \infty \Big\}$$

is independent of Λ, and it coincides with

$$Y := \Big\{ h \in \mathbf{h} \;\Big|\; \sum_{\alpha \in \Delta^+} \dim(\mathbf{g}_\alpha) |e^{-\alpha(h)}| < \infty \Big\}.$$

Y is an open, convex, and $\mathcal{W}$-invariant set, which for every $x \in (X^\vee)^\circ + i\mathbf{h}_\mathbb{R}$ contains tx for sufficiently large $t \in \mathbb{R}^+$. In the affine case $Y = (X^\vee)^\circ + i\mathbf{h}_\mathbb{R} = \{h \in \mathbf{h} \mid \delta(h) > 0\}$, where δ denotes the minimal positive imaginary root.

The following Proposition is an easy consequence:

Proposition 2.2 *Let A be indecomposable and $h \in \mathbf{h}$. For $\Lambda \in P^+ \setminus c$ we can extend $\pi_\Lambda\big(\exp(h)\big)$ to a trace class operator on the Hilbert space completion of $L(\Lambda)$ if and only if $h \in Y$.*

In particular $T^{\mathrm{tr}} = \{\exp(h) \mid h \in Y\}$.

The norms of elements of the groups TU_α, $\alpha \in \Delta_{\mathrm{re}}$, can be traced back to norms of torus elements:

Proposition 2.3 *Let $\Lambda \in P^+$. Let $\alpha \in \Delta_{\mathrm{re}}$ and choose an element $x_\alpha \in \mathbf{g}_\alpha$, such that $[x_\alpha, x_\alpha^*] = h_\alpha$. For $h \in \mathbf{h}$, $s \in \mathbb{C}$ we have:*

$$\left\| \pi_\Lambda\big(\exp(h)\exp(sx_\alpha)\big) \right\| = \begin{cases} \left\| \pi_\Lambda\Big(\exp\Big(h + \ln\big(c_+(h,s)\big)\, h_\alpha \Big) \Big) \right\| & \text{for } \alpha(h_{\mathrm{re}}) \ge 0 \\ \left\| \pi_\Lambda\Big(\exp\Big(h - \ln\big(c_-(h,s)\big)\, h_\alpha \Big) \Big) \right\| & \text{for } \alpha(h_{\mathrm{re}}) \le 0 \end{cases}$$

where

$$c_+(h,s) := \sqrt{\frac{1 + e^{-2\alpha(h_{\mathrm{re}})} + s\bar{s}}{2} + \sqrt{\left(\frac{1 + e^{-2\alpha(h_{\mathrm{re}})} + s\bar{s}}{2}\right)^2 - e^{-2\alpha(h_{\mathrm{re}})}}}$$

$$c_-(h,s) := \sqrt{\frac{1 + e^{2\alpha(h_{\mathrm{re}})}(1 + s\bar{s})}{2} + \sqrt{\left(\frac{1 + e^{2\alpha(h_{\mathrm{re}})}(1 + s\bar{s})}{2}\right)^2 - e^{2\alpha(h_{\mathrm{re}})}}}.$$

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Remark Fix $h \in \mathbf{h}$. Then $c_\pm(h,s)$ is increasing in $|s|$, $c_\pm(h,0) = 1$, and $c_\pm(h,s) \to \infty$ for $|s| \to \infty$.

Proof Let $L(\Lambda) = \bigoplus_{j\in J} \operatorname{Str}_j$ be a decomposition of the TG_α-module $L(\Lambda)$ into irreducible finite dimensional TG_α-submodules, such that Str_j and Str_k are orthogonal for $j \neq k$. For $g \in TG_\alpha$ denote by $\pi_j(g)$ the linear map on Str_j, which is induced by $\pi_\Lambda(g)$. We have:

$$\|\pi_\Lambda(g)\| = \sup\{\|\pi_j(g)\| \mid j \in J\}. \tag{2}$$

Our next aim is to calculate $\left\| \pi_j\big(\exp(h)\exp(sx_\alpha)\big) \right\|$ for a TG_α-module Str_j of dimension $m+1$. Let $v_{\lambda-k\alpha} \in \operatorname{Str}_j \cap L(\Lambda)_{\lambda-k\alpha}$, $k = 0,1,\dots,m$, be an orthonormal base of Str_j, such that $x_\alpha^* v_{\lambda-k\alpha} = \sqrt{(k+1)(m-k)}\, v_{\lambda-(k+1)\alpha}$, $k = 0,1,\dots,m-1$. Note that λ and m are related by $m = \lambda(h_\alpha)$. Define a $\mathrm{GL}(2,\mathbb{C})$-action on the linear space $L(m) := \{p \in \mathbb{C}[x,y] \mid \deg(p) = m\}$ by:

$$(Ap)(x,y) = p\big((x,y)A\big) \qquad p \in L(m), \quad A \in \mathrm{GL}(2,\mathbb{C}).$$

Define a positive definite hermitian form on $L(m)$, by requiring the base $\tilde v_k := \sqrt{\binom{m}{k}}\, x^{m-k} y^k$, $k = 0,1,\dots,m$, to be orthonormal. This form is contravariant with respect to the conjugate transpose of $\mathrm{GL}(2,\mathbb{C})$.

We get an injective homomorphism of groups $\psi\colon \pi_j(TG_\alpha) \to \mathrm{GL}(2,\mathbb{C})$, the involution on $\pi_j(TG_\alpha)$, which is induced by the compact involution, compatible with the conjugate transpose, by requiring:

$$\psi\Big(\pi_j\big(\exp(sx_\alpha)\big)\Big) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad \psi\Big(\pi_j\big(\exp(sx_\alpha^*)\big)\Big) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \quad (s \in \mathbb{C})$$

$$\psi\Big(\pi_j\big(\exp(h)\big)\Big) = e^{\frac{\lambda(h)}{m}} \begin{pmatrix} 1 & 0 \\ 0 & e^{-\alpha(h)} \end{pmatrix} \quad (h \in \mathbf{h}).$$

The linear map $\eta\colon \operatorname{Str}_j \to L(m)$, which maps $v_{\lambda-k\alpha}$ to $\tilde v_k$, $k = 0,1,\dots,m$, is an isometry. Furthermore we have $\psi\big(\pi_j(g)\big)\,\eta(v) = \eta\big(\pi_j(g)v\big)$ for all $g \in TG_\alpha$ and $v \in \operatorname{Str}_j$.

The norm of $\pi_j\big(\exp(h)\exp(sx_\alpha)\big)$ equals the square root of the biggest eigenvalue of the positive definite hermitian map $\pi_j\Big(\big(\exp(h)\exp(sx_\alpha)\big)^* \exp(h)\exp(sx_\alpha)\Big)$. Pulling back to $L(m)$, this map corresponds to the positive definite hermitian matrix

$$\begin{pmatrix} r_1 & r_1 s \\ r_1 \bar s & r_2 + r_1 s\bar s \end{pmatrix},$$

where $r_1 := e^{\frac{2\lambda(h_{\mathrm{re}})}{m}}$ and $r_2 := e^{\frac{2\lambda(h_{\mathrm{re}})}{m} - 2\alpha(h_{\mathrm{re}})}$. Using the Cartan decomposition of $\mathrm{GL}(2,\mathbb{C})$, it is easy to see, that this eigenvalue is the m-th power of the biggest eigenvalue of this matrix. We find:

$$\begin{aligned} &\left\| \pi_j\big(\exp(h)\exp(sx_\alpha)\big) \right\| \\ &= e^{\lambda(h_{\mathrm{re}})} \left(\frac{1 + e^{-2\alpha(h_{\mathrm{re}})} + s\bar s}{2} + \sqrt{\left(\frac{1 + e^{-2\alpha(h_{\mathrm{re}})} + s\bar s}{2} \right)^2 - e^{-2\alpha(h_{\mathrm{re}})}} \right)^{\frac{\lambda(h_\alpha)}{2}}. \end{aligned} \tag{3}$$

In particular for $s = 0$ we get:

$$\left\| \pi_j\big(\exp(h)\big) \right\| = \begin{cases} e^{\lambda(h_{\mathrm{re}})} & \text{for } \alpha(h_{\mathrm{re}}) \geq 0 \\ e^{(\lambda - m\alpha)(h_{\mathrm{re}})} & \text{for } \alpha(h_{\mathrm{re}}) \leq 0. \end{cases} \tag{4}$$

Let $\alpha(h_{\mathrm{re}}) \geq 0$. Then also $\alpha\Big(h_{\mathrm{re}} + \Big(\ln\big(c_+(h,s)\big)\Big) h_\alpha \Big) \geq 0$, and we get from (3) and (4):

$$\left\| \pi_j\big(\exp(h)\exp(sx_\alpha)\big) \right\| = \left\| \pi_j\Big(\exp\Big(h + \ln\big(c_+(h,s)\big) h_\alpha \Big) \Big) \right\|.$$

Using (2) we get the first equation of the proposition. The case $\alpha(h_{\mathrm{re}}) \leq 0$ is proved in a similar way. ∎

Proposition 2.4 *Let A be of indefinite type. Then for every $t \in T^{\mathrm{tr}}$ there exists an element $g \in G$, such that $gtg^{-1} \notin G^b$.*

Proof Let $t = \exp(h)$ with $h \in Y$. Because of $\sum_{\alpha\in\Delta^+}\big(\dim(\mathbf{g}_\alpha)\big)\,|e^{-\alpha(h)}| < \infty$, and $|\Delta^+_{\mathrm{re}}| = \infty$, there exists a real root $\alpha \in \Delta_{\mathrm{re}}$, such that $\alpha(h_{\mathrm{re}}) \neq 0$. We may assume $\alpha(h_{\mathrm{re}}) > 0$, otherwise replace α by $-\alpha$. Let $x_\alpha \in \mathbf{g}_\alpha$ such that $[x_\alpha, x^*_\alpha] = h_\alpha$, and let $s \in \mathbb{C}$. We have

$$\exp(sx_\alpha)\exp(h)\exp(-sx_\alpha) = \exp(h)\exp\big(s(e^{-\alpha(h)} - 1)x_\alpha\big),$$

and due to the last proposition we get

$$\left\| \pi_\Lambda\big(\exp(sx_\alpha)\exp(h)\exp(-sx_\alpha)\big) \right\| = \left\| \pi_\Lambda\Big(\exp\Big(h + \ln\big(c_+(h,t)\big) h_\alpha \Big) \Big) \right\|,$$

where $t := s(e^{-\alpha(h)} - 1)$.

Suppose this norm exists for all s. Then due to the last remark and Proposition 2.1 we have $h_{\mathrm{re}} + \mathbb{R}^+_0 h_\alpha \subseteq \overline{X^\vee}$. Because $\overline{X^\vee}$ is a closed convex cone, it contains the sequence $\frac{h_{\mathrm{re}}}{m} + h_\alpha$, $m \in \mathbb{N}$, and also the limit h_α of this sequence. Since $\overline{X^\vee}$ is σ_α-invariant it contains $\pm h_\alpha$.

Due to [K, Theorem 5.6], there exists a root $\beta \in \Delta^+_{\mathrm{im}} \cap (-C)$. Due to [K, Proposition 5.8], we have $\overline{X^\vee} = \{h \in \mathbf{h}_\mathbb{R} \mid \gamma(h) \geq 0 \text{ for all } \gamma \in \Delta^+_{\mathrm{im}}\}$. Therefore $\beta \in -C$ is fixed by σ_α, which contradicts $\sigma_\alpha \neq id$. ∎

3 The Steinberg Map and the Steinberg Cross Section

To cut short the notation, set:

$$\text{æ}_i(s) := \exp(se_i) \quad i = 1, \ldots, n, \quad s \in \mathbb{C}$$

$$t_i(s) := t_{h_i}(s) \quad i = 1, \ldots, 2n - l, \quad s \in \mathbb{C}^\times$$

$$n_i := n_{\alpha_i} = \exp(-e_i)\exp(f_i)\exp(-e_i) \quad i = 1, \ldots, n.$$

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Generalizing the definitions of the classical and affine case, define the Steinberg map

$$\chi := (\mathrm{Tr}_{\Lambda_1}, \dots, \mathrm{Tr}_{\Lambda_{2n-l}}) \colon G^{\text{alg-tr}} \to \mathbb{C}^{2n-l},$$

define the Steinberg cross section

$$C := U_1 n_1 \cdots U_n n_n T_{\text{rest}},$$

and a parametrization ω of C:

$$\begin{aligned}\omega \colon \mathbb{C}^n \times (\mathbb{C}^\times)^{n-l} &\to C\\ (s_1, \dots, s_{2n-l}) &\mapsto æ_1(s_1) n_1 \cdots æ_n(s_n) n_n t_{n+1}(s_{n+1}) \cdots t_{2n-l}(s_{2n-l}).\end{aligned}$$

In our investigation of the Steinberg map, *i.e.*, the restriction of the adjoint quotient to C, we shall have to study the behaviour of C under conjugation with elements of T. For that purpose, as well as for the functional identity proved later, it is useful to introduce the following abstract T-action on C. Recall the parametrization of the torus T:

$$\begin{aligned}\psi \colon (\mathbb{C}^\times)^{2n-l} &\to T\\ (s_1, \dots, s_{2n-l}) &\mapsto t_1(s_1) \cdots t_{2n-l}(s_{2n-l}).\end{aligned}$$

Multiplying corresponding components, the torus $(\mathbb{C}^\times)^{2n-l}$ acts on $\mathbb{C}^{2n-l}$, and via ψ^{-1} we get an action of T on $\mathbb{C}^{2n-l}$, which we denote by a central dot. This action can be restricted to the T-invariant set $\mathbb{C}^n \times (\mathbb{C}^\times)^{n-l}$. Via ω we get an action of T on the Steinberg cross section.

Next we give a useful description of this action. For an element $\sigma_{i_t} \cdots \sigma_{i_1}$ of the Weyl group (not necessarily reduced) set:

$$\beta_1 := \alpha_{i_1}, \quad \beta_k := \sigma_{i_1} \cdots \sigma_{i_{k-1}} \alpha_{i_k} \quad (k = 2, \dots, t).$$

The following formula, which can be proved by induction over t, generalizes the formula for a reflection at a simple root:

$$\sigma_{i_t} \cdots \sigma_{i_1} h = h - \sum_{k=1}^{t} \beta_k(h) h_{i_k}. \tag{5}$$

Define the linear map

$$\begin{aligned}\phi \colon \mathbf{h} &\to \mathbf{h}'\\ h &\mapsto h - \operatorname{cox} h\end{aligned}$$

where $\operatorname{cox} := \sigma_n \sigma_{n-1} \cdots \sigma_1$ is a Coxeter element of the Weyl group. We have $\beta_1 = \alpha_1$ and, for $i \neq 1$, β_i is of the form α_i + linear terms in $\alpha_1, \dots, \alpha_{i-1}$, in particular $\beta_1, \dots, \beta_n$ are linear independent. Due to (5), the kernel of ϕ equals the center $z(\mathbf{g})$, and ϕ is surjective.

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Lemma 3.1 *Let $h \in \mathbf{h}$ and $h_{\text{rest}} \in \mathbf{h}_{\text{rest}}$. We have:*

$$\omega\big(\exp(\phi(h) + h_{\text{rest}}) \cdot s\big) = \exp(h)\omega(s)\exp(-h)\exp\big(\phi(h) + h_{\text{rest}}\big)\,.$$

Proof Let $\tilde{h} \in \mathbf{h}$, $\sigma \in \mathcal{W}$, $x_\alpha \in \mathbf{g}_\alpha$ where $\alpha \in \Delta_{\text{re}}$, and $s, c \in \mathbb{C}$. We find by checking on the admissible representations:

$$\exp(\tilde{h})n_\sigma = n_\sigma \exp(\sigma^{-1}\tilde{h}) \tag{6}$$

$$\exp(\tilde{h})\exp(x_\alpha) = \exp(e^{\alpha(\tilde{h})}x_\alpha)\exp(\tilde{h}) \tag{7}$$

$$t_{\tilde{h}}(s) = t_{\tilde{h}}(se^c)\exp(-c\tilde{h}). \tag{8}$$

In $\exp(h)æ_1(s_1)n_1\cdots æ_n(s_n)n_n t_{n+1}(s_{n+1})\cdots t_{2n-l}(s_{2n-l})\exp(-h)$ we can move the torus element $\exp(h)$ from the left successively to the right, by using equations (7), (6). We get:

$$æ_1(s_1e^{\beta_1(h)})n_1\cdots æ_n(s_ne^{\beta_n(h)})n_n t_{n+1}(s_{n+1})\cdots t_{2n-l}(s_{2n-l})\exp(\operatorname{cox} h - h).$$

For $j \in \{n+1, \ldots, 2n-l\}$ let $c_j \in \mathbb{C}$. Due to (8) we have $t_i(s_i) = t_i(s_ie^{c_i})\exp(-c_ih_i)$. Inserting in the last expression proves the Lemma. ■

In particular we are interested in the action of the center $Z(G)$ of the Kac-Moody group on the Steinberg cross section. The center $Z(G)$ is contained in T. Its identity component is related to the center

$$\begin{aligned} z(\mathbf{g}) &= \{h \in \mathbf{h} \mid \alpha_i(h) = 0 \text{ for all } i\} \\ &= \{a_1h_1 + \cdots a_nh_n \in \mathbf{h} \mid (a_1, \ldots, a_n)A = 0\} \end{aligned}$$

of the Kac-Moody algebra by $Z(G)^\circ = \exp\big(z(\mathbf{g})\big)$.

First we study the trace functions $\operatorname{Tr}_\Lambda \colon G^{\text{alg-tr}} \to \mathbb{C}$, $\Lambda \in P^+$, on the Steinberg cross section C in a neighborhood of $n_1n_2\cdots n_n \in C$. For an element $\Lambda \in P^+$ let $e_\Lambda \colon T \to \mathbb{C}^\times$ be its T-character, which is defined by:

$$e_\Lambda\big(t_1(s_1)\cdots t_{2n-l}(s_{2n-l})\big) := (s_1)^{\Lambda(h_1)}\cdots(s_{2n-l})^{\Lambda(h_{2n-l})} \quad (s_i \in \mathbb{C}^\times).$$

Theorem 3.2 *Let A be of indefinite type. There exists an open, $Z(G)$-invariant neighborhood $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^{n-l}$ of $(0, \ldots, 0) \times (1, \ldots, 1)$, such that for every $\Lambda \in P^+$ the map $\operatorname{Tr}_\Lambda$ is defined on $\omega(V)$, and*

$$\operatorname{Tr}_\Lambda \circ \omega \colon V \to \mathbb{C}$$

gives an analytical map satisfying the functional identity:

$$(\operatorname{Tr}_\Lambda \circ \omega)(t \cdot s) = e_\Lambda(t)(\operatorname{Tr}_\Lambda \circ \omega)(s) \quad \big(t \in Z(G), s \in V\big)\,. \tag{9}$$

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Proof 1) We first show that there exists an open neighborhood $\tilde{V} \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^{n-l}$ of $(0, \ldots, 0) \times (1, \ldots, 1)$, such that $\omega(\tilde{V}) \subseteq G^{\text{tr-alg}}$:

The transpose of A is of indefinite type. Due to [K, Theorem 4.3], there exists an element $h' \in C^\vee \cap \mathbf{h}'$. Due to [K, Proposition 3.12], we have $h' \in C^\vee \subseteq (X^\vee)^\circ$. Therefore for a sufficiently large $t \in \mathbb{R}^+$ we get $y := \sum_{j=1}^n c_j h_j := th' \in Y$. Recall that Y is open, and 0 is an accumulation point of $(X^\vee)^\circ + i\mathbf{h}_\mathbb{R}$. Therefore there exists an element $\tilde{h} \in (X^\vee)^\circ + i\mathbf{h}_\mathbb{R}$ such that $y - n\tilde{h} \in Y$. Choose an element $h \in \mathbf{h}$ such that $\phi(h) = y$. Using the last Lemma we get for $s \in \mathbb{C}^n \times (\mathbb{C}^\times)^{n-l}$:

$$\begin{aligned}
&\exp(-h)\omega(s)\exp(h)\\
&\quad = \omega\big(\exp(-y)\cdot s\big)\ \exp(y)\\
&\quad = \underbrace{\omega(e^{-c_1}s_1, \ldots, e^{-c_n}s_n, 1, \ldots, 1)\exp(n\tilde{h})}_{(a)}\\
&\qquad\quad \cdot \underbrace{\exp(y - n\tilde{h})\omega(0, \ldots, 0, s_{n+1}, \ldots, s_{2n-l})}_{(b)}.
\end{aligned}$$

Using the equations (6) and (7) it is easy to see, that the factor (a) can be written in the form:

$$\prod_{j=1}^{n} \exp(\sigma_j \cdots \sigma_n \tilde{h}) æ_j(s_j e^{-c_j - j\alpha_j(\sigma_j \cdots \sigma_n \tilde{h})}) n_j.$$

We have $\sigma_j \cdots \sigma_n \tilde{h} \in (X^\vee)^\circ + i\mathbf{h}_\mathbb{R}$, and $(X^\vee)^\circ + i\mathbf{h}_\mathbb{R}$ is open. Due to Propositions 2.1 and 2.3 the factor (a) is an element of G^b for all tuples $(s_1, \ldots, s_n)$, such that $|s_j|$ is sufficiently small, $j = 1, \ldots, n$. The factor (b) can be written in the form:

$$\exp\Big(y - n\tilde{h} + \sum_{j=n+1}^{2n-l} (\ln s_j) h_j\Big)$$

Y is open, and we have $Y + i\mathbf{h}_\mathbb{R} = Y$. Due to Proposition 2.2 this is an element of T^{tr} for all tuples $(s_{n+1}, \ldots, s_{2n-l})$, such that $\big||s_j| - 1\big|$ is sufficiently small, $j = n+1, \ldots, 2n-l$.

Thus, we know that the element $\exp(-h)\omega(s)\exp(h)$ is in G^{tr}, it follows that $\omega(s) \in G^{\text{tr-alg}}$.

2) Let $s \in \mathbb{C}^n \times (\mathbb{C}^\times)^{n-l}$. For $t \in Z(G) \subseteq T$ there exist elements $h \in \mathbf{h}$ and $h_{\text{rest}} \in \mathbf{h}_{\text{rest}}$ such that $t = \exp\big(\phi(h) + h_{\text{rest}}\big)$, and due to the last lemma we have:

$$\omega(t \cdot s) = \exp(h)\omega(s)\exp(-h)t.$$

The element t acts on $L(\Lambda)$ as multiplication by the scalar $e_\Lambda(t)$. Therefore $V := Z(G)\tilde{V}$ is an open neighborhood of $(0, \ldots, 0) \times (1, \ldots, 1)$ such that $\omega(V) \subseteq G^{\text{tr-alg}}$, and the functional identity of the theorem is valid.

3) Let $s \in V$. In the same way as in the classical and affine case, compare [St], [B], we can determine $(\mathrm{Tr}_\Lambda)_\mu\big(\omega(s)\big)$, $\mu \in P(\Lambda)$. Summing up over μ we find, that

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$(\mathrm{Tr}_\Lambda \circ \omega)(s)$ is given by the following absolutely convergent series:

$$\sum_{\mu \in P(\Lambda) \cap P^+} c_\mu s_1^{\mu(h_1)} \cdots s_{2n-l}^{\mu(h_{2n-l})} \tag{10}$$

where c_μ is the trace of the linear map $\prod_{j=1}^n \frac{\pi_\Lambda((e_j)^{\mu(h_j)})}{\mu(h_j)!} \pi_\Lambda(n_j) \colon L(\Lambda)_\mu \to L(\Lambda)_\mu$. This series is a power series in $s_1, \ldots, s_n$, and a Laurent series in $s_{n+1}, \ldots, s_{2n-l}$. Due to Hartog's theorem it defines an analytical map on V. ∎

Next we study the Steinberg map $\chi \colon G^{\text{alg-tr}} \to \mathbb{C}^{2n-l}$ on the Steinberg cross section C in a neighborhood of $n_1 n_2 \cdots n_n \in C$. We equip the target space $\mathbb{C}^{2n-l}$ of χ with the $Z(G)$-action, introduced at the beginning of this section.

Theorem 3.3 *Let A be of indefinite type. There exists an open, $Z(G)$-invariant neighborhood $W \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^{n-l}$ of $(0, \ldots, 0) \times (1, \ldots, 1)$, such that the Steinberg map χ is defined on $\omega(W)$, and*

$$\chi \circ \omega \colon W \to \mathbb{C}^{2n-l}$$

gives an analytical regular $Z(G)$-equivariant map.

Proof Set $s_0 := (0, \ldots, 0) \times (1, \ldots, 1) \in \mathbb{C}^n \times (\mathbb{C}^\times)^{n-l}$ for short. Let V be a neighborhood of s_0 as given by the last theorem. The analytical map $\chi \circ \omega \colon V \to \mathbb{C}^{2n-l}$ is $Z(G)$-equivariant, because for $s \in V$ and $t = t_1(c_1) t_2(c_2) \cdots t_{2n-l}(c_{2n-l}) \in Z(G)$ we have due to (9):

$$(\mathrm{Tr}_{\Lambda_j} \circ \omega)(t \cdot s) = c_j (\mathrm{Tr}_{\Lambda_j} \circ \omega)(s) \quad j = 1, \ldots, 2n-l. \tag{11}$$

Denote the determinant of the Jacobian matrix of $\chi \circ \omega$ in s by $D(s)$. We have $D(t \cdot s) = D(s)$, which follows by taking partial derivates in (11):

$$\partial_i (\mathrm{Tr}_{\Lambda_j} \circ \omega)(t \cdot s) = \frac{c_j}{c_i} \partial_i (\mathrm{Tr}_{\Lambda_j} \circ \omega)(s) \quad i, j = 1, \ldots, 2n-l.$$

The map $\chi \circ \omega$ is regular in a point $s \in V$, if $D(s) \neq 0$. It is sufficient to show $D(s_0) \neq 0$, because then D doesn't vanish on an open neighborhood $\tilde{W} \subseteq V$ of s_0, and it also doesn't vanish on $W := Z(G)\tilde{W}$.

For $j = n+1, \ldots, 2n-l$ the module $L(\Lambda_j)$ is one-dimensional. We find:

$$\partial_i (\mathrm{Tr}_{\Lambda_j} \circ \omega)(s_0) = \delta_{ij} \quad \text{for } i = 1, \ldots, 2n-l,\ j = n+1, \ldots, 2n-l.$$

Therefore $D(s_0)$ is the determinant of the matrix $\left(\partial_i (\mathrm{Tr}_{\Lambda_j} \circ \omega)(s_0) \right)_{j,i=1,\ldots,n}$.

Define an order relation I by:

$$i \preceq j :\Longleftrightarrow \Lambda_i \in P(\Lambda_j).$$

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This relation is transitive, because due to [M, Proposition 1.1 b)], $\Lambda_i \in P(\Lambda_j)$ implies $P(\Lambda_i) \subseteq P(\Lambda_j)$. It is antisymmetric, because $\Lambda_i \in P(\Lambda_j)$ implies $\Lambda_i \leq \Lambda_j$. Now let $i, j \in I$. Due to (10), $\partial_i(\mathrm{Tr}_{\Lambda_j} \circ \omega)(s)$ is given by the series:

$$\sum_{\lambda \in P(\Lambda_j) \cap P^+, \lambda(h_i) \neq 0} c_\lambda^{(j)} \lambda(h_i) s_1^{\lambda(h_1)} \cdots s_{i-1}^{\lambda(h_{i-1})} s_i^{\lambda(h_i)-1} s_{i+1}^{\lambda(h_{i+1})} \cdots s_{2n-l}^{\lambda(h_{2n-l})}.$$

Inserting $s = s_0$ we find:

$$\partial_i(\mathrm{Tr}_{\Lambda_j} \circ \omega)(s_0) = \begin{cases} c_{\Lambda_i}^{(j)} & \text{if } i \preceq j \\ 0 & \text{else.} \end{cases} \tag{12}$$

For $M \subseteq I$ denote by $\max(M)$ the set of maximal elements of M. Define a sequence recursively by

$$I_1 := \max(I), \quad I_{k+1} := \max\bigl(I \setminus (I_1 \cup \cdots \cup I_k)\bigr) \quad (k \in \mathbb{N}),$$

and let m be the smallest positive integer such that $I_m \neq \varnothing$, $I_{m+1} = \varnothing$. We have $I = I_1 \,\dot\cup\, I_2 \,\dot\cup\, \cdots \,\dot\cup\, I_m$. After a suitable permutation of the index set, we can write the matrix $\bigl(\partial_i(\mathrm{Tr}_{\Lambda_j} \circ \omega)(s_0)\bigr)_{j,i=1,\ldots,n}$ in block form relative to $I_1, I_2, \ldots, I_m$. Due to (12), we get an upper triangular matrix with diagonal elements $c_{\Lambda_1}^{(1)}, \ldots, c_{\Lambda_n}^{(n)}$. Therefore $D(s_0) = c_{\Lambda_1}^{(1)} c_{\Lambda_2}^{(2)} \cdots c_{\Lambda_n}^{(n)}$.

Let $j \in I$. The element $c_{\Lambda_j}^{(j)}$ is the trace of the linear map $\pi_{\Lambda_j}(e_j)\pi_{\Lambda_j}(n_j)\colon L(\Lambda_j)_{\Lambda_j} \to L(\Lambda_j)_{\Lambda_j}$. The space $L(\Lambda_j)_{\Lambda_j}$ is spanned by a highest weight vector v. Because the α_j-string of Λ_j contains only the elements $\Lambda_j, \Lambda_j - \alpha_j$, we get:

$$e_j n_j v = e_j \exp(-e_j) f_j v = \exp(-e_j)(e_j f_j - f_j e_j) v = v.$$

Therefore $c_{\Lambda_j}^{(j)} = 1$. ∎

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