Canad. Math. Bull. Vol. **22** (3), 1979

PERMUTING THE ELEMENTS OF A FINITE SOLVABLE GROUP†

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ABSTRACT. The main result in this note is the following

THEOREM. *Let G be a finite solvable group. There exists a permutation σ of the set G such that $\{g \cdot \sigma(g);\ g \in G\} = G$ if and only if the Sylow 2-subgroup of G is non-cyclic or trivial.*

§1. **Introduction.** Let G be a finite group, σ a permutation of the set G and $C_\sigma = \{g \cdot \sigma(g);\ g \in G\}$. We say σ covers G if $C_\sigma = G$. In conversation with the second author D. Solitar has raised the problem of deciding which finite groups can be covered by suitable σ. If G has a non-trivial Sylow 2-subgroup that is cyclic then no permutation of G can cover G (Lemma 5). We conjecture that G can be covered in all other cases and prove this for solvable groups. Of course if the order of G is odd then the identity permutation covers G. In general, however, a group may be covered by some permutation but not by any automorphism of G. We do not know of any non-abelian simple group that can be covered by an automorphism. The explicit statement of the main result is as follows.

THEOREM. *Let G be a finite solvable group. If the Sylow 2-subgroup of G is trivial or non-cyclic, then there exists a permutation σ of the set G such that $\{g \cdot \sigma(g);\ g \in G\} = G$. If the Sylow 2-subgroup of G is non-trivial cyclic, then $\{g \cdot \sigma(g);\ g \in G\} \neq G$ for any permutation σ of G.*

§2. **Proofs.** We begin with a few observations and the proof of the main result for some special cases.

LEMMA 1. *If a permutation σ covers G then there exists a permutation σ' of G that covers G and $\sigma'(e) = e$.*

Proof. Define $\sigma'(g) = \sigma(g) \cdot (\sigma(e))^{-1}$. Then

$$G = \{g \cdot \sigma(g);\ g \in G\} = \{g \cdot \sigma(g) \cdot (\sigma(e))^{-1};\ g \in G\}.$$

CONVENTION. If σ covers G then we shall assume $\sigma(e) = e$.

Received by the editors May 18, 1978.
† Research partially supported by the National Research Council of Canada.
AMS(MOS) *subject classification* (1970). Primary: 20D10.
Key words and phases: Finite solvable group, permutation.

LEMMA 2. *If $H \lhd G$ and both H and G/H can be covered then so can G.*

Proof. $H=\{e, h_2, \ldots, h_r\}=C_\sigma$, σ a permutation of H. $G/H=\{He, Hg_2, \ldots, Hg_s\}=C_\pi$, π a permutation of G/H. Define a permutation ρ of G as follows: $\rho(hg_i)=(\sigma(h))^{g_i}\pi(g_i)$ where $h \in H$ and $\pi(g_i)$ is the coset representative of $\pi(Hg_i)$. Then $C_\rho=\{hg_i \cdot (\sigma(h))^{g_i}\pi(g_i)\}=\{h\sigma(h) \cdot g_i\pi(g_i)\}=G$.

LEMMA 3. *The following groups can be covered.* (i) $C_2 \times C_2$; (ii) $C_2 \times C_2 \times C_2$; (iii) $C_{2^n} \times C_2$, $n>1$ *where C_r is the cyclic group of order r.*

Proof. (i) Let $G=\langle a\rangle \times \langle b\rangle$ where $\langle a\rangle$ and $\langle b\rangle$ are cyclic of order two. Take σ to be the automotphism: $\sigma(a)=b$, $\sigma(b)=ab$. Then $C_\sigma=\{e \cdot e,\ a \cdot b,\ b \cdot ab,\ ab \cdot a\}=G$.

(ii) Let $G=\langle a\rangle \times \langle b\rangle \times \langle c\rangle$ where $\langle a\rangle$, $\langle b\rangle$, $\langle c\rangle$ are cyclic of order two. Then the automorphism σ given by $\sigma(a)=b$, $\sigma(b)=c$, $\sigma(c)=ac$ covers G.

(iii) Let $G=\langle a\rangle \times \langle b\rangle$ where $\langle a\rangle$ is cyclic of order 2^n, $n>1$, and $\langle b\rangle$ is cyclic of order two. Write c to denote $a^{2^{n-1}}$ and consider the map σ of G given by: $\sigma(e)=e$,

$$\sigma(a^r)=\begin{cases} a^{r-1}cb; & r=1, \ldots, 2^{n-2}. \\ a^{r-1}c; & r=2^{n-2}+1, \ldots, 2^{n-1}. \\ a^r c; & r=2^{n-1}+1, \ldots, 3 \cdot 2^{n-2}-1. \quad * \\ a^r cb; & r=3 \cdot 2^{n-2}, \ldots, 2^n-1. \end{cases}$$

$$\sigma(a^r b)=\begin{cases} a^r b & r=1, \ldots, 2^{n-2}-1. \quad * \\ a^r; & r=2^{n-2}, \ldots, 2^{n-1}-1. \\ b; & r=2^{n-1}. \\ a^{r-1}; & r=2^{n-1}+1, \ldots, 3 \cdot 2^{n-2}. \\ a^{r-1}b; & r=3 \cdot 2^{n-2}+1, \ldots, 2^n. \end{cases}$$

It is straightforward to verify that σ is a permutation of G and $C_\sigma=G$. Note that unlike cases (i) and (ii), no automorphism τ of G can cover G. For $\tau(c)=\tau(a^{2^{n-1}})=(\tau(a))^{2^{n-1}}=c$, and hence $c\tau(c)=e=e \cdot \tau(e)$.

LEMMA 4. *Let $G=\langle a, b\rangle$ be a non-abelian 2-group such that $A=\langle a\rangle$ is normal in G and $b^2 \in A$. Then G can be covered.*

Proof. From the hypotheses, $\langle a\rangle$ is cyclic of order 2^n, $n>1$, and $a^b=a^{-1}$ or $a^{2^{n-1}\pm 1}$. Define a permutation σ of G as follows:

$$\sigma(a^r)=\begin{cases} a^r; & r=0, 2, \ldots, 2^{n-1}-2,\ 2^{n-1}+1,\ 2^{n-1}+3, \ldots, 2^n-1. \\ a^r b; & r=1, 3, \ldots, 2^{n-1}-1,\ 2^{n-1},\ 2^{n-1}+2, \ldots, 2^n-2. \end{cases}$$

$$\sigma(a^r b)=\begin{cases} a^{2^{n-1}-1-r}; & r=0, 2, \ldots, 2^{n-1}-2,\ 2^{n-1}+1,\ 2^{n-1}+3, \ldots, 2^n-1. \\ a^{2^{n-1}-1-r}b; & r=1, 3, \ldots, 2^{n-1}-1,\ 2^{n-1},\ 2^{n-1}+2, \ldots, 2^n-2. \end{cases}$$

* These two lines do not occur if $n=2$.

Once again, it can easily be verified that σ is a permutation of G and $C_\sigma = G$. Note again that no automorphism τ of G can cover G since $\tau(a^{2^{n-1}}) = a^{2^{n-1}}$.

LEMMA 5. *If G is a group of even order and the Sylow 2-subgroup of G is cyclic, then no permutation of G can cover G.*

Proof. In any abelian group A of even order containing a unique element t or order two, the product $\prod_{a \in A} a$ of all the elements of A equals t. Under the hypothesis of the Lemma, Burnside's Theorem [1, Theorem 7.6.1] implies that G' is of odd order and G/G' is of even order containing a unique element of order two. Thus $\prod_{g \in G} g \notin G'$ but $(\prod_{g \in G} g)^2 \in G'$. Thus $C_\sigma \neq G$ for any permutation σ of G.

LEMMA 6. *Every non-cyclic 2-group G can be covered.*

Proof. Assume, by way of induction, that every non-cyclic 2-group of order less than $|G|$ can be covered. By Lemma 2 we can assume that for every proper, non-trivial normal subgroup H of G either H or G/H is cyclic. The only abelian groups G with this property are the ones in the class $C_2 \times C_2 \times C_2$ or $C_{2^n} \times C_2$ and Lemma 3 provides the result.

In the non-abelian case, G/G^2 is not cyclic, where $G^2 = \langle g^2; g \in G\rangle$. Therefore G^2 is cyclic. Similarly $Z(G)$ is cyclic. We claim that $Z(G) \leq G^2$; if not, $Z(G)G^2$ cannot be cyclic, so $G/Z(G)G^2$ is cyclic, generated by the image of some $g \in G$. Then $G = \langle g, Z(G), G^2\rangle = \langle g, Z(G)\rangle$, since G^2 is the Frattini subgroup of G. But then G is abelian, which is not so.

Let A be a maximal cyclic subgroup of G containing G^2. Then G/A is either cyclic of order two or elementary abelian of order four. In the former case use Lemma 4 to obtain the result. The latter case does not occur. To see this, suppose that G/A is elementary abelian of order four. Then $G = \langle a, b, c\rangle$, with b^2, c^2, and $(bc)^2$ in $A = \langle a\rangle$. A cannot be central, since $Z(G) \leq G^2$ and A is maximal cyclic. Now $\langle a^2, b\rangle$ and $\langle a^2, c\rangle$ must be cyclic, since the corresponding factor groups are not. Therefore precisely one of b, c, bc centralizes a, say b, and $a^c = at$ where t is the unique involution of $\langle a\rangle$. If $b^c \neq b$, then $b \in Z(G) \leq A$. If $b^c \neq b$, then $b^c = bt$, since $\langle c, a^2\rangle$ is cyclic, so $(ab) \in Z(G) \leq A$, and $b \in A$. This contradiction completes the proof.

Proof of the theorem. The second part is covered by Lemma 5. To show the first part, let G be a minimal counterexample and let S be a Sylow 2-subgroup of G. By hypothesis S is non-cyclic and by minimality of G and Lemma 2, G has no normal subgroup of odd order. Let A be the maximal normal 2-subgroup of G and let B/A be the maximal normal subgroup of G/A of odd order. Then by the Schur–Zassenhaus Theorem [1, Theorem 6.2.1], B is the split extension of A by K where $|K|$ is odd. A is not cyclic since G has no normal subgroup of odd order. Thus by Lemmas 6 and 2, B can be covered. Hence $B \neq G$. Now the Sylow 2-subgroup of G/B must be non-trivial cyclic for

https://doi.org/10.4153/CMB-1979-040-6 Published online by Cambridge University Press

otherwise, by our choice of G, G/B and hence G can be covered. By Burnside's Theorem, G/B has a normal 2-complement. Since G/B has no normal subgroup of odd order, G/B is a cyclic 2-group. Thus G has the following proper invariant series: $\langle e\rangle < A < AK < G$ with A a non-cyclic 2-group, $B = AK$, K a subgroup of odd order, G/B a cyclic 2-group and $S = \langle A, x\rangle$ where S is the Sylow 2-subgroup of G. Now by the conjugacy part of the Schur–Zassenhaus Theorem, $K^x = K^a$ for some $a \in A$. Let $c = xa^{-1}$. Then $S = \langle A, c\rangle$ and $K^c = K$.

Since S is non-cyclic, it can be covered by a permutation τ. Every element $g \in G$ can be represented uniquely in the form $g = sk$ for some $s \in S$, $k \in K$; and every $s \in S$ can be represented uniquely in the form $s = ac^i$, $a \in A$, $0 \le i < m$ where m is the index of A in S. Define a permutation σ of G as follows: for $g = sk^{-1}$, $s \in S$, $k \in K$ then $\sigma(g) = k\tau(s)k^{c^n}$ where $\tau(s) \in Ac^n$, $0 \le n < m$.

We now show that σ is one-to-one. Suppose that $\sigma(s_1k_1^{-1}) = \sigma(s_2k_2^{-1})$. Then $k_1(a_1c^r)k_1^{c^r} = k_2(a_2c^s)k_2^{c^s}$ where $\tau(s_1) = a_1c^r$, $\tau(s_2) = a_2c^s$; $0 \le r \le s < m$. Thus $k_1a_1k_1c^r = k_2a_2k_2c^s$. From this it follows that $s = r$ and $k_1^2 = k_2^2$. But $|K|$ is odd, thus $k_1 = k_2$ and so $a_1 = a_2$ and $s_1k_1^{-1} = s_2k_2^{-1}$.

To see that the map $g \to g\sigma(g)$ is one-to-one, suppose that $s_1k_1^{-1}\sigma(s_1k_1^{-1}) = s_2k_2^{-1}\sigma(s_2k_2^{-1})$. Then $s_1a_1k_1c^r = s_2a_2k_2c^s$ where $\sigma(s_1) = a_1c^r$, $\sigma(s_2) = a_2c^s$, $0 \le r \le s < m$. Thus $k_1^{c^r} = k_2^{c^s}$ and $s_1a_1c^r = s_2a_2c^s$ or $s_1\tau(s_1) = s_2\tau(s_2)$. Since τ covers S, $s_1 = s_2$ and hence $r = s$ and $k_1 = k_2$. Thus σ covers G. This completes the proof.

Reference

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