ON A RESULT OF M. HEINS

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Some years ago Heins (**1**) proved that a Riemann surface which can be conformally imbedded in every closed Riemann surface of a fixed positive genus g is conformally equivalent to a bounded plane domain. In the proof the main effort is required to prove that a surface satisfying this condition is schlichtartig. Heins gave quite a simple proof of the remaining portion (**1**; Lemma 1). The main part of the proof depended on exhibiting a family of surfaces of genus g such that a surface which could be conformally imbedded in all of them was necessarily schlichtartig. Another proof using a different construction was recently given by Rochberg (**2**). We will give here a further proof based on the method of the extremal metric and using a further construction which is in some ways more direct than those previously given.

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Theorem 1. *A Riemann surface $\mathscr{R}$ which can be conformally imbedded in every closed Riemann surface of a fixed positive genus g is schlichtartig.*

We begin the proof by constructing a family of closed Riemann surfaces of genus g. Let S denote the domain on the sphere bounded by the circles $|z-(4n+1)| = 1$, $|z+(4n+1)| = 1$, $n = 1, \ldots, g$. Let $\Delta_t^{(n)}$ denote the ring domains $t<|z_n|<1$, $n = 1, \ldots, g$. Let $\mathscr{S}_t$ be the Riemann surface whose elements are the points of Cl S and $\Delta_t^{(n)}$, $n = 1, \ldots, g$, with topology to correspond to identifying $z = -(4n+1)+e^{i\theta}$ with $z_n = -te^{-i\theta}$, $0 \leqq \theta < 2\pi$ and $z = (4n+1)+e^{i\theta}$ with $z_n = e^{i\theta}$, $0 \leqq \theta < 2\pi$ and with the evident choice of local uniformising parameters so that S, $\Delta_t^{(n)}$, $n = 1, \ldots, g$, are conformally imbedded in $\mathscr{S}_t$ and the curves $\mathscr{S}_t$ corresponding to the boundary circles on S are analytic. The domains on $\mathscr{S}_t$ corresponding to the $\Delta_t^{(n)}$, $n = 1, \ldots, g$, will be called handles, the curves in them corresponding to $|z_n| = r$, $n = 1, \ldots, g$, $t<r<1$, level lines.

If $\mathscr{R}$ were not schlichtartig it would have a finite subsurface of positive genus $\hat{\mathscr{R}}$ with analytic boundary curves c_j, $j = 1, \ldots, N$. Let c_j', $j = 1, \ldots, N$, be disjoint analytic curves on $\hat{\mathscr{R}}$ so that c_j, c_j' bound a doubly-connected subdomain D_j of $\hat{\mathscr{R}}$ and c_j', $j = 1, \ldots, N$, bound a finite surface $\mathscr{R}^*$ in $\hat{\mathscr{R}}$. If $\mathscr{R}$ is conformally imbedded in $\mathscr{S}_t$ we may use all the same terms for the images there. It is clear that the module of a doubly-connected domain non-trivially imbedded in $\hat{\mathscr{R}}$ is bounded.

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The level lines λ on each handle belong to one of the disjoint categories given by the following conditions:

(i) λ contains an open subarc lying in and joining distinct boundary components of some D_j, $j = 1, \ldots, N$;

(ii) λ lies in $\hat{\mathscr{R}}$;

(iii) λ does not meet $\mathscr{R}^*$.

If each handle contained a level line in (iii), $\mathscr{R}^*$ would lie in a schlichtartig subdomain of $\mathscr{S}_t$ and thus be itself schlichtartig, a contradiction. Thus there is at least one handle, say $\Delta_t^{(n)}$, for which the level lines all belong to (i) or (ii). Those in (i) can be distributed into disjoint classes Λ_j, $j = 1, \ldots, N$, such that for $\lambda \in \Lambda_j$, λ contains an open subarc in D_j joining the two boundary components. *A priori* a certain ambiguity might be present but since c_j, c_j', $j = 1, \ldots, N$, are analytic the choices could be made so that each Λ_j would be composed of level curves corresponding to values of r in a finite number of intervals in $(t, 1)$. On a set T of level curves with the corresponding set of values of r, τ, we have the logarithmic measure $L(T) = \int_\tau r^{-1} dr$. Let M_j, $j = 1, \ldots, N$, be the module of D_j. At the points in $\lambda \in \Lambda_j$, we define the metric $\rho_j(z)|\,dz\,|$ by $\rho_j(z)|\,dz\,| = (L(\Lambda_j))^{-1}\,|\,z_n\,|^{-1}\,|\,dz_n\,|$ provided Λ_j is not void. Then defining on D_j the metric $\rho(z)|\,dz\,|$ by

$$\begin{aligned} \rho(z) &= \rho_j(z), \quad z \in \lambda,\ \lambda \in \Lambda_j \\ &= 0, \qquad \text{otherwise} \end{aligned}$$

we obtain an admissible metric for the problem determining the module M_j of D_j. Thus

$$M_j \leqq 2\pi (L(\Lambda_j))^{-1}$$

or (a result trivially valid also if Λ_j is void)

$$L(\Lambda_j) \leqq 2\pi M_j^{-1}. \tag{1}$$

Adding (1) for $1 \leqq j \leqq N$ we have

$$L(\text{i}) \leqq 2\pi \sum_{j=1}^{N} M_j^{-1}$$

the right-hand side being a bound independent of t. Thus a set of level lines on $\Delta_t^{(n)}$ of logarithmic measure at least

$$\log(1/t) - 2\pi \sum_{j=1}^{N} M_j^{-1}$$

consists of curves in $\hat{\mathscr{R}}$. Consider the components of the union of this set of level lines in $\hat{\mathscr{R}}$. Two level lines can belong to different components only if the domain they bound on $\Delta_t^{(n)}$ contains a boundary contour of $\hat{\mathscr{R}}$. Thus there are at most $N+1$ such components and one of them has logarithmic measure at least

$$(N+1)^{-1}\left(\log(1/t)-2\pi\sum_{j=1}^{N} M_j^{-1}\right).$$

For small positive t this gives a contradiction. Hence $\mathscr{R}$ must be schlichtartig.

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Both Heins and Rochberg remarked that for the result of Theorem 1 it is sufficient for $\mathscr{R}$ to admit an imbedding only in a certain subclass of closed Riemann surfaces of genus g. The same is evidently true in the present proof. Further, it is clear that in the construction of Section 2 we could have taken S as bounded by any $2g$ disjoint mutually exterior circles and could have allowed their radii to vary in diverse ways with t.

REFERENCES

(1) M. Heins, A problem concerning the continuation of Riemann surfaces, *Contributions to the Theory of Riemann Surfaces* (Annals of Mathematics Studies, No. 30, Princeton University Press, 1953), 55-62.

(2) R. Rochberg, Continuation of Riemann surfaces, *to appear*.

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